COFIBRANT SIMPLICIAL SETS

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Abstract.

1. Elegant Reedy categories

1.1. **Reedy categories.** Recall that a *Reedy structure* on a category $\mathcal{C}$ is given by: a pair of identity-on-objects subcategories $\mathcal{C}_-$ and $\mathcal{C}_+$ of $\mathcal{C}$ together with a degree function $|{-}|\colon \mathrm{ob}\,\mathcal{C} \to \mathbb{N}$ such that:

- For all non-identity $\sigma\colon c \to d$ in $\mathcal{C}_-$, we have $|c| > |d|$;
- For all non-identity $\delta\colon c \to d$ in $\mathcal{C}_+$, we have $|c| < |d|$.
- Every map $\gamma\colon c \to d$ of $\mathcal{C}$ admits a unique factorisation $\gamma = \delta\sigma$ where $\sigma \in \mathcal{C}_-$ and $\delta \in \mathcal{C}_+$.

1.2. **Presheaves and degenerate elements.** Given a presheaf X on a Reedy category $\mathcal{C}$, we will write $x\gamma$ for the action of a map $\gamma\colon d \to c$ of $\mathcal{C}$ on an element $x \in X(c)$; thus $x\gamma = (X\gamma)(x) \in X(d)$. We say that $x \in X(c)$ is *non-degenerate* if, whenever $x = y\sigma$ with $\sigma \in \mathcal{C}_-$, we have $\sigma = 1_c$, and say that it is *degenerate* if $x = y\sigma$ for some non-identity $\sigma\colon d \to c$ in $\mathcal{C}_-$. We write $X_{\mathrm{nd}}(c)$ and $X_{\mathrm{d}}(c)$ for the sets of non-degenerate and degenerate elements of $X(c)$.

1.3. **Elegant Reedy categories.** A Reedy category $\mathcal{C}$ is called *elegant* ([]) if, for every presheaf $X\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and $x \in X(c)$, there is a unique pair $(\sigma_x \in \mathcal{C}_-(d,c),\ \bar{x} \in X_{\mathrm{nd}}(d))$ with $x = \bar{x}\sigma_x$.

Proposition 1. *$\mathcal{C}$ is an elegant Reedy category if and only if every span of maps in $\mathcal{C}_-$ can be completed to a commutative square in $\mathcal{C}_-$ which is an absolute pushout in $\mathcal{C}$. It follows that every map in $\mathcal{C}_-$ is a split epimorphism.*

Proof. See Proposition 3.8 of [] and the remarks following. □

2. Categories of non-degenerate elements

2.1. **Degeneracy-reflecting maps.** It is immediate that a map of presheaves $f\colon X \to Y$ on a Reedy category $\mathcal{C}$ preserves degeneracy and reflects non-degeneracy, in the sense that $f(X_{\mathrm{d}}(c)) \subseteq Y_{\mathrm{d}}(c)$ and $f^{-1}(Y_{\mathrm{nd}}(c)) \subseteq X_{\mathrm{nd}}(c)$ for all $c \in \mathcal{C}$. We say that a map of presheaves $f\colon X \to Y$ on a Reedy category *reflects degeneracy* or *preserves non-degeneracy* if one of the two equivalent conditions

$$f(X_{\mathrm{nd}}(c)) \subseteq Y_{\mathrm{nd}}(c) \quad \text{and} \quad f^{-1}(Y_{\mathrm{d}}(c)) \subseteq X_{\mathrm{d}}(c)$$

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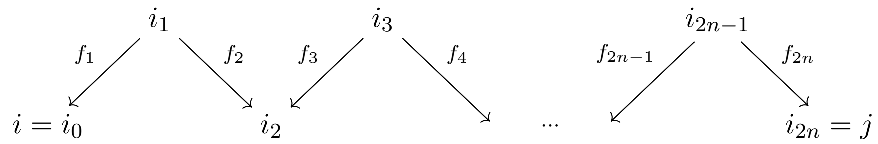
holds for all $c \in \mathcal{C}$. We write $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}}$ for the subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ containing every object but only the degeneracy-reflecting maps.

