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Ideals, Radicals, and Structure of Additive Categories

ROSS STREET
Mathematics Department, Macquarie University, New South Wales 2109, Australia (Email: street@mpce.mq.edu.au)

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Abstract. Simple and semisimple additive categories are studied. We prove, for example, that an artinian additive category is (semi)simple iff it is Morita equivalent to a division ring(oid). Semiprimitive additive categories (that is, those with zero radical) are those which admit a *noether full*, faithful functor into a category of modules over a division ringoid.

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The idea that additive categories are rings with several objects was developed convincingly by Barry Mitchell [8] who showed that it is unusual for a theorem of (non-commutative) ring theory not to carry over to additive categories. Here we would like to further argue that insight and efficiency (in concepts, statements, and proofs) are to be gained by dealing with additive categories throughout, and that familiar theorems for rings come out of the natural development of category theory. In other words, we attempt to apply additive category theory to ring theory rather than to generalize ring theory to additive categories. As a typical advantage to this approach we point to the fact that a ring and its category of finitely generated projective modules can be treated on an equal footing.

The radical of an additive category was defined by G. M. Kelly [7]. One of our purposes is to analyse this radical in more detail and to investigate its relation to a notion of semisimple additive category. An additive category is called semiprimitive when its radical is zero, and we provide a characterization of these categories in terms of a categorical concept called *noether fullness*. We were inspired by a preprint of Karlheinz Baumgartner [2], and, while we claim little in the present paper is really new, the results seem largely unknown and without a uniformly categorical published treatment.

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Let **Ab** denote the category of (small) abelian groups. All categories and functors will be additive (meaning **Ab**-enriched) without further mention. So a category with only one object amounts to a ring (with identity).

For any category $\mathcal{A}$, a *(right) $\mathcal{A}$-module* is a functor $M : \mathcal{A}^{op} \to \mathbf{Ab}$; if $f \in \mathcal{A}(A, B)$ and $m \in M(B)$, we write mf for $M(f)(m) \in M(A)$. We write $\mathcal{M}od\mathcal{A}$ for the category of $\mathcal{A}$-modules and natural transformations between them. There is a fully functor $\mathcal{Y}_{\mathcal{A}} : \mathcal{A} \to \mathcal{M}od\mathcal{A}$, called the Yoneda embedding, which takes each object X of $\mathcal{A}$ to the *representable* module $\mathcal{Y}_{\mathcal{A}}(X) = \mathcal{A}_X$ where $\mathcal{A}_X(A) = \mathcal{A}(A, X)$. Let $\mathcal{Q}\mathcal{A}$ denote the full subcategory of $\mathcal{M}od\mathcal{A}$ consisting of those modules which are retracts of finite direct sums of representable modules $\mathcal{A}_X, X \in \mathcal{A}$. A module M is in $\mathcal{Q}\mathcal{A}$ iff the representable functor $(\mathcal{M}od\mathcal{A})(M, -) : \mathcal{M}od\mathcal{A} \to \mathbf{Ab}$ preserves colimits. The category $\mathcal{Q}\mathcal{A}$ is called the *projective (or Cauchy) completion* of $\mathcal{A}$. If $\mathcal{A}$ is a ring, $\mathcal{Q}\mathcal{A}$ is the category of finitely generated, projective $\mathcal{A}$-modules. A category $\mathcal{A}$ is called *projectively complete* when it has finite direct sums and splittings for all idempotents; this holds iff $\mathcal{Y}_{\mathcal{A}} : \mathcal{A} \to \mathcal{Q}\mathcal{A}$ is an equivalence of categories. Clearly $\mathcal{Q}\mathcal{A}$ is projectively complete for all $\mathcal{A}$. Categories $\mathcal{A}, \mathcal{B}$ are called *Morita equivalent* when $\mathcal{M}od\mathcal{A}$ is equivalent to $\mathcal{M}od\mathcal{B}$. A basic Morita-style theorem (true very generally for enriched categories; see [9] for example) is that $\mathcal{A}, \mathcal{B}$ are Morita equivalent iff $\mathcal{Q}\mathcal{A}$ is equivalent to $\mathcal{Q}\mathcal{B}$. For any $\mathcal{A}$, clearly $\mathcal{A}$ and $\mathcal{Q}\mathcal{A}$ are Morita equivalent.

An object of a category $\mathcal{X}$ is called *artinian* (respectively, *noetherian*) when every descending (respectively, ascending) chain of subobjects is finite. Call $\mathcal{A}$ *artinian* when each $\mathcal{A}_B$ is an artinian object of $\mathcal{M}od\mathcal{A}$. In this case, each object of $\mathcal{Q}\mathcal{A}$ is an artinian $\mathcal{A}$-module, and $\mathcal{Q}\mathcal{A}$ is artinian.

