FACTORIZATIONS IN BICATEGORIES

by

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Abstract. For bicategories, the notion of "monic" and "epic" maps are relativized to a suitable set of "weights" used as indexing types for limits. In this way, known factorization systems are recovered and conditions are given in order to analyse them by the process of taking a kernel followed by a quotient.

1. Introduction

Many functorial factorizations are known in *Cat* and in other bicategories (see e.g. Street and Walters [11], Johnstone [7] and [8], Bénabou [2], Bousfield [4], Makkai [9]). The aim of this paper is twofold: first, to investigate the common features of these factorizations, second to study the conditions under which they correspond to the general pattern, known from the 1-dimensional (regular) case (see [1]), of taking a "kernel" process followed by a "quotient".

In between the two processes, a notion of "monomorphism" and a correspondig notion of "strong epimorphism" remain defined. This is the first problem: what is it a "kernel", what a "quotient" and, mainly, which is the correct notion of monomorphism for an arrow in a bicategory?

It is known that in different bicategories, different and useful factorizations are used. In [5] e.g. the choice of the conservative arrows to mimic monomorphisms leads to the notion of "conservational bicategory", while in other cases the most natural generalization of the epi-mono factorization in *Cat* utilizes the fully-faithful functors in place of monomorphims.

Following an idea of the unpublished paper [10], the notion of monomorphism is here relativized to a suitable set $\mathcal{W}$ of functors $\alpha : \mathbf{C} \longrightarrow \mathbf{D}$ in *Cat*, called *weights*, which are used as indexing types for limits. In this way, one can also study how the notion of monomorphism varies when the set of weights is varying.

The set $\mathcal{W}$ of weights is regarded as a full sub-bicategory of $Hom(\mathbf{2}, Cat)$, the bicategory of homomorphisms, strong transformations and modifications, where **2** is the category with two objects $\{0, 1\}$ and one arrow $i : 0 \to 1$. The *kernel* of an arrow in a bicategory $\mathcal{X}$ is the functor $\mathcal{W}^{op} \to \mathcal{X}$, obtained by the limit process indexed by $\mathcal{W}$, monic arrows are defined by trivial kernels, and strong epic arrows by the *orthogonality condition* which is appropriate to bicategories.

We then define *quotients* by the corresponding colimits and prove that the process of taking kernels is right biadjoint to that of taking quotients:

$$Hom(\mathbf{2}, \mathcal{X}) \underset{Q}{\overset{K}{\rightleftarrows}} Hom(\mathcal{W}^{op}, \mathcal{X})$$

This is a basic result. Under the mild assumptions that $\mathcal{W}$ contains the representable weight $\mathbf{2}(0, -) : \mathbf{2} \to Cat$, any arrow can be factored canonically through the quotient of its kernel: lax descent results, such as those of Zawadowski [12] and [13] and of Makkai [9], can be understood in the context of this adjointness, with respect to a suitable bicategory $\mathcal{W}$ (see also Betti [3]).

Quotients are easily shown to be strong epic, but the second arrow in the canonical factorization needs not to be monic. This is true provided $\mathcal{X}$ satisfies suitable conditions which take into account a form of $\mathcal{W}$-*regularity* and moreover it satisfies a finitary condition with respect to $\mathcal{W}$. Under those assumptions, we have a factorization structure for the bicategory $\mathcal{X}$, which extends the regular one and comprehends known factorizations for Cat and other bicategories.

Two generalizations are possible (which however we don't follow): first to consider a monoidal 2-category instead of Cat, i.e. a base 2-category whose underlying category is equipped with a monoidal structure for which the tensor product is a 2-functor and associativity and unity conditions are 2-natural. However, contrary to this level of generality, here we deal as if bicategories were 2-categories, in order to avoid unnecessary coherence problems.

Yet another generalization regards the fact that the arrows in a bicategory $\mathcal{X}$ are usefully regarded as homomorphisms $\mathbf{2} \to \mathcal{X}$: in many cases, considering a general (small) bicategory $\mathbf{A}$ instead of $\mathbf{2}$, provided other aspects are accordingly adjusted, leads to general, still valid, properties.

The indexed limits and colimits of the paper are defined up to equivalence in $\mathcal{X}$ and should better be called bilimits and bicolimits: when convenient, we omit the prefix "bi". To fix terminology, next section introduces main definitions: as already said, for convenience we often suppress the 2-cells relating to associativity and unit laws.

The second author is indebted to R. Paré for pointing out early in this work that taking the quotient should be and indeed is left adjoint to taking the kernel.

2. Definitions relevant to bicategories and factorization structures

Recall that a *homomorphism* of bicategories $F : \mathcal{B} \to \mathcal{X}$ is given by a function on objects, together with functors

$$F_{a,b} : \mathcal{B}(a,b) \to \mathcal{X}(Fa, Fb)$$

(indexed by pairs of objects in $\mathcal{B}$) and with invertible 2-cells in $\mathcal{X}$:

$$I_{Fa} \cong F(I_a)$$

$$Fg \cdot Fh \cong F(g \cdot h)$$

(as usual, we omit subscripts when not necessary) respectively indexed by objects and composable pairs of arrows in $\mathcal{B}$. These invertible 2-cells are subject to appropriate functoriality and coherence conditions.

Given two homomorphisms $F, G : \mathcal{B} \to \mathcal{X}$, a *strong natural transformation* $\sigma : F \stackrel{\cdot}{\longrightarrow} G$ assigns to each object a of $\mathcal{B}$ an arrow $\sigma_a : Fa \to Ga$, and to each arrow $h : a \to b$ in $\mathcal{B}$ an invertible 2-cell σ_h as in the following diagram:

$$\begin{array}{ccc} Fa & \xrightarrow{\sigma_a} & Ga \\ {\scriptstyle Fh}\downarrow & \overset{\sigma_h}{\cong} & \downarrow{\scriptstyle Gh} \\ Fb & \xrightarrow{\sigma_b} & Gb \end{array}$$

These data are subject to three axioms, expressing the compatibility of σ_h and σ_k for a 2-cell $h \to k$ in $\mathcal{B}$, the behavior of σ on identities, and the behavior of σ with respect to composition of arrows in $\mathcal{B}$.

A *modification* $\mu : \sigma \to \tau : F \to G$ of strong natural transformations consists of 2-cells $\mu_a : \sigma_a \to \tau_a$ for each object a in $\mathcal{B}$, such that the following diagram of 2-cells commutes, for any $f : a \to b$ in $\mathcal{B}$

$$\begin{array}{ccc} Gh\cdot\sigma_a & \xrightarrow{Gh\cdot\mu_a} & Gh\cdot\tau_a \\ {\scriptstyle\sigma_h}\downarrow & & \downarrow{\scriptstyle\tau_h} \\ \sigma_b\cdot Fh & \xrightarrow{\mu_b\cdot Fh} & \tau_b\cdot Fh \end{array}$$

From bicategories $\mathcal{B}$ and $\mathcal{X}$ we can form the bicategory $Hom(\mathcal{B},\mathcal{X})$ of homomorphisms, strong natural transformations and modifications.

Given homomorphisms $\alpha : \mathbf{A}^{op}\times\mathcal{B}\to Cat$ and $F:\mathcal{B}\to\mathcal{X}$ (with $\mathcal{B}$ small, or finite, and in such cases we say that α is a *small, or finite, indexing type with extra variables* $\mathbf{A}$), the *bilimit of F indexed by* α, if it exists, is a homomorphism $\{\alpha,F\}:\mathbf{A}\to\mathcal{X}$ which birepresents

$$Hom(\mathbf{A}^{op}\times\mathcal{B},Cat)(\alpha,\mathcal{X}(-,F)):Hom(\mathbf{A},\mathcal{X})\to Cat$$

in the sense that, for any homomorphism $H:\mathbf{A}\to\mathcal{X}$, there is an equivalence of categories

$$Hom(\mathbf{A},\mathcal{X})(H,\{\alpha,F\})\simeq Hom(\mathbf{A}^{op}\times\mathcal{B},Cat)(\alpha,\mathcal{X}(H,F)).$$

More properly, the indexed bilimit is given by the pair $(\{\alpha,F\},p)$, where $p:\alpha\to Hom(\mathbf{A},\mathcal{X})(\{\alpha,F\},F)$ is the unit of the above birepresentation.