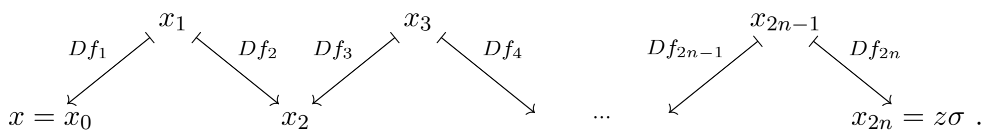
Proposition 2. *If $\mathcal{C}$ is an elegant Reedy category, then the inclusion functor $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ creates small colimits.*

Proof. Let $D\colon \mathcal{I} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}}$ be a diagram, and $(p_i\colon Di \to V)_{i \in \mathcal{I}}$ a colimiting cocone for D in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. We first show that this cocone lies in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}}$, which is to say that each p_i reflects degeneracies.

So let $x \in (Di)(c)$ be such that $p_i(x)$ is degenerate in $V(c)$; we will show that x is itself degenerate. Since $\mathcal{C}$ is elegant, we can write $p_i(x) = y\sigma$ for a unique non-identity $\sigma \in \mathcal{C}_-(d,c)$ and $y \in V_{\mathrm{nd}}(d)$. Since the p_i's are jointly surjective, there exists $j \in \mathcal{I}$ and $z \in (Dj)(d)$ such that $p_j(z) = y$; and thus we have $p_j(z\sigma) = p_j(z)\sigma = y\sigma = p_i(x) \in V(c)$. Due to the way that colimits are computed in $\mathbf{Set}$, this means that there exists a zig-zag



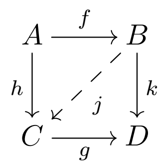
in $\mathcal{I}$ and elements $x_k \in Di_k$ with



But now as $x_{2n} = z\sigma$ is degenerate, and each Df_k preserves and reflects degeneracies, each x_k must be degenerate; in particular, $x = x_0$ is degenerate as required.

It remains to show that the cocone $(p_i\colon Di \to V)$ is colimiting in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}}$. Given any other cocone $(q_i\colon Di \to W)$ under D, we have, because V is colimiting in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, a unique induced map $q\colon V \to W$ with $q_k = qp_k$ for all $k \in \mathcal{I}$. It suffices to show that q in fact lies in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}}$. So let $x \in V(c)$ with $q(x) \in W(c)$ degenerate. Because the p_i's are jointly surjective, there is $j \in \mathcal{I}$ and $y \in (Dj)(c)$ with $p_j(y) = x$. Now $q_j(y) = qp_j(y) = q(x)$ is degenerate, and so y is too, since q_j reflects degeneracies. Thus also $x = p_j(y)$ is degenerate as required. □

2.2. **The Reedy factorisation system.** Recall that a pair of maps (f,g) in a category are said to be *orthogonal*—written $f \perp g$—if, for every commuting square as in the solid part of the diagram



there exists a unique filler $j\colon B \to C$ as indicated making both triangles commute. Given a class of maps $\mathcal{J}$, we write $\mathcal{J}^{\perp}$ for the class of maps k such that $j \perp k$ for all $j \in \mathcal{J}$; dually, we write ${}^{\perp}\mathcal{J}$ for the class of all k such that $k \perp j$ for all $j \in \mathcal{J}$. A pair of classes $(\mathcal{E}, \mathcal{M})$ is a *factorisation system* if $\mathcal{E} = {}^{\perp}\mathcal{M}$ and $\mathcal{M} = \mathcal{E}^{\perp}$ and every map factorises as an $\mathcal{E}$-map followed by an $\mathcal{M}$-map; such factorisations are necessarily unique up to unique isomorphism.