For additive categories $\mathcal{A}, \mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is the additive category whose objects are pairs (A, B) of objects $A \in \mathcal{A}$, $B \in \mathcal{B}$, and whose homs are given by

$$(\mathcal{A} \otimes \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \otimes \mathcal{B}(B, B').$$

For each $\mathcal{A}$, there is a hom functor $\mathcal{H}_{\mathcal{A}} : \mathcal{A}^{op} \otimes \mathcal{A} \to \mathbf{Ab}$ given by $\mathcal{H}_{\mathcal{A}}(A, B) = \mathcal{A}(A, B)$. In other words, $\mathcal{H}_{\mathcal{A}}$ is an $(\mathcal{A} \otimes \mathcal{A}^{op})$-module.

Before discussing the radical, we make some remarks about general ideals. Given an object $X \in \mathcal{A}$, a *right X-ideal* R of $\mathcal{A}$ is a submodule of $\mathcal{A}_X \in \mathcal{M}od\mathcal{A}$. It can be regarded as a set R of arrows into X such that, if $f, g : A \to X$ are in R and $v : C \to A$, then $(f + g)v$ is in R. This agrees with the definition of right ideal when $\mathcal{A}$ is a ring. Note that $R = \mathcal{A}_X$ iff R contains a retraction.

An $\mathcal{A}$-module M is called *simple* (or "irreducible") when it has precisely two distinct submodules $0 \subseteq M$ and $M \subseteq M$. An $\mathcal{A}$-module M is called *semisimple* (or "completely reducible") when it is a direct sum of simple modules. Recall ([3] Ch. 1, Proposition 4.1) that a module is semisimple iff each submodule is a direct summand (the proof there works for $\mathcal{A}$-modules without change).

An *ideal* $\mathcal{K}$ in a category $\mathcal{A}$ is a submodule of $\mathcal{H}_{\mathcal{A}} \in \mathcal{M}od(\mathcal{A}^{op} \otimes \mathcal{A})$. We can identify an ideal $\mathcal{K}$ with the union of all the sets $\mathcal{K}(A, B)$ of arrows in $\mathcal{A}$; a set $\mathcal{K}$ of arrows in $\mathcal{A}$ is an ideal iff, for all $f, g : A \to B$ in $\mathcal{K}$, the arrow $u(f + g)v : X \to Y$ is in $\mathcal{K}$ for all $u : B \to Y$, $v : X \to A$. We recapture the submodule via $\mathcal{K}(A, B) = \mathcal{A}(A, B) \cap \mathcal{K}$. Each $\mathcal{K}(A, A)$ is an ideal of the endomorphism ring $\mathcal{A}(A, A)$. For any functor $T : \mathcal{A} \to \mathcal{B}$, the *kernel* ker$T$, consisting of the arrows f in $\mathcal{A}$ with $T(f) = 0$, is an ideal in $\mathcal{A}$. A functor is faithful iff its kernel is 0 (that is, contains only the zero arrows). If $T : \mathcal{A} \to \mathcal{B}$ is a functor then each ideal $\mathcal{L}$ in $\mathcal{B}$ has a *restriction ideal* $T^{-1}(\mathcal{L})$ in $\mathcal{A}$ with $f \in T^{-1}(\mathcal{L})$ iff $T(f) \in \mathcal{L}$.

A category $\mathcal{A}$ is called *simple* when it has precisely two ideals 0 and $\mathcal{A}$. Any nonzero full subcategory of a simple category is simple since any proper ideal in the subcategory would generate a proper ideal in the whole category.

PROPOSITION 1. *For any division ring D, the category VectD (= $\mathcal{M}odD$) of vector spaces over D is not simple; however, the (VectD)-module represented by D is simple.*

Proof. The ideal of $VectD$ consisting of the arrows with finite dimensional image is neither 0 nor $VectD$. A submodule R of the $(VectD)$-module M represented by D is a right D-ideal in $VectD$; if R contains a non-zero arrow $F : A \to D$ then f is a retraction and so $R = M$. □

PROPOSITION 2. *Restriction of ideals along $\mathcal{Y}_\mathcal{A} : \mathcal{A} \to \mathcal{Q}\mathcal{A}$ provides a bijection between ideals in $\mathcal{Q}\mathcal{A}$ and ideals in $\mathcal{A}$.*

