Amplification of the Proof of Proposition 5 in "Variation through enrichment"

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There are more details needed to help the reader through the proof of Proposition 5 in the paper [2].

A fourth sentence is required in Lemma 4 so that it becomes:

Lemma 4. *Suppose* $F, G : \mathscr{A} \to \mathscr{B}$ *are monoid arrows in* $\mathscr{W}$-Mat(X, X) *and let* $H : \mathscr{B} \to \mathscr{C}$ *be the coequalizer of* F, G *in* $\mathscr{W}$-Mat(X, X). *The* $\mathscr{W}$*-graph* $\mathscr{C}$ *possesses a unique monoid structure such that* H *becomes a monoid arrow if and only if*

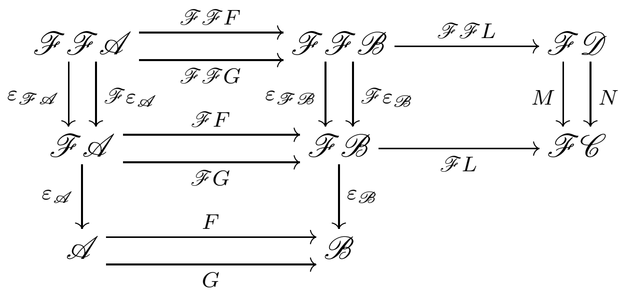
$$H \cdot \mu \cdot \mathscr{B}F = H \cdot \mu \cdot \mathscr{B}G \text{ and } H \cdot \mu \cdot F\mathscr{B} = H \cdot \mu \cdot G\mathscr{B} \,.$$

Furthermore, in this case, this monoid arrow is a coequalizer of F, G *in* $|\mathscr{W}$-Cat$|$. *If there exists an arrow* $T : \mathscr{B} \to \mathscr{A}$ *in* $\mathscr{W}$-Mat(X, X) *such that* $FT = 1_{\mathscr{B}} = GT$ *then the displayed equations above hold.*

The fourth sentence is proved by two similar calculations. Here is the calculation for the second displayed equation:

$$\begin{aligned} H \cdot \mu \cdot F\mathscr{B} &= H \cdot \mu \cdot FF \cdot \mathscr{A}T = H \cdot F \cdot \mu \cdot \mathscr{A}T \\ &= H \cdot G \cdot \mu \cdot \mathscr{A}T = H \cdot \mu \cdot GG \cdot \mathscr{A}T = H \cdot \mu \cdot G\mathscr{B} \,. \end{aligned}$$

The diagram in the Proof of Proposition 5 of [2] (top of page 115), in more detail, is as follows.



It would be hard for the reader to see why Lemma 4, in its old form, could be applied. The diagram suppresses the use of the underlying functor which is right adjoint to $\mathscr{F}$. So the $\mathscr{F}$ appearing is really the comonad generated by that adjunction. Thus the comultiplication $\delta : \mathscr{F} \to \mathscr{F}\mathscr{F}$ has $\varepsilon : \mathscr{F} \to 1$ as counit. So the parallel pair on the left side of the top right square have a common right inverse $\delta_{\mathscr{A}}$. Similarly for the top middle vertical parallel with common right inverse $\delta_{\mathscr{B}}$. Since δ is natural, we induce a common right inverse T to the parallel pair M, N of the right column. Now the revised Lemma 4 applies and the proof continues as stated.

Remark In [2] we refer to and adapt Harvey Wolff's proof of Proposition 2.11 in [1] but he does have some confusing mistakes apart from the fact that he is dealing with a symmetric closed monoidal category $\mathscr{V}$ and not a bicategory $\mathscr{W}$ as base. Corollary 2.9 (ii) about quasi-split seems false: step 4 of the displayed proof on page 131 is wrong. So things look bad for the proof of Proposition 2.11 since he quotes the quasi-split part of Corollary 2.9 on page 132. He says something is quasi-split by $UFU\eta_A$ which is not even well formed. It should be $UF\eta_{UA}$ and that satisfies the correct Corollary 2.9 (i).

References

[1] Harvey Wolf, V-cat and V-graph, *J. Pure Appl. Algebra* **4** (1974) 123–135.

[2] R. Betti, A. Carboni, R. Street and R. Walters, Variation through enrichment, *J. Pure Appl. Algebra* **29** (1983) 109–127; MR85e:18005.