Proposition 3. *If $\mathcal{E}$ is a locally presentable category (so in particular, a presheaf category) and $\mathcal{J}$ is any small class of maps in $\mathcal{E}$, then $({}^{\perp}(\mathcal{J}^{\perp}), \mathcal{J}^{\perp})$ is a factorisation system.*

Proof. Well-known. □

Suppose now that $\mathcal{C}$ is a Reedy category. Applying the preceding result to the class of maps $\mathcal{J} = y(\mathcal{C}_-)$ (where $y\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding) yields a factorisation system $(\mathcal{E}, \mathcal{M})$ on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. Explicitly, this factorisation system $(\mathcal{E}, \mathcal{M})$ has classes

$$\mathcal{M} = \{g\colon X \to Y \mid y(\sigma) \perp g \text{ for all } \sigma \in \mathcal{C}_-\}$$
$$\mathcal{E} = \{f\colon X \to Y \mid f \perp g \text{ for all } g \in \mathcal{M}\}$$

Proposition 4. *When $\mathcal{C}$ is elegant, every $\mathcal{E}$-map is an epimorphism.*

Proof. Every map in $y(\mathcal{C}_-)$ is a split epimorphism. Since epis and monos are orthogonal in presheaf categories, every monomorphism is an $\mathcal{M}$-map; whence every $\mathcal{E}$-map is an epimorphism. □

Proposition 5. *For any Reedy category $\mathcal{C}$, the $\mathcal{M}$-maps in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ reflect degeneracies; if $\mathcal{C}$ is elegant, then every degeneracy-reflecting map is in $\mathcal{M}$.*

Proof. We first show that any $\mathcal{M}$-map $f\colon X \to Y$ reflects degeneracies. Let $x \in X(c)$ such that $f(x)$ is degenerate in $Y(c)$; thus $f(x) = z\sigma$ for some non-identity $\sigma \in \mathcal{C}_-(d, c)$ and $z \in Y(d)$, and so we have a commutative square

(*)
$$\begin{array}{ccc} y(c) & \xrightarrow{y(\sigma)} & y(d) \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle z} \\ X & \xrightarrow[f]{} & Y \end{array}$$

in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. Since $f \in \mathcal{M}$, there is a filler $w\colon y(d) \to X$, and so an element $w \in X(d)$ with $w\sigma = x$. Thus x is degenerate as required.

Suppose now that $\mathcal{C}$ is elegant; we show that every degeneracy-reflecting map $f\colon X \to Y$ is an $\mathcal{M}$-map. Thus, we must show that every diagram (*) admits a unique filler. Because $\mathcal{C}$ is elegant, $y(\sigma)$ is (split) epimorphic, and so uniqueness is forced; thus we need only show existence. Thus, given elements $x \in X(c)$ and $z \in Y(d)$ with $f(x) = z\sigma$, we must find $w \in X(d)$ with $w\sigma = x$ and $f(w) = z$. Since $\mathcal{C}$ is elegant, we can write $x = \bar{x}\tau$ and $z = \bar{z}\rho$ with $\tau, \rho \in \mathcal{C}_-$ and $\bar{x}, \bar{z}$ non-degenerate. Since f reflects degeneracies, it preserves non-degeneracies, and so $f(\bar{x})$ is non-degenerate; and now

$$\bar{z}\rho\sigma = z\sigma = f(x) = f(\bar{x})\tau$$

exhibits $f(x)$ as the image of a non-degenerate element in two different ways; whence, by elegance of $\mathcal{C}$, we have $f(\bar{x}) = \bar{z}$ and $\tau = \rho\sigma$. Now taking $w = \bar{x}\rho$, we have $w\sigma = \bar{x}\rho\sigma = \bar{x}\tau = x$ and $f(w) = f(\bar{x})\rho = \bar{z}\rho = z$, as required. □

2.3. **$[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}}$ is a presheaf category.** Let $\mathcal{C}$ be an elegant Reedy category, and let $\mathcal{K}$ be a set of isomorphism-class representatives of objects $X \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ which admit an $\mathcal{E}$-map from a representable; note that $\mathcal{K}$ is indeed only a set, since there are only a set of representables, each $\mathcal{E}$-map is an epimorphism, and a presheaf topos is well-copowered.