Bicolimits in $\mathcal{X}$ are just bilimits in $\mathcal{X}^{op}$, the bicategory obtained by reversing all arrows of $\mathcal{X}$. The *bicolimit of G: $\mathcal{B}\to\mathcal{X}$ indexed by $\beta:\mathcal{B}^{op}\times\mathbf{A}\to Cat$* (with *extra variables in* $\mathbf{A}$) is denoted by $\beta * G$, and is a birepresentation:

$$Hom(\mathcal{B}^{op}\times\mathbf{A},Cat)(\beta,\mathcal{X}(G,K))\simeq Hom(\mathbf{A},\mathcal{X}(\beta * G,K))$$

for every $K:\mathbf{A}\to\mathcal{X}$, with counit $q:\beta\to Hom(\mathbf{A},\mathcal{X}(G,\beta * G))$.

We now give the definition of a *factorization structure*, which extends the known one to bicategories, as in Johnstone [6].

A *factorization structure* in a bicategory $\mathcal{X}$ consists of two classes of morphisms ($\mathcal{E}$,$\mathcal{M}$), both closed under composition and containing all equivalences, and such that

i) for every arrow $f:A\to B$ in $\mathcal{X}$ there exist $m\in\mathcal{M}$ and $e\in\mathcal{E}$, and an invertible 2-cell $m\cdot e\cong f$.

ii) The elements of $\mathcal{E}$ are *orthogonal* to those of $\mathcal{M}$, in the sense that given a square

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle h}\downarrow & \cong & \downarrow{\scriptstyle k} \\ A & \xrightarrow{m} & B \end{array}$$

commuting up to an invertible 2-cell $\alpha : ke \xrightarrow{\cong} mh$, with $m \in \mathcal{M}$ and $e \in \mathcal{E}$, there exist a *diagonal arrow* $t : Y \to A$, unique up to an invertible 2-cell, and invertible 2-cells $\beta : te \xrightarrow{\cong} h$ and $\gamma : k \xrightarrow{\cong} mt$, such that $\alpha = (m \cdot \beta)(\gamma \cdot e)$.

The following properties can be easily proved, as is usual for factorization structures in categories.

1. $\mathcal{E} \cap \mathcal{M}$ consists precisely of equivalences in $\mathcal{X}$.

2. Any arrow which is orthogonal to every $m \in \mathcal{M}$ is in $\mathcal{E}$.

3. The factorization $f \simeq m \cdot e$ is unique up to an equivalence which is uniquely determined up to a unique invertible 2-cell. In such cases, we loosely say "determined up to equivalence".

3. Kernels and monic maps

Let $\mathcal{X}$ be a finitely complete and cocomplete bicategory. This means that $\mathcal{X}$ admits indexed limits and colimits of homomorphisms $\mathbf{A} \to \mathcal{X}$ when $\mathbf{A}$ is a finite bicategory. In particular, for limits, this means that $\mathcal{X}$ has a biterminal object, bipullbacks (preserved by representables) and admits cotensoring by finite categories.

The notion of "kernel" depends on a suitable set $\mathcal{W}$ of "weights", i.e. arrows in Cat. More precisely, we consider a full sub-bicategory $\mathcal{W}$ of $Hom(\mathbf{2}, Cat)$, taking value in finite categories.

For any arrow $f : A \to B$ in $\mathcal{X}$, regarded as a homomorphism $f : \mathbf{2} \to \mathcal{X}$, consider the homomorphism $\{-, f\} : \mathcal{W}^{op} \to \mathcal{X}$ such that $\{\alpha, f\}$ is the limit of f indexed by the weight $\alpha : \mathbf{2} \to Cat$.

It is known that, when $\mathcal{X}$ admits all indexed limits, the homomorphism $\{-, f\}$ can be defined on $Hom(\mathbf{2}, Cat)^{op}$ and provides the right Kan extension of f through the Yoneda embedding:

$$\begin{array}{ccc} \mathbf{2} & \longrightarrow & Hom(\mathbf{2}, Cat)^{op} \\ & {\scriptstyle f} \searrow & \Big\downarrow {\scriptstyle \{-,f\}} \\ & & \mathcal{X} \end{array}$$

Thus, in this case, $\{-, f\}$ admits as a right biadjoint the homomorphism $\mathcal{X}(-, f-)$:

$$\mathcal{X}(y, \{\alpha, f\}) \simeq Hom(\mathbf{2}, Cat)^{op}(\mathcal{X}(y, f-), \alpha)$$

for any α in $Hom(\mathbf{2}, Cat)$.

Definition 3.1

Denote by K_f *the homomorphism* $\{-, f\}$ *defined on* $\mathcal{W}^{op}$ *and call it the* **kernel** *of f.*

Clearly, $K_f(\alpha)$ is defined up to equivalence.

Remarks 3.2

1. $K_f(\alpha)$ can be calculated by bipullbacks and cotensors in $\mathcal{X}$:

(1)
$$\begin{array}{ccc} K_f(\alpha) & \longrightarrow & B^{\mathbf{D}} \\ \Big\downarrow & p.b. & \Big\downarrow {\scriptstyle B^{\alpha}} \\ A^{\mathbf{C}} & \xrightarrow{f^{\mathbf{C}}} & B^{\mathbf{C}} \end{array}$$

for $\alpha : \mathbf{C} \to \mathbf{D}$ in $\mathcal{W}$.

Denote by I_A the identity of A in $\mathcal{X}$. There is a canonical comparison map

(2)
$$K_{I_A}(\alpha) \to K_f(\alpha)$$

uniquely defined up to invertible 2-cells, induced by the universal property of $K_f(\alpha)$. In particular $K_{I_A}(\alpha) \simeq A^{\mathbf{D}}$.

2. The comparison map $K_{I_A}(\alpha) \to K_f(\alpha)$ can be seen also in the following way: any object f in $Hom(\mathbf{2}, \mathcal{X})$ comes equipped with an arrow $I_A \xrightarrow{\widehat{f}} f$, precisely

$$\begin{array}{ccc} A & \xrightarrow{I_A} & A \\ {\scriptstyle I_A}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

Taking indexed limits amounts to a homomorphism:

$$Hom(\mathbf{2}, \mathcal{X}) \xrightarrow{K} Hom(\mathcal{W}^{op}, \mathcal{X})$$

and the comparison $K_{I_A}(\alpha) \to K_f(\alpha)$ is the image of $\widehat{f}$ under this homomorphism.