Proof. Let $\mathcal{R}\mathcal{A}$ denote the category of modules which are finite direct sums of $\mathcal{A}_X$'s. Then restriction along $\mathcal{Y}_\mathcal{A} : \mathcal{A} \to \mathcal{R}\mathcal{A}$ provides a bijection between ideals in $\mathcal{R}\mathcal{A}$ and ideals in $\mathcal{A}$ [K; Lemma 2]. It remains to examine restriction under the inclusion $\mathcal{R}\mathcal{A} \hookrightarrow \mathcal{Q}\mathcal{A}$. Given an ideal $\mathcal{K}$ in $\mathcal{R}\mathcal{A}$, we extend it to an ideal $\mathcal{L}$ in $\mathcal{Q}\mathcal{A}$ by defining $f : M \to N$ in $\mathcal{Q}\mathcal{A}$ to be in $\mathcal{L}$ when $jfr \in \mathcal{K}$ where $j : N \to Y$ is a coretraction (= split monic), where $r : X \to M$ is a retraction (= split epic), and where $X, Y \in \mathcal{R}\mathcal{A}$. It is routine to check that this criterion is independent of the choice of r, j, and that $\mathcal{L}$ is the unique ideal in $\mathcal{Q}\mathcal{A}$ whose restriction is $\mathcal{K}$. □

The category $\mathcal{Q}D$ of finite dimensional D-vector spaces is thus simple since the division ring D is.

PROPOSITION 3. *An additive category $\mathcal{A}$ is simple iff it is Morita equivalent to a simple ring E. In fact, the ring E can be taken to be $\mathcal{A}(X, X)$ for any non-zero object X of $\mathcal{A}$. If E contains a minimal right ideal then $\mathcal{A}$ is Morita equivalent to the category $\mathcal{Q}D$ of finite dimensional vector spaces over a division ring D. Hence, a category is simple artinian iff it is Morita equivalent to a division ring.*

Proof. We shall use the following (general enriched) categorical lemma: *if $U : \mathcal{E} \to \mathcal{A}$ is a fully faithful functor whose composite $\mathcal{Y}_\mathcal{A} U : \mathcal{E} \to \mathcal{M}od\mathcal{A}$ is dense then $\mathcal{Q}U : \mathcal{Q}\mathcal{E} \to \mathcal{Q}\mathcal{A}$ is an equivalence of categories.* We shall supply a proof. The denseness of $\mathcal{Y}_\mathcal{A} U$ means that restriction along U^{op} is a fully faithful functor $R : \mathcal{M}od\mathcal{A} \to \mathcal{M}od\mathcal{E}$, while it is standard that left kan extension along a fully faithful U^{op} is a left adjoint functor $K : \mathcal{M}od\mathcal{E} \to \mathcal{M}od\mathcal{A}$ with $RK \cong 1$.

Since R is faithful, it follows that K is an inverse equivalence for R. By Morita theory, QU is an equivalence.

From Proposition 2 it follows that, if $\mathcal{B}$ is simple, so it $Q\mathcal{B}$. So "if" follows. Suppose $\mathcal{A}$ is simple and take any non-zero object X of $\mathcal{A}$. We prove that the representable module $\mathcal{A}_X$ is a generator in the category $\mathcal{M}od\mathcal{A}$; that is, if $\theta : M \to N$ is a non-zero module map then we must show there exists a map $\xi : \mathcal{A}_X \to M$ with $\theta\xi \neq 0$. By Yoneda's Lemma, this means we must show that the component $\theta_X : M(X) \to N(X)$ is non-zero. Let L be the image of θ; it is non-zero since θ is. Since $\mathcal{A}$ is simple, the kernel of $L : \mathcal{A}^{op} \to \mathbf{Ab}$ is zero; so L is faithful. Thus $X \neq 0$ implies $L(X) \neq 0$. So the image of θ_X is non-zero, as required.

Take $E = \mathcal{A}(X, X)$ which is a simple ring since it is a non-zero full subcategory of $\mathcal{A}$. We have a fully faithful functor $U : E \to \mathcal{A}$ which composes with $\mathcal{Y}_{\mathcal{A}}$ to give a fully faithful functor $T : E \to \mathcal{M}od\mathcal{A}$ picking out the $\mathcal{A}$-module $\mathcal{A}_X$; we have proved that the image of T is a (strongly) generating full subcategory. It follows from [4] that T is a dense functor. The above quoted lemma yields that E and $\mathcal{A}$ are Morita equivalent.

Consider the third sentence of the Proposition. A minimal right ideal I of E is a simple E-module and is easily shown to be finitely generated projective (see [6], p. 171–2). So $I \in QE$ and $D = (QE)(I, I) = (\mathcal{M}odE)(I, I)$ is a division ring by Schur's Lemma. By the above applied to QE in place of $\mathcal{A}$, we see that E and D, and hence $\mathcal{A}$ and D are Morita equivalent. The last sentence of the Proposition follows since $\mathcal{A}$ artinian implies E artinian, so certainly E has a minimal right ideal. □

It follows that, if $\mathcal{A}$ is simple artinian, then $Q\mathcal{A}$ is an abelian category in which all short exact sequences split. Also, in that case, by ([3], Theorem 4.2, p.11), every $\mathcal{A}$-module is projective and injective.

DEFINITION 1. The *radial* rad($\mathcal{A}$) of an additive category $\mathcal{A}$ consists of those arrows $f : A \to B$ in $\mathcal{A}$ such that $1_A - gf : A \to A$ is a retraction for all arrows $g : B \to A$.