Proposition 6. *For any $K \in \mathcal{K}$ the functor $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](K, \text{–}) \colon [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$ preserves colimits of diagrams of $\mathcal{M}$-maps.*

Proof. Given a diagram $D \colon \mathcal{I} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ of $\mathcal{M}$-maps, let $(p_i \colon Di \to V)$ be a colimiting cocone; by Proposition 2, each p_i is again an $\mathcal{M}$-map. We must show that the induced map of sets

$$\mathrm{colim}_{i \in \mathcal{I}} [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](K, Di) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](K, V)$$

is invertible, or equally that the fibre of this map over each $f \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](K, V)$ is a singleton. This fibre is the set of connected components of the category $\mathbf{Fact}(f)$ whose objects are factorisations $(f = p_i g \colon K \to Di \to V)$ and whose morphisms $(p_i, g) \to (p_j, k)$ are commutative diagrams

$$\begin{tikzcd} & Di \arrow[dd, "Df"] \arrow[dr, "p_i"] & \\ K \arrow[ur, "g"] \arrow[dr, "k"'] & & V \\ & Dj \arrow[ur, "p_j"'] & \end{tikzcd} .$$

Thus we must show that $\mathbf{Fact}(f)$ is connected. Note that this will be so in the special case where $K = y(c)$, since then $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](y(c), \text{–})$ is the cocontinuous functor given by evaluation at c. For a general $K \in \mathcal{K}$, we first choose some $\mathcal{E}$-map $q \colon y(c) \twoheadrightarrow K$; now $\mathbf{Fact}(fq)$ is connected by the above, so it suffices to show that the functor $q^* \colon \mathbf{Fact}(f) \to \mathbf{Fact}(fq)$ sending (p_i, g) to (p_i, gq) is an isomorphism of categories. But given (h, p_i) in $\mathbf{Fact}(fq)$, we have a square as in the solid part of the diagram:

$$\begin{tikzcd} y(c) \arrow[r, "q"] \arrow[d, "h"'] & H \arrow[d, "f"] \arrow[dl, dashed, "j"] \\ Di \arrow[r, "p_i"'] & V \end{tikzcd} ;$$

since q is in $\mathcal{E}$, and p_i in $\mathcal{M}$, we conclude that there is a unique filler j making both triangles commute. So q^* is bijective on objects, and clearly is faithful; for fullness, suppose that (p_i, g) and $(p_j, k) \in \mathbf{Fact}(f)$ and that we have a commuting diagram

$$\begin{tikzcd} & Di \arrow[dd, "Df"] \arrow[dr, "p_i"] & \\ y(c) \arrow[ur, "gq"] \arrow[dr, "kq"'] & & V \\ & Dj \arrow[ur, "p_j"'] & \end{tikzcd} .$$

We must show that in fact $Df \circ g = k$. For this, it suffices to show the equality on precomposition with the $\mathcal{E}$-map q and postcomposition with the $\mathcal{M}$-map p_j; and the first of these is true by assumption, and the latter by the calculation $p_j \circ Df \circ g = p_i \circ g = f = p_j \circ k$. Thus q^* is fully faithful, and hence an isomorphism of categories as claimed. □

Proposition 7. *The set of objects $\mathcal{K}$ is dense in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}}$.*

Proof. Let $\mathcal{K} \downarrow X$ be the full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}}/X$ on objects of the form $(K \in \mathcal{K}, m\colon K \to X \in \mathcal{M})$, and let $\Phi\colon \mathcal{K} \downarrow X \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}}$ send (K, m) to K. We must show that the cocone $\theta\colon \Phi \Rightarrow \Delta X$ with $\theta_{(K,m)} = m$ is colimiting. Since $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{nd}}$ is closed under colimits in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, it suffices to show that the cocone is colimiting in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$; and as colimits in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ are pointwise, it suffices for this to show that $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](y(c), –)$ sends θ to a colimit in $\mathbf{Set}$.