3. For any finite category $\mathbf{V}$ and for any arrow $f : A \to B$ in $\mathcal{X}$ there is a canonical natural equivalence of homomorphisms $\mathcal{W}^{op} \to \mathcal{X}$:

$$(K_f(-))^{\mathbf{V}} \simeq K_{f^{\mathbf{V}}}(-)$$

For any $\alpha : \mathbf{C} \to \mathbf{D}$ in $\mathcal{W}$, the bipullback (1) defines $K_f(\alpha)$. Exponentiation to $\mathbf{V}$ preserves bipullbacks, hence

$$\begin{array}{ccc} K_f(\alpha)^{\mathbf{V}} & \longrightarrow & (B^{\mathbf{V}})^{\mathbf{D}} \\ \downarrow & & \downarrow{\scriptstyle (B^{\mathbf{V}})^{\alpha}} \\ (A^{\mathbf{V}})^{\mathbf{C}} & \xrightarrow{(f^{\mathbf{V}})^{\mathbf{C}}} & (B^{\mathbf{V}})^{\mathbf{C}} \end{array}$$

is a bipullback and

$$K_f(\alpha)^{\mathbf{V}} \simeq (K_{f^{\mathbf{V}}})(\alpha).$$

Definition 3.3

An arrow $f : A \to B$ in $\mathcal{X}$ is said to be **monic** *(relative to $\mathcal{W}$) when, for all $\alpha \in \mathcal{W}$, the comparison map of Remark 3.2.1 is an equivalence*

$$K_{I_A}(\alpha) \simeq K_f(\alpha).$$

In other words, f is monic when the image $K\widehat{f}$ of the canonical $\widehat{f} : I_A \to f$ in previous Remark 3.2.2 is an equivalence in $Hom(\mathcal{W}^{op}, \mathcal{X})$.

This is clearly equivalent also to the fact that, for every $\alpha : \mathbf{C} \to \mathbf{D}$ in $\mathcal{W}$, the following is a bipullback :

$$\begin{array}{ccc} A^{\mathbf{D}} & \xrightarrow{f^{\mathbf{D}}} & B^{\mathbf{D}} \\ {\scriptstyle A^{\alpha}}\downarrow & p.b. & \downarrow{\scriptstyle B^{\alpha}} \\ A^{\mathbf{C}} & \xrightarrow{f^{\mathbf{C}}} & B^{\mathbf{C}} \end{array}$$

Remarks 3.4

1. Monic maps can be characterized also by bipullbacks and cotensors in Cat. Namely, for any object X in $\mathcal{X}$ and any $\mathbf{C} \xrightarrow{\alpha} \mathbf{D}$ in $\mathcal{W}$:

$$\begin{array}{ccc} \mathcal{X}(X,A)^{\mathbf{D}} & \xrightarrow{\mathcal{X}(X,f)^{\mathbf{D}}} & \mathcal{X}(X,B)^{\mathbf{D}} \\ {\scriptstyle \mathcal{X}(X,A)^{\alpha}}\downarrow & p.b. & \downarrow{\scriptstyle \mathcal{X}(X,B)^{\alpha}} \\ \mathcal{X}(X,A)^{\mathbf{C}} & \xrightarrow{\mathcal{X}(X,f)^{\mathbf{C}}} & \mathcal{X}(X,B)^{\mathbf{C}} \end{array}$$

is a bipullback. This uses the universal property of the cotensor and that $\mathcal{X}(X,-)$ preserves bipullbacks.

2. For any composable pair $A \xrightarrow{f} B \xrightarrow{g} X$ in $\mathcal{X}$, the object $g \cdot f$ in $Hom(\mathbf{2}, \mathcal{X})$ comes equipped with an arrow $\widehat{g}_f : f \to g \cdot f$, given by

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle I_A}\downarrow & \cong & \downarrow{\scriptstyle g} \\ A & \xrightarrow{g \cdot f} & X \end{array}$$

By the universal property of $K_{g \cdot f}(\alpha)$, there is a canonical comparison map

$$K_f(\alpha) \to K_{g \cdot f}(\alpha)$$

generalizing that of Remark 3.2.1. This comparison map can be seen as the image under the homomorphism

$$Hom(\mathbf{2}, \mathcal{X}) \xrightarrow{K} Hom(\mathcal{W}^{op}, \mathcal{X})$$

of the canonical $\widehat{g}_f$. Moreover, when g is monic, the comparison $K_f \to K_{g \cdot f}$ is an equivalence in $Hom(\mathcal{W}^{op}, \mathcal{X})$. For this, given any weight $\alpha : \mathbf{C} \to \mathbf{D}$, consider the bipullback

$$\begin{array}{ccccc} K_{g\cdot f}(\alpha) & \longrightarrow & B^{\mathbf{D}} & \xrightarrow{g^{\mathbf{D}}} & X^{\mathbf{D}} \\ \big\downarrow & & \big\downarrow{\scriptstyle B^{\alpha}} & & \big\downarrow{\scriptstyle X^{\alpha}} \\ A^{\mathbf{C}} & \xrightarrow{f^{\mathbf{C}}} & B^{\mathbf{C}} & \xrightarrow{g^{\mathbf{C}}} & X^{\mathbf{C}} \end{array}$$

which provides the definition of $K_{g\cdot f}(\alpha)$. The right hand square is a bipullback because g is monic. Hence the left hand one is, proving that $K_{g\cdot f}(\alpha) \simeq K_f(\alpha)$.

Theorem 3.5

1) Monic maps are closed under cotensoring with finite categories, composition and bipullback.

2) Equivalences are monic maps.

3) f is monic in $\mathcal{X}$ if and only if $\mathcal{X}(X, f)$ is monic in Cat, for all objects X in $\mathcal{X}$.

4) If $g \cdot f$ is monic and g is monic, also f is.

Proof. 1) By Remark 3.2.3, for any weight α we have

$$K_{f^{\mathbf{V}}}(\alpha) \simeq K_f(\alpha)^{\mathbf{V}} \simeq K_{I_A}(\alpha)^{\mathbf{V}} \simeq K_{I_A^{\mathbf{V}}}(\alpha)$$

when f is monic. This proves closure under cotensoring.

Suppose now that $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are both monic. Then, in $Hom(\mathbf{2}, \mathcal{X})$, we have a composite

$$I_A \xrightarrow{\widehat{f}} f \xrightarrow{\widehat{g_f}} g \cdot f$$

and, applying the functor K, by previous Remark 3.4.2 we have

$$K_{I_A} \simeq K_f \simeq K_{g\cdot f}$$

Hence $g \cdot f$ is monic.

Consider now the closure with respect to bipullbacks. Suppose that

$$\begin{array}{ccc} P & \xrightarrow{\bar{f}} & Q \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & B \end{array} \qquad (3)$$

is a bipullback in $\mathcal{X}$ and f is monic. Then, we get a bipullback

$$\begin{array}{ccc}
P^{\mathbf{D}} & \xrightarrow{\bar{f}^{\mathbf{D}}} & Q^{\mathbf{D}} \\
{\scriptstyle \pi^{\mathbf{D}}}\downarrow & & \downarrow{\scriptstyle g^{\mathbf{D}}} \\
A^{\mathbf{D}} & \xrightarrow{f^{\mathbf{D}}} & B^{\mathbf{D}} \\
{\scriptstyle A^{\alpha}}\downarrow & & \downarrow{\scriptstyle B^{\alpha}} \\
A^{\mathbf{C}} & \xrightarrow{f^{\mathbf{C}}} & B^{\mathbf{C}}
\end{array}$$

by composition of two bipullbacks: the bottom square is a bipullback because f is monic, the upper one because it is obtained by exponentiating diagram (3). Hence also the following is a bipullback:

$$\begin{array}{ccc}
P^{\mathbf{D}} & \xrightarrow{\bar{f}^{\mathbf{D}}} & Q^{\mathbf{D}} \\
{\scriptstyle P^{\alpha}}\downarrow & & \downarrow{\scriptstyle D^{\alpha}} \\
P^{\mathbf{C}} & \xrightarrow{\bar{f}^{\mathbf{C}}} & Q^{\mathbf{C}} \\
{\scriptstyle \pi^{\mathbf{C}}}\downarrow & & \downarrow{\scriptstyle g^{\mathbf{C}}} \\
A^{\mathbf{C}} & \xrightarrow{f^{\mathbf{C}}} & B^{\mathbf{C}}
\end{array}$$

Here, the bottom square is a bipullback, so the upper one is and this proves that $\bar{f}$ is monic.