This definition agrees with the Jacobson radical for a ring ([6] Theorem 4.1, p. 196).

PROPOSITION 4. *The radical is an ideal.*

Proof. First we show rad($\mathcal{A}$) is closed under addition. Take $f, f' : A \to B$ in rad($\mathcal{A}$) and take any $g : B \to A$. There exists h such that $(1_A - gf)h = 1_A$. But then

$$1_A - g(f + f') = (1_A - gf)(1_A - hgf')$$

is a composite of retractions and so a retraction. So $f + f'$ is in rad($\mathcal{A}$).

No trick is required to see that rad($\mathcal{A}$) is a left ideal. Take $f : A \to B$ in rad($\mathcal{A}$), $u : B \to C$, and $g : C \to A$. Then $1_A - g(uf) = 1_A - (gu)f$ is a retraction. So $uf \in \text{rad}(\mathcal{A})$.

To see that rad($\mathcal{A}$) is a right ideal take $f : A \to B$ in rad($\mathcal{A}$) and any $v : X \to A$. For any $g : B \to X$ we know that $1_A - vgf$ is a retraction; so we have h such that $(1_A - vgf)h = 1_A$. Then (motivated by a "geometric series formula" for h) we have

$$\begin{aligned}(1_X - gfv)(1_X + gfhv) &= 1_x - gfv + gfhv - gfvgfhv \\ &= 1_X - gfv + gf(1_A - vgf)hv \\ &= 1_X - gfv + gf1_Av \\ &= 1_X.\end{aligned}$$

So $1_X - gfv$ is a retraction. So $fv \in \text{rad}(\mathcal{A})$. □

Now we can show that our radical agrees with Kelly's ([7] Lemma 6) and is selfdual.

PROPOSITION 5. *An arrow* $f : A \to B$ *is in* rad($\mathcal{A}$) *iff* $1_A - gf$ *is invertible for all* $g : B \to A$.

Proof. If $f \in \text{rad}(\mathcal{A})$, we have $(1_A - gf)h = 1_A$ for some h. Then $h = 1_A + fh$. By Proposition 4, we have $-fh \in \text{rad}(\mathcal{A})$, so $1_A - (-fh) = h$ is a retraction. So h is a retraction and a coretraction. So h is invertible. So $1_A - gf = h^{-1}$ is invertible. The converse is obvious. □

For any ideal $\mathcal{K}$ in $\mathcal{A}$, we write $\mathcal{A}/\mathcal{K}$ for the category with the same objects as $\mathcal{A}$ and with homs given by the quotient groups $(\mathcal{A}/\mathcal{K})(A, B) = \mathcal{A}(A, B)/\mathcal{K}(A, B)$. There is a canonical quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{K}$ which is the identity on objects and takes each arrow to its $\mathcal{K}$-coset. The following characterization was taken as the definition in [7]; we include a proof for completeness.

PROPOSITION 6. *The radical* rad($\mathcal{A}$) *is the largest ideal of* $\mathcal{A}$ *such that the quotient functor* $\mathcal{A} \to \mathcal{A}/\text{rad}(\mathcal{A})$ *is conservative (= reflects isomorphisms). The quotient functor also reflects retractions and coretractions.*

Proof. We first prove that the quotient functor reflects retractions. Suppose f becomes a retraction in $\mathcal{A}/\text{rad}(\mathcal{A})$. Then there exists $g : B \to A$ such that $1 - fg$ is in rad($\mathcal{A}$). So $1 - (1 - fg)$ is a retraction. So fg is a retraction. So f is a retraction. Dually, the quotient functor reflects coretractions. As an arrow is invertible iff it is both a retraction and coretraction, the quotient functor reflects isomorphisms.

Suppose $\mathcal{K}$ is an ideal of $\mathcal{A}$ for which $\mathcal{A} \to \mathcal{A}/\mathcal{K}$ is conservative. We shall show that $\mathcal{K} \subseteq \text{rad}(\mathcal{A})$. Take $f \in \mathcal{K}$. We must see that $1 - gf$ is invertible for all g. Since $\mathcal{A} \to \mathcal{A}/\mathcal{K}$ is conservative, it suffices to see that $1 - gf$ becomes invertible in $\mathcal{A}/\mathcal{K}$. But $gf \in \mathcal{K}$ since $\mathcal{K}$ is an ideal. So 1 and $1 - gf$ become equal in $\mathcal{A}/\mathcal{K}$, yet 1 is already invertible in $\mathcal{A}$. □

PROPOSITION 7. *An arrow* $f : A \to B$ *is in* $\mathrm{rad}(\mathcal{A})$ *iff* $gf \in R(A)$ *for all objects* X*, all maximal right X-ideals R, and all arrows* $g : B \to X$.