As in the previous proof, this is equally to show that, for each map $f\colon y(c) \to X$, the category $\mathbf{Fact}(f)$ of factorisations of f through maps $m\colon K \to X$ in $\mathcal{K} \downarrow X$ is connected. To see this, form an $(\mathcal{E}, \mathcal{M})$ factorisation $f = me\colon y(c) \to K \to X$; we claim that (m, e) is an initial object in $\mathbf{Fact}(f)$, which immediately implies its connectedness. Indeed, for any $(m', e') \in \mathbf{Fact}(f)$, we have the diagram:

$$\begin{array}{ccccc} & & y(c) & & \\ & \swarrow^{e} & & \searrow^{e'} & \\ K & & \overset{j}{\dashrightarrow} & & K' \\ & \searrow_{m} & & \swarrow_{m'} & \\ & & X & & \end{array}$$

with e an $\mathcal{E}$-map and m an $\mathcal{M}$-map, whence there is a unique filler as indicated. Thus (m, e) is initial in $\mathbf{Fact}(f)$, which is therefore connected as required. $\square$

2.4. **The $\mathcal{E}$-map classifier.** Let $\mathcal{C}$ be an elegant Reedy category. For each object $X \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, write $\mathrm{Qu}(X)$ for the set of $\mathcal{E}$-quotients of X: isomorphism classes of $\mathcal{E}$-maps $q\colon X \twoheadrightarrow Q$, where we identify two such maps just when they are isomorphic in the coslice $X/\mathcal{C}$. Note that $\mathrm{Qu}(X)$ really is a set, as every $\mathcal{E}$-map is an epimorphism (by elegance) and a presheaf has only a mere set of epimorphic quotients. Now note that any $f\colon X \to Y$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ induces a function $\mathrm{Qu}(f)\colon \mathrm{Qu}(Y) \to \mathrm{Qu}(X)$ by sending $q\colon Y \twoheadrightarrow Q$ to the $\mathcal{E}$-part of the $(\mathcal{E}, \mathcal{M})$-factorisation of qf. In this way we obtain a functor

$$\mathrm{Qu}\colon [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{op}} \to \mathbf{Set} \ .$$

Proposition 8. Qu *preserves small limits.*

Proof. Given $D\colon \mathcal{I} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ a diagram and $(p_i\colon Di \to V)_{i \in \mathcal{I}}$ a colimiting cocone, we must show that the induced cone

$$\big(\mathrm{Qu}(p_i)\colon \mathrm{Qu}(V) \to \mathrm{Qu}(Di)\big)_{i \in \mathcal{I}}$$

is limiting. So suppose that we are given a compatible family of elements of the $\mathrm{Qu}(Di)$'s: thus we have $\mathcal{E}$-quotient maps $q_i\colon Di \twoheadrightarrow Qi$ for each $i \in \mathcal{I}$ such that, for each $f\colon i \to j$ in $\mathcal{I}$, we have a (necessarily unique) factorisation as on the left in:

$$\begin{array}{ccc} Di & \xrightarrow{Df} & Dj \\ {\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle q_j} \\ Qi & \underset{Qf \in \mathcal{M}}{\dashrightarrow} & Qj \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Di & \xrightarrow{p_i} & V \\ {\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle q} \\ Qi & \underset{\ell_i \in \mathcal{M}}{\dashrightarrow} & W \end{array} \ .$$

We must show that there is a unique $\mathcal{E}$-quotient $q\colon V \twoheadrightarrow W$ such that for each $i \in \mathcal{I}$, there is a (necessarily unique) factorisation as on the right above. For existence of

q, let $(\ell_i \colon Qi \to W)$ be a colimiting cocone for Q and let q be the unique map with $qp_i = \ell_i q_i$ for all $i \in \mathcal{I}$. Thus q is the colimit of the q_i's in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]^{\mathbf{2}}$, and so is an $\mathcal{E}$-map since each q_i is so. Moreover, because each Qf is in $\mathcal{M}$, so too is each ℓ_i by Propositions 2 and 5.