2) is obvious.

3) We have already seen that, when f is monic, also

$$\mathcal{X}(X,f) : \mathcal{X}(X,A) \to \mathcal{X}(X,B)$$

is monic, for any X in $\mathcal{X}$ (Remark 3.4.1). Conversely, if $\mathcal{X}(X,f)$ is monic in *Cat* for any X, we have a bipullback for any weight α:

$$\begin{array}{ccc}
\mathcal{X}(X,A^{\mathbf{D}}) & \longrightarrow & \mathcal{X}(X,B^{\mathbf{D}}) \\
\downarrow & & \downarrow{\scriptstyle \mathcal{X}(X,B^{\alpha})} \\
\mathcal{X}(X,A^{\mathbf{C}}) & \xrightarrow{\mathcal{X}(X,f^{\mathbf{C}})} & \mathcal{X}(X,B^{\mathbf{C}})
\end{array}$$

and this proves that f is monic.

4) Consider the following commutative diagram in $Hom(\mathbf{2}, \mathcal{X})$:

$$\begin{array}{ccc} I_A & \xrightarrow{\ \widehat{f}\ } & f \\ & \widehat{gf} \searrow & \Big\downarrow \widehat{g}_f \\ & & gf \end{array}$$

Applying the bifunctor K, in $Hom(\mathcal{W}^{op}, \mathcal{X})$ we get the diagram:

$$\begin{array}{ccc} K_{I_A} & \xrightarrow{\ K\widehat{f}\ } & K_f \\ & \searrow & \Big\downarrow K\widehat{g}_f \\ & & K_{g \cdot f} \end{array}$$

commuting up to an invertible 2-cell. Here two sides are equivalences so also the third one, namely $K\widehat{f}$, is an equivalence and f is monic. □

Examples 3.6

We now show that known notions of "monomorphism" can be easily obtained as in Definition 3.3.

1. The simplest, original case, is obtained for categories (= discrete bicategories), when $\mathcal{W}$ consists only of the arrow $\alpha : 2 \to 1$, where 2 is the discrete two objects category. In this case, $K_f(\alpha)$ is the domain of the kernel pair of f, and "monic" has the usual meaning.

The following examples are relative to Cat. Observe however that they remain true for the sub-bicategory Lex, of categories with finite limits and left exact functors and for other sub-bicategories.

2. Suppose $\mathcal{W}$ consists of the only weight $\alpha : 2 \to \mathbf{2}$. For a functor $F : \mathbf{A} \to \mathbf{B}$, the category $K_F(\alpha)$ is (equivalent to) the comma category F/F, whose objects are triples $(a, Fa \to Fa', a')$. Now, $K_{I_\mathbf{A}}(\alpha) \simeq \mathbf{A^2}$ and an easy calculation proves that the condition $K_F(\alpha) \simeq \mathbf{A^2}$ means exactly that F is a fully faithful functor.

3. Considering only the weight $\beta : \Pi \to \mathbf{2}$ where Π denotes the category $\cdot \rightrightarrows \cdot$ with two objects and two distinct parallel arrows between them (no 2-cell), it is easy to check that $K_F(\beta)$ is equivalent to the category of those parallel arrows in $\mathbf{A}$ which become equal under F. Thus the equivalence $K_F(\beta) \simeq K_{I_\mathbf{A}}$ provides faithful functors.

4. Now consider in $\mathcal{W}$ the only weight $\alpha : \mathbf{2} \to \widetilde{\mathbf{2}}$, where $\widetilde{\mathbf{2}}$ denotes the category with two objects and an isomorphism between them. In this case, $K_F(\alpha)$ gives the domain of the kernel pair for the functor $F : A \to B$, namely the category whose objects are triples $\{(a, h : a \to a', a') | Fh \ \text{ iso}\}$. The condition $K_F(\alpha) \simeq K_{I_A}(\alpha) \simeq A^{\widetilde{\mathbf{2}}}$ thus says that F is conservative (i.e. reflects isomorphisms).

5. As a further example, take the weight $\alpha : 1 \to \mathbf{2}$ which associates to the only object of 1 the domain of the non trivial arrow in $\mathbf{2}$. Then, for a functor $F : \mathbf{A} \to \mathbf{B}$, the kernel $K_F(\alpha)$ is the comma category $F/I_\mathbf{B}$, whose objects are triples $(a, Fa \to b, b)$. So, $K_{I_\mathbf{A}}(\alpha) \simeq \mathbf{A}^\mathbf{2}$ and $K_F(\alpha) \simeq K_{I_\mathbf{A}}(\alpha)$ means that F is a discrete cofibration.

4. Quotients and strong epimorphisms

Strong epimorphisms in $\mathcal{X}$ are defined by the "orthogonal property" with respect to the monomorphisms. For a finitely complete category $\mathcal{X}$, the orthogonality condition can be expressed by a bipullback in *Cat*, as in the following definition.

Definition 4.1
An arrow $e : X \to Y$ in $\mathcal{X}$ is said to be **strong epic** *relatively to $\mathcal{W}$, when the following square is a bipullback, for all monic maps $m : A \to B$:*

$$\begin{array}{ccc} \mathcal{X}(Y,A) & \xrightarrow{\mathcal{X}(Y,m)} & \mathcal{X}(Y,B) \\ {\scriptstyle \mathcal{X}(e,A)}\downarrow & p.b. & \downarrow{\scriptstyle \mathcal{X}(e,B)} \\ \mathcal{X}(X,A) & \xrightarrow{\mathcal{X}(X,m)} & \mathcal{X}(X,B) \end{array} \tag{4}$$

In the situation of diagram (4), we say that the arrow e is *above* m, or equivalently that m is *below* e.

Theorem 4.2
0) An arrow which is both strong epic and monic is an equivalence.
1) Strong epic maps are closed under composition.
2) Equivalences are strong epic maps.

3) If the composite $k \cdot h$ is strong epic and h is strong epic, also k is.
4) Each α in $\mathcal{W}$ is strong epic in Cat.

Proof. 0) is obvious. 1) Suppose that $h : X \to Y$ and $k : Y \to Z$ are strong epic maps and $f : A \to B$ is monic. In the following diagram, both squares are bipullbacks, so the diagram is a bipullback:

$$(5)\qquad \begin{array}{ccc} \mathcal{X}(Z,A) & \xrightarrow{\mathcal{X}(Z,f)} & \mathcal{X}(Z,B) \\ {\scriptstyle \mathcal{X}(k,A)}\downarrow & & \downarrow{\scriptstyle \mathcal{X}(k,B)} \\ \mathcal{X}(Y,A) & \xrightarrow{\mathcal{X}(Y,f)} & \mathcal{X}(Y,B) \\ {\scriptstyle \mathcal{X}(h,A)}\downarrow & & \downarrow{\scriptstyle \mathcal{X}(h,B)} \\ \mathcal{X}(X,A) & \xrightarrow{\mathcal{X}(X,f)} & \mathcal{X}(X,B) \end{array}$$

2) Obvious. 3) Consider again diagram (5). Now, the diagram is a bipullback and the bottom square is a bipullback. Hence the above square is, and this proves that k is a strong epic map.