Proof. Take $f \in \mathrm{rad}(\mathcal{A})$. Suppose R is a maximal right X-ideal and $g : B \to X$. Suppose $gf \notin R(A)$. Since R is maximal, R and gf generate $\mathcal{A}_X$. In particular, $1_X = gfk + r$ where $r \in R(X)$. By Propositions 4 and 5, $1 - gfk = r \in R(X)$ is invertible. So $R = \mathcal{A}_X$, a contradiction. So $gf \in R(A)$ as required.

Conversely, suppose $f \notin \mathrm{rad}(\mathcal{A})$. Then there exists $g : B \to A$ such that $1 - gf$ is not a retraction. Then $1 - gf$ generates a proper right A-ideal $S \subset \mathcal{A}_A$. By Zorn's Lemma, there is a maximal right A-ideal R containing S. Then $(1 - gf) + gf = 1 \notin R(A)$, so $gf \notin R(A)$. □

LEMMA 8. *A module M is simple iff* $M \cong \mathcal{A}_X/R$ *for some maximal right ideal* R.

Proof. Suppose M is simple. Then there exists a non-zero element $x \in M(X)$ for some X. The natural transformation $\xi : \mathcal{A}_X \to M$, determined by $\xi_X(1_X) = x$, is epic since M is simple. Let $R \subseteq \mathcal{A}_X$ be the kernel of ξ. If $R \subseteq S \subset \mathcal{A}_X$ then $\mathcal{A}_X/S \subseteq \mathcal{A}_X/R = M$; so the simplicity of M gives $S = R$. So R is maximal.

Conversely, suppose we have a short exact sequence

$$0 \to R \to \mathcal{A}_X \to M \to 0$$

with R maximal. Take any submodule N of M. The kernel S of the composite

$$\mathcal{A}_X \to M \to M/N$$

has $R \subseteq S \subseteq \mathcal{A}_X$, and so either $R = S$ or $S = \mathcal{A}_X$. So either $N = 0$ or $N = M$. So M is simple. □

PROPOSITION 9. *An arrow f is in* $\mathrm{rad}(\mathcal{A})$ *iff* $M(f) = 0$ *for all simple modules* M.

Proof. Suppose $\xi : \mathcal{A}_X \to M$ is epic with kernel R. For any arrow $f : A \to B$,

$$\begin{aligned} M(f) = 0 \quad &\text{iff} \quad M(f)\xi_B = 0 \\ &\text{iff} \quad \xi_A \mathcal{A}_X(f) = 0 \\ &\text{iff} \quad \mathcal{A}_X(f)(g) \in R(A) \text{ for all } g : B \to X \\ &\text{iff} \quad gf \in R(A) \text{ for all } g : B \to X. \end{aligned}$$

The result now follows from Proposition 7 and Lemma 8. □

PROPOSITION 10. *The restriction of the radical of* $\mathcal{QA}$ *is the radical of* $\mathcal{A}$*. That is,* $\mathrm{rad}(\mathcal{QA})$ *and* $\mathrm{rad}(\mathcal{A})$ *correspond under the bijection between ideals in Proposition 2.*

Proof. The Morita theory for enriched categories gives that restriction along $\mathcal{Y}_{\mathcal{A}} : \mathcal{A} \to \mathcal{Q}\mathcal{A}$ provides an equivalence of categories $\mathcal{M}od\mathcal{Q}\mathcal{A} \xrightarrow{\sim} \mathcal{M}od\mathcal{A}$. So simple $\mathcal{A}$-modules are all obtained as restriction of simple $\mathcal{Q}\mathcal{A}$-modules. So the result follows from Proposition 9. □

An $\mathcal{A}$-module M is called *faithful* when $M : \mathcal{A}^{op} \to \mathbf{Ab}$ is a faithful functor; that is, when $M(f) = 0$ implies $f = 0$.

DEFINITION 2. An additive category $\mathcal{A}$ is called *primitive* when it has a faithful simple module. It is called *semiprimitive* when, for each arrow $f \neq 0$ in $\mathcal{A}$, there exists a simple module M such that $M(f) \neq 0$.

The category $VectD$ of vector spaces of a division ring D is primitive but not simple (Proposition 1).

Suppose $(\mathcal{A}_\alpha : \alpha \in \Lambda)$ is a family of catgories all with the same set of objects. The *local product* of this family is the category $\mathcal{B}$ with the same objects as all the $\mathcal{A}_\alpha$ and with hom groups $\mathcal{B}(A, B)$ given by the product over $\alpha \in \Lambda$ of all the hom groups $\mathcal{A}_\alpha(A, B)$: composition is componentwise. There are local projection functors $P_\alpha : \mathcal{B} \to \mathcal{A}_\alpha$ which are the identity on objects. A category $\mathcal{A}$ is said to be a *local subproduct* of the family when there exists a faithful functor $\mathcal{A} \to \mathcal{B}$ which is the identity on objects and whose composite with each P_α is full. Clearly, $\mathcal{A}$ is a local subproduct of the family $(\mathcal{A}_\alpha : \alpha \in \Lambda)$ iff there exists a family $(\mathcal{K}_\alpha : \alpha \in \Lambda)$ of ideals of $\mathcal{A}$ whose intersection is 0 and for which there are isomorphisms $\mathcal{A}/\mathcal{K}_\alpha \cong \mathcal{A}_\alpha$.