To show uniqueness of q, suppose that $q' \colon V \twoheadrightarrow W'$ is another $\mathcal{E}$-quotient for which we have factorisations $\ell'_i q_i = q' p_i$ with $\ell'_i \in \mathcal{M}$. It follows from the fact that each q_i is epimorphic that we have a cocone of maps $(p_i, \ell'_i) \colon q_i \to q'$; since q is a colimit of the q_i's, we induce a map

$$\begin{array}{ccc} V & = & V \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ W & \xrightarrow[h]{} & W' \end{array}$$

in $V/[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. Now h is an $\mathcal{E}$-map because q and q' are; but it also an $\mathcal{M}$-map, because each Qf and each ℓ'_i is in $\mathcal{M}$. It is therefore invertible and so $q = q'$ in $\mathrm{Qu}(V)$. □

Since Qu preserves small limits, it is representable; more explicitly, we have:

Proposition 9. Qu *is represented by the object* L *with* $L(c) = \mathrm{Qu}(y(c))$ *and* $L(\gamma) = \mathrm{Qu}(y(\gamma))$*; the natural isomorphisms*

$$\theta_X \colon [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](X, L) \to \mathrm{Qu}(X)$$

witnessing the representation send $f \colon X \to L$ *to the* $\mathcal{E}$*-part of its* $(\mathcal{E}, \mathcal{M})$ *factorisation. The inverse of* θ_X *sends an* $\mathcal{E}$*-quotient* $q \colon X \twoheadrightarrow Q$ *to the map* $X \to L$ *which sends* $x \in X(c)$ *to* $\mathrm{Qu}(\bar{x})(q)$*.*

Proof. Clearly the maps θ_X are natural in X; we must show they are invertible. Since both $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](-, L)$ and Qu send colimits in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ to limits in $\mathbf{Set}$, and since every presheaf is a colimit of representables, it suffices to show that each $\theta_{y(c)}$ is invertible with the given inverse. That inverse sends $q \in \mathrm{Qu}(y(c))$ to the map $\bar{q} \colon y(c) \to L$ corresponding to it under the Yoneda lemma; hence we must show that, for every $q \in \mathrm{Qu}(y(c))$, the $(\mathcal{E}, \mathcal{M})$ factorisation of $\bar{q} \colon y(c) \to L$ is of the form

$$\bar{q} = y(c) \xrightarrow{q} Q \xrightarrow{\ell} L \ .$$

First we show that $\bar{q}$ factors through q; since q is regular epi, this is equally to show that whenever $f, g \colon y(d) \rightrightarrows y(c)$ satisfy $qf = qg$, we also have $\bar{q}f = \bar{q}g$. But $\bar{q}f$ and $\bar{q}g$ are the elements of $\mathrm{Qu}(y(d))$ obtained as the $\mathcal{E}$-parts of the respective $(\mathcal{E}, \mathcal{M})$ factorisation of qf and qg; thus they will certainly agree if $qf = qg$. Thus we have a factorisation $\bar{q} = \ell q$; and it remains to show that ℓ is an $\mathcal{M}$-map.

So let $x \in Q(d)$ be a non-degenerate element. Since q is surjective, we can choose some $\gamma \in y(c)(d) = \mathcal{C}(d, c)$ with $q(\gamma) = x$. Now $\ell(x) = \bar{q}(\gamma) = \mathrm{Qu}(\gamma)(q) \in \mathrm{Qu}(y(d))$, and we have a commutative diagram as on the left in:

$$\begin{array}{ccc} y(d) & \xrightarrow{y(\gamma)} & y(c) \\ {\scriptstyle \ell(x)}\downarrow & & \downarrow{\scriptstyle q} \\ Q' & \dashrightarrow_{h \in \mathcal{M}} & Q \end{array} \quad \text{and} \quad \begin{array}{ccc} y(d) & \xrightarrow{y(\sigma)} & y(d') \\ {\scriptstyle \ell(x)}\downarrow & & \downarrow{\scriptstyle z} \\ Q' & = & Q' \end{array} .$$

Suppose that $\ell(x)$ were a degenerate element of L, $\ell(x) = z\sigma$, say. Then we would have a commutative diagram as on the right above; consequently, $\ell(x)$ sends $1_d \in y(d)(d)$ to a degenerate element of $Q'(d)$, and so the composite $q \circ y(\gamma)$ sends 1_d to a degenerate element of $Q(d)$; but $(q \circ y(\gamma))(1_d) = q(\gamma) = x$, contradicting non-degeneracy of x. □

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