4) Let $f : \mathbf{A} \to \mathbf{B}$ be any monic in Cat. By definition, this means that for all weights $\alpha : \mathbf{C} \to \mathbf{D}$ in $\mathcal{W}$, the following is a bipullback:

$$\begin{array}{ccc} \mathbf{A}^{\mathbf{D}} & \xrightarrow{f^{\mathbf{D}}} & \mathbf{B}^{\mathbf{D}} \\ {\scriptstyle A^{\alpha}}\downarrow & & \downarrow{\scriptstyle \mathbf{B}^{\alpha}} \\ \mathbf{A}^{\mathbf{C}} & \xrightarrow{f^{\mathbf{C}}} & \mathbf{B}^{\mathbf{C}} \end{array}$$

and this means exactly that α is epic in Cat. □

Definition 4.3
For any homomorphism $G : \mathcal{W}^{op} \to \mathcal{X}$, define the **quotient** *of G as the homomorphism $Q_G : \mathbf{2} \to \mathcal{X}$ given by the indexed colimit*

$$Q_G(a) = ev(-,a) * G$$

for a in $\mathbf{2}$, where $ev(-,a) : \mathcal{W} \to Cat$ denotes the homomorphism of "evaluation at a": $ev(\alpha,a) = \alpha(a)$.

Say that an arrow $q : A \to B$ is a *quotient* if, regarded as a homomorphism $\mathbf{2} \to \mathcal{X}$, it is equivalent to Q_G for some $G : \mathcal{W}^{op} \to \mathcal{X}$.

Remark 4.4

Observe that, using the evaluation homomorphism $ev(\alpha,-) : \mathbf{2} \to Cat$, the kernel $K_f(\alpha)$ can be described as the indexed limit $\{ev(\alpha,-), f\}$. Moreover, considering the evaluation homomorphism in two arguments

$$ev : \mathcal{W} \times \mathbf{2} \to Cat$$

and taking indexed limits and colimits with extra variables, we have

$$K_f \simeq \{ev, f\}$$
$$Q_G \simeq ev * G$$

Lemma 4.5

The arrow $e : X \to Y$ in $\mathcal{X}$ is strong epic if and only if for any monic $f : A \to B$, any arrow $H : e \to f$ in $Hom(\mathbf{2}, \mathcal{X})$ factors uniquely, up to an invertible 2-cell, through the canonical arrow $\widehat{f} : I_A \to f$.

Proof. If e is a strong epic map, the factorization of the arrow $H = (u, \tau, v)$ in $Hom(\mathbf{2}, \mathcal{X})$, given by

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & Y \\ {\scriptstyle u}\downarrow & \overset{\tau}{\cong} & \downarrow{\scriptstyle v} \\ A & \xrightarrow{\;f\;} & B \end{array}$$

is the following

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & Y \\ {\scriptstyle te}\downarrow & & \downarrow{\scriptstyle t} \\ A & \xrightarrow{\;I_A\;} & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\;f\;} & B \end{array}$$

where t is the diagonal.

Conversely, by the factorization of H through I_A one recovers easily the diagonal t. □

Theorem 4.6

Q is left biadjoint to K. In symbols: $Q \dashv\mid K$.

Proof. By definition of the indexed limits and colimits which are involved, we have the natural equivalences of categories:

$$Hom(\mathcal{W}^{op}, \mathcal{X})(G, K_f) \simeq Hom(\mathcal{W} \times \mathbf{2}, Cat)(ev, \mathcal{X}(G, f)) \simeq$$
$$\simeq Hom(\mathbf{2}, \mathcal{X})(Q_G, f)$$

for any homomorphism $G : \mathcal{W}^{op} \to \mathcal{X}$ and any homomorphism $f : \mathbf{2} \to \mathcal{X}$. □

Theorem 4.7
Quotients are strong epic maps.

Proof. Consider Lemma 4.5: let f be monic and let be $H : Q_G \to f$ be an arrow in $Hom(\mathbf{2}, \mathcal{X})$. By adjointness we recover an arrow $\bar{H} : G \to K_f$ in $Hom(\mathcal{W}^{op}, \mathcal{X})$ and we have the situation

$$G \xrightarrow{\bar{H}} K_f \xleftarrow{K\widehat{f}} K_{I_A}$$

where $\widehat{f}$ denotes, as before, the canonical arrow $I_A \to f$ in $Hom(\mathbf{2}, \mathcal{X})$. Here, $K\widehat{f}$ is an equivalence because f is monic and, again by adjointness, we recover the requested factorization of H. □

We represents simply by

$$\begin{array}{cc} G \longrightarrow K_f & (Hom(\mathcal{W}^{op}, \mathcal{X})) \\ \hline ev \longrightarrow \mathcal{X}(G, f) & (Hom(\mathcal{W} \times \mathbf{2}, Cat)) \\ \hline Q_G \longrightarrow f & (Hom(\mathbf{2}, \mathcal{X})) \end{array}$$

the correspondences on objects of the equivalences involved in the biadjointness $Q \dashv\mid K$ of Theorem 4.6.

Observe that these correspondences are natural in f and in G. In particular, unit and counit of the adjointness correspond to the canonical injection in a colimit q_G (resp. projection from a limit p_f):

$$\begin{array}{c} G \xrightarrow{\eta_G} KQ_G \\ \hline ev \xrightarrow{q_G} \mathcal{X}(G, Q_G) \\ \hline Q_G \xrightarrow{1} Q_G \end{array}$$

and

$$\begin{array}{c} QK_f \xrightarrow{\epsilon_f} f \\ \hline ev \xrightarrow{p_f} \mathcal{X}(K_f, f) \\ \hline K_f \xrightarrow{1} K_f \end{array}$$

The following Corollary of Theorem 4.6 is proved by naturality of the above correspondences.

Corollary 4.8
The following triangles in $Hom(\mathcal{W} \times \mathbf{2}, Cat)$ commute up to invertible 2-cells:

$$\begin{array}{ccc} ev & \xrightarrow{p_{Q_G}} & \mathcal{X}(KQ_G, Q_G) \\ & \searrow^{q_G} & \downarrow{\scriptstyle \mathcal{X}(\eta_G, Q_G)} \\ & & \mathcal{X}(G, Q_G) \end{array}$$

and

$$\begin{array}{ccc} ev & \xrightarrow{q_{K_f}} & \mathcal{X}(K_f, QK_f) \\ & \searrow^{p_f} & \downarrow{\scriptstyle \mathcal{X}(K_f, \epsilon_f)} \\ & & \mathcal{X}(K_f, f) \end{array}$$

□

From the previous Corollary in particular we obtain the *triangular isomorphisms*

$$\begin{array}{ccc} K_f & \xrightarrow{\eta_{K_f}} & KQK_f \\ & \searrow^{I_{K_f}} & \downarrow{\scriptstyle K\epsilon_f} \\ & & K_f \end{array}$$

and

$$\begin{array}{ccc} Q_G & \xrightarrow{Q\eta_G} & QKQ_G \\ & \searrow^{I_{Q_G}} & \downarrow{\scriptstyle \epsilon_{Q_G}} \\ & & Q_G \end{array}$$

5. Examples

We refer to the examples 3.6. Let $\mathcal{E}$ denote the class of all arrows in *Cat* which are essentially surjective on objects (*eso*). In other words, $\mathcal{E}$ consists of all functors $E : \mathbf{A} \to \mathbf{B}$ such that any object b in $\mathbf{B}$ is isomorphic with an object of type Fa, with a in $\mathbf{A}$.

Theorem 5.1
A functor $F : \mathbf{X} \to \mathbf{Y}$ is fully faithful if and only if it is below each E in $\mathcal{E}$.

Proof. Given a natural isomorphism

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{E} & \mathbf{B} \\ {\scriptstyle H}\downarrow & \cong & \downarrow{\scriptstyle K} \\ \mathbf{X} & \xrightarrow{F} & \mathbf{Y} \end{array}$$

it is easy to define a diagonal $D : \mathbf{B} \to \mathbf{X}$ and to check that it is uniquely defined, up to a unique natural isomorphism, when F is fully faithful.