PROPOSITION 11. *The following conditions on an additive category $\mathcal{A}$ are equivalent:*

(i) *$\mathcal{A}$ is semiprimitive;*
(ii) $\mathrm{rad}(\mathcal{A}) = 0$*;*
(iii) *$\mathcal{A}$ has a faithful semisimple module;*
(iv) *$\mathcal{A}$ is a local subproduct of primitive categories.*

Proof. Proposition 9 immediately shows the equivalence of (i) and (ii). For (i) ⇒ (iii), select a simple $\mathcal{A}$-module M_f for each non-zero arrow f in $\mathcal{A}$; then the direct sum of these M_f is semisimple and faithful. For (iii) ⇒ (iv), suppose $(M_\alpha : \alpha \in \Lambda)$ is a family of simple $\mathcal{A}$-modules whose direct sum M is semisimple and faithful. Let $\mathcal{A}_\alpha = \mathcal{A}/\ker M_\alpha$ and let $\mathcal{B}$ be their local product. Certainly each $\mathcal{A}$-module M_α induces a faithful simple $\mathcal{A}_\alpha$-module, so $\mathcal{A}_\alpha$ is primitive. The functor $\mathcal{A} \to \mathcal{B}$ determined by the quotient functors $\mathcal{A} \to \mathcal{A}_\alpha$ has the same kernel as M and so is faithful. Finally, for (iv) ⇒ (ii), suppose there exists a family $(\mathcal{K}_\alpha : \alpha \in \Lambda)$ of ideals of $\mathcal{A}$ whose intersection is 0 and such that each $\mathcal{A}/\mathcal{K}_\alpha$ has a faithful simple module M_α. If $f \in \mathrm{rad}(\mathcal{A})$ then it is taken to the radical of $\mathcal{A}/\mathcal{K}_\alpha$ by the canonical quotient functor. So $M_\alpha(f) = 0$ by Proposition 9, whence $f \in \mathcal{K}_\alpha$ since M_α is faithful. Since $f \in \mathcal{K}_\alpha$ for all α, we deduce that $f = 0$, as required. □

For any additive category $\mathcal{A}$, the category $\mathcal{A}/\mathrm{rad}(\mathcal{A})$ is semiprimitive, since $\mathrm{rad}(\mathcal{A}/\mathrm{rad}(\mathcal{A})) = 0$.

PROPOSITION 12. *Every simple category is primitive.*

Proof. If $\mathcal{A}$ is simple then it has a non-zero object A; then we can use Zorn's Lemma to obtain a maximal right A-ideal R. By Lemma 8, we have a simple $\mathcal{A}$-module $M = \mathcal{A}_A/R$ whose kernel is not the whole of $\mathcal{A}$. Since $\mathcal{A}$ is simple, $\ker M = 0$. So M is faithful and $\mathcal{A}$ is primitive. □

DEFINITION 3. A functor $T : \mathcal{C} \to \mathcal{X}$ is called *noether full* when, for all objects $A, B \in \mathcal{C}$, all noetherian subobjects V of $T(A) \in \mathcal{X}$, and all arrows $t : T(A) \to T(B)$ in $\mathcal{X}$, there exists an arrow $f : A \to B$ in $\mathcal{C}$ such that the restrictions of $T(f)$ and t to V are equal. Every full functor is noether full. Conversely, if each $T(A)$ is noetherian, then noether full implies full.

PROPOSITION 13. (i) *The radical of a product of categories is the product of the radicals.*

(ii) *Suppose $\mathcal{X}$ is a Grothendieck abelian category in which all monics split and every object is a filtered colimit of its noetherian subobjects. If $T : \mathcal{A} \to \mathcal{X}$ is noether full then the radical of $\mathcal{A}$ is contained in the restriction of the radical of $\mathcal{X}$.*

Proof. (i) Let $\mathcal{A}$ be the product of categories $\mathcal{A}_\alpha$. Let $f = (f_\alpha)$ be an arrow of $\mathcal{A}$. If each f_α is in $\mathrm{rad}(\mathcal{A}_\alpha)$ then clearly f is in $\mathrm{rad}(\mathcal{A})$. Conversely, if f is in $\mathrm{rad}(\mathcal{A})$, take any g in $\mathcal{A}_\alpha$ for which $1 - gf_\alpha$ makes sense. Define h in $\mathcal{A}$ to be 0 in all components except α where it is g. Then $1 - hf$ is invertible; so $1 - gf_\alpha$ is invertible. So $f_\alpha \in \mathrm{rad}(\mathcal{A}_\alpha)$.