Conversely, it is enough to observe, as already implicit in example 2 of 3.6, that a functor $F : \mathbf{A} \to \mathbf{B}$ is fully-faithful if it is below the eso functor $\alpha : 2 \to \mathbf{2}$. $\square$

The above proof clarifies the rôle of the weights. Here a functor $F : \mathbf{A} \to \mathbf{B}$ is said to be a *projection* when $\mathbf{B} \simeq \mathbf{A}(\Sigma^{-1})$ is equivalent with a category of fractions of $\mathbf{A}$, and recall that a functor $E : \mathbf{A} \to \mathbf{B}$ is said *initial* when any category E/b is non empty and pathwise connected (see Street and Walters [9]).

Formally with the same proof as above, in relation to the corresponding examples of section 3, we have:

Corollary 5.2.
A functor F is faithful (conservative, discrete cofibration) if and only if it is below all eso and full (projection, initial) functors.

It follows that every eso (eso and full, projection, initial) functor is above all fully-faithful (faithful, conservative, discrete cofibration). If an arrow f is above all monics of a factorization system, then it is in particular above its monic factor with respect to this factorization system. Which renders this factor an equivalence. In other words:

Corollary 5.3

In Cat, a functor $E : \mathbf{A} \to \mathbf{B}$ is eso (eso and full, initial) if and only if it is above all fully-faithful (faithful, discrete cofibration) functors. In Lex, it is a projection if and only if it is above all conservative functors. □

6. The canonical factorization

So far we don't have yet any factorization. From now on we assume that the given sub-bicategory $\mathcal{W}$ of weights satisfies the following assumption:

Representability axiom. The *representable weight* $\rho = \mathbf{2}(0, -) : \mathbf{2} \to Cat$ is in $\mathcal{W}$.

Observe that this assumption does not change the notion of monomorphisms. However it has an immediate consequence:

Lemma 6.1
For any $f : \mathbf{2} \to \mathcal{X}$, the components at ρ and 0 of the following natural transformations:

$$1) \quad ev \xrightarrow{p_f} \mathcal{X}(K_f, f)$$
$$2) \quad ev \xrightarrow{q_{K_f}} \mathcal{X}(K_f, QK_f)$$

are equivalences in $\mathcal{X}$. Respectively:

$$ev(\rho, 0) = \mathbf{1} \xrightarrow{p_f(\rho,0)} \mathcal{X}(K_f(\rho), f(0))$$
$$ev(\rho, 0) = \mathbf{1} \xrightarrow{q_{K_f}(\rho,0)} \mathcal{X}(K_f(\rho), QK_f(0))$$

Moreover, for any $G : \mathcal{W}^{op} \to \mathcal{X}$, the component at 0 of the natural transformations

$$3) \quad QK_f \xrightarrow{\epsilon_f} f$$
$$4) \quad QKQ_G \xrightarrow{\epsilon_{Q_G}} Q_G \xrightarrow{Q\eta_G} QKQ_G$$

are equivalences in $\mathcal{X}$.

Proof. 1) The projection $p_f(\rho, 0)$ is a "Yoneda equivalence" connecting limits indexed by representables with values taken at the representing object.

2) The indexing functor

$$ev(-,0) : \mathcal{W} \to Cat$$

of the colimit $QK_f(0) = ev(-,0) * K_f$ is representable. Indeed, for any weight α in $\mathcal{W}$, by Yoneda, we have the natural equivalence (in Cat):

$$ev(\alpha, 0) \simeq \mathcal{W}(\rho, \alpha).$$

Hence, the injection $q_{K_f}(\rho, 0)$ is again a "Yoneda equivalence" connecting colimits indexed by representables with values taken at the representing object.

3) follows from Corollary 4.9. Consider the component at ρ and 0 of the natural transformations involved in the second diagram:

$$\begin{array}{ccc} ev(\rho,0) = \mathbf{1} & \xrightarrow{q_{K_f}(\rho,0)} & \mathcal{X}(K_f(\rho), QK_f(0)) \\ & \searrow^{p_f(\rho,0)} & \downarrow{\scriptstyle \mathcal{X}(K_f(\rho), \epsilon_f(0))} \\ & & \mathcal{X}(K_f(\rho), f(0)) \end{array}$$

and recall that $q_{K_f}(\rho, 0)$ and $p_f(\rho, 0)$ are equivalences by 1), 2) above.

4) Consider the triangular isomorphism

$$\epsilon_{Q_G} \cdot Q\eta_G \cong I_{Q_G}$$

and take the component at 0. We have

$$\epsilon_{Q_G}(0) \cdot Q\eta_G(0) \cong I_{Q_G(0)}$$

and we know that $\epsilon_{Q_G}(0)$ is an equivalence (by 3) above), so also $Q\eta_G(0)$ is an equivalence and

$$Q\eta_G(0) \cdot \epsilon_{Q_G}(0) \cong I_{QKQ_G(0)}.$$

$\square$

Given any arrow $f : \mathbf{2} \to \mathcal{X}$, now consider the quotient QK_f and the counit $\epsilon_f : QK_f \to f$:

$$\begin{array}{ccc} QK_f(0) & \xrightarrow{QK_f(i)} & QK_f(1) \\ {\scriptstyle \epsilon_f(0)}\downarrow & \cong & \downarrow{\scriptstyle \epsilon_f(1)} \\ f(0) = A & \xrightarrow{f} & B = f(1) \end{array}$$

By Lemma 6.1 we have the factorization $f \simeq \epsilon_f(1)\cdot q$, where q is the composite of $Q\eta_G(0)$ with $QK_f(i)$, the image under QK_f of the only arrow i of $\mathbf{2}$. Observe that q is epic by Theorem 4.7 and Theorem 4.2.

Definition 6.2. *We refer to the factorization $f \simeq \epsilon_f(1)\cdot q$ previously obtained as to the* **canonical factorization** *of f (relative to the given weights $\mathcal{W}$).*

Example 6.3: the lax descent problem.

The *lax descent problem* (see [12]) arises with the following weights, together with all arrows between them in $Hom(\mathbf{2}, Cat)$:

$$\rho : \mathbf{1} \to \mathbf{1}$$
$$\alpha : 2 \to \mathbf{2}$$
$$\beta : 3 \to \mathbf{3}$$

(here 3 is the discrete three object category and $\mathbf{3}$ is the category generated by $\cdot \to \cdot \to \cdot$).

For a morphism $f : A \to B$ in $\mathcal{X}$, one takes the *resolution* $K_f : \mathcal{W}^{op} \to \mathcal{X}$, usually represented in the form:

$$f/f/f \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; f/f \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; A \xrightarrow{\;\;f\;\;} B$$

(here $f/f/f \simeq K_f(\beta)$, $f/f \simeq K_f(\alpha)$ and $A \simeq K_f(\rho)$).

By taking the colimit of this resolution, one obtains the *descent object* $Des\ (f)$ and a canonical arrow $q : A \to Des\ (f)$, through which f can be factored: $f \simeq jq$.

This is the canonical factorization relative to the given $\mathcal{W}$.

Now, the problem is to characterize those f's for which the object B can be recovered back from the resolution. These are the *effective descent morphisms* in $\mathcal{X}^{op}$. In this case the comparison j is an equivalence. With the terminology of Theorem 4.6, the problem is that of characterizing those f's for which the counit $\epsilon_f : QK_f \to f$ is an equivalence (see Betti [3]).

7. $\mathcal{W}$-regularity for bicategories

If $\alpha : \mathbf{C} \to \mathbf{D}$ and $\beta : \mathbf{D} \to \mathbf{E}$ is a composable pair of weights, then the notion of monomorphism determined in any category $\mathcal{X}$ does not change by adjoining the composite $\beta \cdot \alpha$ to the given set $\mathcal{W}$ of weights. Indeed, if $f : A \to B$ is monic (with respect to $\mathcal{W}$), then

$$K_f(\beta \cdot \alpha) \simeq K_{I_A}(\beta \cdot \alpha) \simeq A^{\mathbf{E}}$$

as results from the following composition of bipullbacks:

$$\begin{array}{ccc}
A^{\mathbf{E}} & \xrightarrow{f^{\mathbf{E}}} & B^{\mathbf{E}} \\
{\scriptstyle A^{\beta}}\downarrow & & \downarrow{\scriptstyle B^{\beta}} \\
A^{\mathbf{D}} & \xrightarrow{f^{\mathbf{D}}} & B^{\mathbf{D}} \\
{\scriptstyle A^{\alpha}}\downarrow & & \downarrow{\scriptstyle B^{\alpha}} \\
A^{\mathbf{C}} & \xrightarrow{f^{\mathbf{C}}} & B^{\mathbf{C}}
\end{array}$$

Analogously, if α is in $\mathcal{W}$ and f is monic, then it is easy to see that

$$K_f(\alpha \times \mathbf{V}) \simeq K_f(\alpha)^{\mathbf{V}} \simeq K_{I_A}(\alpha)^{\mathbf{V}} \simeq A^{\mathbf{D} \times \mathbf{V}}$$

for any finite category $\mathbf{V}$.

Consider now the product $\alpha \times \beta$ of two weights $\alpha : \mathbf{C} \to \mathbf{D}$ and $\beta : \mathbf{E} \to \mathbf{F}$ as the composite

$$\mathbf{C} \times \mathbf{E} \xrightarrow{\alpha \times \mathbf{E}} \mathbf{D} \times \mathbf{E} \xrightarrow{\mathbf{D} \times \beta} \mathbf{D} \times \mathbf{F}.$$

For any monic arrow f in $\mathcal{X}$, we have:

$$K_f(\alpha \times \beta) \simeq A^{\mathbf{D} \times \mathbf{F}}$$

hence the class of monics is not altered if we assume that $\mathcal{W}$ is closed under products. In conclusion, *henceforth we assume that $\mathcal{W}$ is closed under equivalences, composition and products.*

Definition 7.1

A finitely complete and cocomplete bicategory $\mathcal{X}$ is said to be **regular** *relative to $\mathcal{W}$, or $\mathcal{W}$-***regular***, if strong epic arrows are stable under bipullback and cotensoring.*

Observe that the regularity condition implies the following property which in fact is used in the proof of Theorem 7.3:

Lemma 7.2
In a $\mathcal{W}$-regular bicategory, for any strong epic arrow $q : A \to Q$ and any arrow $h : Q \to B$, the comparison map $q_h(\alpha) : K_{hq}(\alpha) \to K_h(\alpha)$ is strong epic, for every weight $\alpha \in \mathcal{W}$.

To prove this property is enough to consider the following composition of bipullbacks:

$$\begin{array}{ccccc} K_{hq}(\alpha) & \xrightarrow{q_h(\alpha)} & K_h(\alpha) & \longrightarrow & B^{\mathbf{D}} \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle B^{\alpha}} \\ A^{\mathbf{C}} & \xrightarrow{q^{\mathbf{C}}} & Q^{\mathbf{C}} & \xrightarrow{h^{\mathbf{C}}} & B^{\mathbf{C}} \end{array}$$

In order to ensure the second component of the canonical factorization to be monic, we want another assumption:

Assumption: finite monicness
Let $f : A \to B$ be any arrow in $\mathcal{X}$ that we regard as usual via the canonical arrow $\widehat{f} : I_A \to f$ in $Hom(\mathbf{2}, \mathcal{X})$. By applying the functor K, one obtains a comparison arrow $K\widehat{f}(\alpha) : K_{I_A}(\alpha) \to K_f(\alpha)$ for any weight α. Denote simply by f_α the comparison $K\widehat{f}(\alpha)$.

We assume that going on taking successive comparison arrows, the process eventually arrives to a monomorphism after a finite number of steps.

This means that $f_{\alpha\beta\ldots\gamma}$ is an equivalence independently from the weights α, $\beta \ldots \gamma$, provided they are sufficiently enough.

The idea behind this assumption is that taking a comparison increases the "degree of monicness" of arrows. The least n such that $f_{\alpha\beta\ldots\gamma}$ is a monomorphism after n steps provides a sort of a measure of how much f is far from beeing monic.

Observe that the assumption is trivially true in the original case of regular categories: one step is enough.

Theorem 7.3
If $f \simeq j \cdot q$ is any factorization in $\mathcal{X}$ and j is monic, then $K_f \simeq K_q$.
Conversely, in a regular bicategory $\mathcal{X}$ which satisfies the finite monicness assumption, let $f \simeq j \cdot q$ be the canonical factorization, then $K\epsilon_f : KQK_f \simeq K_f$ implies that j is monic.

We postpone the proof of Theorem 7.3. It will be done after a discussion of the finite monicness assumption.

Examples 7.4

We have already remarked that the finite monicness assumption holds true for the regular factorization of discrete bicategories. Here are two more cases in which the assumption is true.

1. Suppose that $X^\alpha : X^{\mathbf{D}} \to X^{\mathbf{C}}$ is a monomorphism for any object X in $\mathcal{X}$ and any weight $\alpha : \mathbf{C} \to \mathbf{D}$. This means equivalently that for any other weight $\beta : \mathbf{E} \to \mathbf{F}$ the following is a bipullback in $\mathcal{X}$:

$$\begin{array}{ccc} X^{\mathbf{D}\times\mathbf{F}} & \xrightarrow{X^{\mathbf{D}\times\beta}} & X^{\mathbf{D}\times\mathbf{E}} \\ {\scriptstyle X^{\alpha\times\mathbf{F}}}\downarrow & \cong & \downarrow{\scriptstyle X^{\alpha\times\mathbf{E}}} \\ X^{\mathbf{C}\times\mathbf{F}} & \xrightarrow{X^{\mathbf{C}\times\beta}} & X^{\mathbf{C}\times\mathbf{E}} \end{array}$$

Consider now the comparison f_α:

$$\begin{array}{ccccc} K_{I_A}(\alpha) \simeq A^{\mathbf{D}} & \xrightarrow{f_\alpha} & K_f(\alpha) & \longrightarrow & B^{\mathbf{D}} \\ & & {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle B^\alpha} \\ & & A^{\mathbf{C}} & \xrightarrow{f^{\mathbf{C}}} & B^{\mathbf{C}} \end{array}$$

Here, B^α and $\sigma \cdot f_\alpha \simeq A^\alpha$ are monic arrows. Hence by the cancellation property of monomorphisms (4 of Theorem 3.5) also f_α is.

The above condition holds true in the examples 3 and 4 of 3.6. It is not true however in the important case 2 (again of Examples 3.6). This case needs a lenght three process of comparison which we illustrate next.

2. For this case we introduce a weakening of the notion of monomorphism.

Say that an arrow $f : A \to B$ in $\mathcal{X}$ is *faithful* when for any object X the composition with f is a functor $\mathcal{X}(X, A) \to \mathcal{X}(X, B)$ which is faithful in Cat.

Now, define an arrow $f : A \to B$ in $\mathcal{X}$ to be *premonic* (relatively to $\mathcal{W}$) if for any weight $\alpha : \mathbf{C} \to \mathbf{D}$ the comparison $f_\alpha : A^{\mathbf{D}} \to K_f(\alpha)$ is faithful.