(ii) Take $f \in \mathrm{rad}(\mathcal{A})$. We must prove $1 - T(f)h$ is monic for all h. Let V be any noetherian subobject of the domain of h. Then, using noether fullness, there exists g such that h and $T(g)$ have the same restriction to V. Then $1 - T(f)h$ and $T(1 - fg)$ have the same restriction to V. Since $1 - fg$ is invertible, certainly the restriction of $1 - T(f)h$ to V is monic. But the domain of h is the filtered colimit of such subobjects V. So $1 - T(f)h$ is indeed (split) monic. So $T(f) \in \mathrm{rad}(\mathcal{X})$. □

A *division ringoid* is an additive category in which each non-zero arrow is invertible. Each division ringoid $\mathcal{D}$ is equivalent to a coproduct (in the category of additive categories) of division rings; hence, $\mathcal{M}od\mathcal{D}$ is equivalent to a product of categories of vector spaces over division rings. Every division ringoid is artinian since the representable modules are simple.

The *weak product* of a family $(\mathcal{A}_\alpha : \alpha \in \Lambda)$ of additative categories $\mathcal{A}_\alpha$ with zero objects (but this time with no restriction that the object sets should be the same) is the full subcategory of the cartesian product

$$\prod_\alpha \mathcal{A}_\alpha$$

consisting of those objects $(A_\alpha : \alpha \in \Lambda)$ such that $A_\alpha = 0$ for all but a finite number of α.

Notice that, if $\mathcal{D}$ is a division ringoid, the projective completion $\mathcal{QD}$ of $\mathcal{D}$ is equivalent to the weak product of categories of finite dimensional vector spaces over division rings.

We now give the easy proof of our version of the so-called Chevalley–Jacobson Density Lemma ([5] Theorem 2.1.2, pp. 41–2), ([6] Chapter 4 Section 3).

PROPOSITION 14. *For any additive category $\mathcal{A}$, let $\mathcal{D}^{op}$ be a full subcategory of $\mathcal{M}od\mathcal{A}$ consisting of simple $\mathcal{A}$-modules. Then the functor $T : \mathcal{A}^{op} \to \mathcal{M}od\mathcal{D}$ given by $T(A)(M) = M(A)$ is noether full. If $\mathcal{A}$ is artinian then T is full and factors through $\mathcal{QD}$.*

Proof. Let $\mathcal{E}$ denote the full subcategory of $\mathcal{M}od\mathcal{A}$ consisting of the finite direct sums of modules in $\mathcal{D}$. Clearly $\mathcal{E}$ is the projective completion of the division ringoid $\mathcal{D}$; so restriction is an equivalence of categories $\mathcal{M}od\mathcal{E} \xrightarrow{\sim} \mathcal{M}od\mathcal{D}$, and we have $T : \mathcal{A}^{op} \to \mathcal{M}od\mathcal{E}$ given by the same formula as in the Proposition. Suppose $t : T(B) \to T(A)$ is an arrow of $\mathcal{M}od\mathcal{E}$ and V is a noetherian subobject of $T(B)$. Then there are modules $M_1, ..., M_n \in \mathcal{D}$ and elements $x_1 \in V(M_1), ..., x_n \in V(M_n)$ which generate V. Put $M = M_1 \oplus ... \oplus M_n \in \mathcal{E}$ and $x = x_1 + ... + x_n \in V(M_1) \oplus ... \oplus V(M_n) = V(M) \subseteq T(B)(M) = M(B)$. Let $\xi : \mathcal{A}_B \to M$ be the $\mathcal{A}$-module map determined (via Yoneda's Lemma) by $\xi_B(1_B) = x$. Let X be the image of ξ with inclusion $\imath : X \to M$. Clearly $X \in \mathcal{E}$, so we have a commutative square:

$$\begin{array}{ccc} X(B) & \xrightarrow{t_X} & X(A) \\ {\scriptstyle \imath_B}\downarrow & & \downarrow{\scriptstyle \imath_A} \\ M(B) & \xrightarrow[t_M]{} & M(A) \end{array}$$

it follows that $t_M(x) \in X(A)$. But $X(A)$ is the image of ξ_A, so there exists $f : A \to B$ such that $t_M(x) = \xi_A(f) = xf = T(f)(x)$. Since x generates V, it follows that t and $T(f)$ agree on V.

Now suppose $\mathcal{A}$ is artinian. We prove the last sentence of the Proposition by showing that each $T(A)$ is noetherian. If not, there is a strictly increasing infinite chain $V_1 \subset V_2 \subset V_3 \subset ...$ of noetherian subobjects of $T(A)$. Let R_n be the right A-ideal consisting of those arrows $f : X \to A$ with $V_n f = 0$. Since $\mathcal{A}$ is artinian, we have $R_n = R_{n+1}$ for some n. Since $V_n \neq V_{n+1}$, there exists $M \in \mathcal{D}$ and $x \in V_{n+1}(M)$ with $x \notin V_n(M)$. Let $t : T(A) \to T(A)$ be any $\mathcal{D}$-module map which has zero components at all $N \in \mathcal{D}$ not isomorphic to M, and whose component at M is zero on $V_n(M)$ but non-zero at $x \in V_{n+1}(M)$. Since T is noether full, there exists $f : A \to A$ such that $T(f)$ agrees with t on V_{n+1}. So

$xf \neq 0$ and $V_n f = 0$. So f is in R_n but not R_{n+1}, a contradiction. □

THEOREM 15. *An additive category $\mathcal{A}$ is semiprimitive iff there exists a division ringoid $\mathcal{D}$ and a faithful, noether full functor $\mathcal{A}^{op} \to \mathcal{M}od\mathcal{D}$. If $\mathcal{A}$ is primitive then there exists a division ring D and a faithful, noether full functor $\mathcal{A}^{op} \to VectD$.*

Proof. "If" follows from Propositions 1 and 13. Let $\mathcal{A}$ be semiprimitive and let $\mathcal{D}^{op}$ be the full subcategory of $\mathcal{M}od\mathcal{A}$ consisting of one representative for each isomorphism class of simple modules; so, by Schur's Lemma, $\mathcal{D}$ is a division ringoid. Let $T : \mathcal{A}^{op} \to \mathcal{M}od\mathcal{D}$ be the noether full functor of Proposition 14. By Proposition 9, T is also faithful. If $\mathcal{A}$ is primitive, we can take $D = \mathcal{D}$ in Proposition 14 to have a single faithful simple module as its only object. □

The additive category form of the Artin–Wedderburn Structure Theorem becomes:

THEOREM 16. *An additive category is semiprimitive artinian iff it is Morita equivalent to a division ringoid.*

Proof. Suppose $\mathcal{A}$ is semiprimitive artinian. We can suppose $\mathcal{A}$ is projectively complete since we are only interested in it up to Morita equivalence. From Proposition 14 and Theorem 15, we can suppose $\mathcal{A}$ is a full subcategory of a weak product $\mathcal{V}$ of categories QD_i where each D_i, $i \in I$, is a division ring. If there is an $i \in I$ such that the i-th component of A is zero for all $A \in \mathcal{A}$ then that i can be discarded and so we can assume that I contains no such elements. Let V_i be the object of $\mathcal{V}$ which is zero in all components except the i-th where it is D_i. There exists some $A \in \mathcal{A}$ such that the i-th component of A is non-zero. That component of A contains a 1-dimensional subspace as a retract. So A has V_i as a retract. Since $\mathcal{A}$ is projectively complete, $\mathcal{A}$ contains V_i; and since we have this for all i, it follows that $\mathcal{A} = \mathcal{V}$. This proves "only if", while "if" is clear. □

DEFINITION 4. An additive category is called *semisimple* when it is a local subproduct of a family of simple categories.

Every division ringoid $\mathcal{D}$ is semisimple. For, take Λ to be the set of connected components of $\mathcal{D}$ and, for each $\alpha \in \Lambda$, let $\mathcal{K}_\alpha$ be the ideal whose non-zero arrows are all those between objects not in the component α. Then $\mathcal{D}/\mathcal{K}_\alpha$ is equivalent to a division ring, and so simple; and the intersection of all the $\mathcal{K}_\alpha$ is zero.

From Definition 4 and Proposition 11, every semisimple category is certainly semiprimitive. Theorem 16 gives the converse for artinian categories.

PROPOSITION 17. *If a semisimple category satisfies the descending chain condition on ideals then it is a local product of a finite family of simple categories.*

Hence such a category is Morita equivalent to a finite product of categories of finite dimensional vector spaces over division rings.

Proof. We use an easy lemma of ([6], p. 202) (the proof there is quite categorical and applies to modules over a category as readily as over a ring): *If an artinian module is a subdirect product of simple modules then it is a finite direct sum of simple submodules.* Let $\mathcal{A}$ be our category satisfying the hypotheses of the Proposition. Then the $(\mathcal{A} \otimes \mathcal{A}^{op})$-module $\mathcal{H}_{\mathcal{A}} : \mathcal{A}^{op} \otimes \mathcal{A} \to \mathbf{Ab}$ satisfies the hypotheses of the quoted lemma since its submodules are the ideals of $\mathcal{A}$. So $\mathcal{H}_{\mathcal{A}}$ is a finite direct sum of simple modules. The conclusion of the lemma gives the first sentence of our Proposition. For the second sentence, observe that the local product is a full subcategory of the product, and we can proceed as in the proof of Theorem 16. □

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