The assumption we make is now that pairs of weights are *jointly above* faithful functors, in the following sense: for any pair of weights $\alpha : \mathbf{C} \to \mathbf{D}$ and $\beta : \mathbf{E} \to \mathbf{F}$ and any faithful functor $F : \mathbf{A} \to \mathbf{B}$ consider the diagram:

$$\begin{array}{ccc}
\mathbf{C}\times\mathbf{E} & \xrightarrow{\mathbf{C}\times\beta} & \mathbf{C}\times\mathbf{F} \\
{\scriptstyle \alpha\times\mathbf{E}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha\times\mathbf{F}} \\
\mathbf{D}\times\mathbf{E} & \xrightarrow{\mathbf{D}\times\beta} & \mathbf{D}\times\mathbf{F} \\
{\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle K} \\
\mathbf{A} & \xrightarrow{F} & \mathbf{B}
\end{array}$$

where $G : \mathbf{D}\times\mathbf{E} \to \mathbf{A}$, $H : \mathbf{C}\times\mathbf{F} \to \mathbf{A}$ and $K : \mathbf{D}\times\mathbf{F} \to \mathbf{B}$ are endowed with compatible isomorphisms:

$$\begin{aligned}
H\cdot(\mathbf{C}\times\beta) &\cong G\cdot(\alpha\times\mathbf{E}) \\
K\cdot(\alpha\times\mathbf{F}) &\cong F\cdot H \\
K\cdot(\mathbf{D}\times\beta) &\cong F\cdot G
\end{aligned}$$

We assume that, up to invertible 2-cells, there exists a unique functor $L : \mathbf{D}\times\mathbf{F} \to \mathbf{A}$ and suitable isomorphisms which make commute all the diagrams involved. Namely:

$$\begin{aligned}
F\cdot L &\cong K \\
L\cdot(\mathbf{D}\times\beta) &\cong G \\
L\cdot(\alpha\times\mathbf{F}) &\cong H
\end{aligned}$$

It is easy to see that this condition is satisfied by the example 2 of 3.6, when monic arrows are the fully faithful functors in *Cat*. Observe moreover that this condition is entirely expressed within *Cat* and does not involve the bicategory $\mathcal{X}$ where it is applied.

Theorem 7.5 *If pair of weights are jointly above faithful functors, then for any arrow f in $\mathcal{X}$ and any weight α:*
i) the comparison f_α is premonic,
ii) if f is faithful, the comparison f_α is monic.

Proof. *i)* We have to prove that, for any weight $\beta : \mathbf{E} \to \mathbf{F}$ the comparison

$$f_{\alpha\beta} : A^{\mathbf{D}\times\mathbf{F}} \longrightarrow K_{f_\alpha}(\beta)$$

is faithful.

As a consequence of the assumption that pairs of weights are jointly above faithful functors, it is easy to see that the functor:

$$P = (\alpha \times \mathbf{F}) + (\mathbf{D} \times \beta) : (\mathbf{C} \times \mathbf{F}) + (\mathbf{D} \times \mathbf{E}) \longrightarrow \mathbf{D} \times \mathbf{F}$$

is co-faithful, i.e. composition with P is a faithful functor $- \cdot P$ in Cat. By exponentiating one has that

$$A^{\mathbf{D}\times\mathbf{F}} \xrightarrow{A^P} A^{\mathbf{C}\times\mathbf{F}} \times A^{\mathbf{D}\times\mathbf{E}}$$

is faithful in $\mathcal{X}$. Now it is enough to observe that A^P factors through $f_{\alpha\beta}$:

$$\begin{array}{ccc}
A^{\mathbf{D}\times\mathbf{F}} & \xrightarrow{A^P} & A^{\mathbf{C}\times\mathbf{F}} \times A^{\mathbf{D}\times\mathbf{E}} \\
{\scriptstyle f_{\alpha\beta}}\downarrow & \nearrow & \\
K_{f_\alpha}(\beta) & &
\end{array}$$

ii) Since f is faithful, each $\mathcal{X}(X, f)$ is such in Cat. By the assumption, expressing the fact that pairs of weights are jointly above this functor and using cotensoring to eliminate the object X from the condition, we obtain that $A^{\mathbf{D}\times\mathbf{F}}$ is a limit that can be obtained by the following diagram of bipullbacks:

$$\begin{array}{ccccc}
A^{\mathbf{D}\times\mathbf{F}} & \xrightarrow{f_\beta^{\mathbf{D}}} & K_f(\beta)^{\mathbf{D}} & \xrightarrow{\lambda_\beta^{\mathbf{D}}} & A^{\mathbf{D}\times\mathbf{E}} \\
{\scriptstyle f_\alpha^{\mathbf{F}}}\downarrow & & \downarrow & & \downarrow{\scriptstyle f^{\mathbf{D}\times\mathbf{E}}} \\
K_f(\alpha)^{\mathbf{F}} & \xrightarrow{\lambda_\alpha^{\mathbf{F}}} & B^{\mathbf{D}\times\mathbf{F}} & \xrightarrow{B^{\mathbf{D}\times\beta}} & B^{\mathbf{D}\times\mathbf{E}} \\
\downarrow & & \downarrow{\scriptstyle B^{\alpha\times\mathbf{F}}} & & \\
A^{\mathbf{C}\times\mathbf{F}} & \xrightarrow{f^{\mathbf{C}\times\mathbf{F}}} & B^{\mathbf{C}\times\mathbf{F}} & &
\end{array}$$

The top and right diagram yeld the pullback:

$$\begin{array}{ccccc}
A^{\mathbf{D}\times\mathbf{F}} & \xrightarrow{A^{\mathbf{D}\times\beta}} & A^{\mathbf{D}\times\mathbf{E}} & \xrightarrow{1} & A^{\mathbf{D}\times\mathbf{E}} \\
{\scriptstyle f_\alpha^{\mathbf{F}}}\downarrow & & {\scriptstyle f_\alpha^{\mathbf{E}}}\downarrow & & \downarrow{\scriptstyle f^{\mathbf{D}\times\mathbf{E}}} \\
K_f(\alpha)^{\mathbf{F}} & \xrightarrow{K_f(\alpha)^\beta} & K_f(\alpha)^{\mathbf{E}} & \xrightarrow{\lambda_\alpha^{\mathbf{E}}} & B^{\mathbf{D}\times\mathbf{E}}
\end{array}$$

Since $\lambda_\alpha^{\mathbf{E}}$ is a pullback of $f^{\mathbf{C}\times\mathbf{E}}$ and f is faithful, also $\lambda_\alpha^{\mathbf{E}}$ is faithful and the right square is a bipullback. Hence also the left square is and f_α is monic. $\square$

It is now clear that, if pairs of weights are above faithful functors, the finite monicness assumption is true with lenght three: for any f, f_α is premonic, $f_{\alpha\beta}$ is faithful, $f_{\alpha\beta\gamma}$ is monic.

Proof of Theorem 7.3. Suppose that $f \cong j \cdot q$. It is not difficult to check that canonical arrows form a pullback diagram in $Hom(\mathbf{2}, \mathcal{X})$:

$$\begin{array}{ccc} q & \longrightarrow & f \\ \big\downarrow & & \big\downarrow \\ I_C & \longrightarrow & j \end{array}$$

where $j : C \to B$. By applying K and recalling that it is a right biadjoint, we have the pullback square in $Hom(\mathcal{W}^{op}, \mathcal{X})$:

$$\begin{array}{ccc} K_q & \longrightarrow & K_f \\ \big\downarrow & & \big\downarrow \\ K_{I_C} & \longrightarrow & j \end{array}$$

Now, if j is monic, $K_{I_C} \simeq K_j$ and the bipullback of an equivalence is again an equivalence: $K_q \simeq K_f$.

Conversely, for a given weight $\alpha : \mathbf{C} \to \mathbf{D}$, consider the following diagram of bipullbacks in a regular bicategory $\mathcal{X}$:

$$\begin{array}{ccccc} K_q(\alpha) & \xrightarrow{\ \sigma\ } & Q^{\mathbf{D}} & & \\ {\scriptstyle K\epsilon_f(\alpha)}\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle j_\alpha} & \searrow^{j^{\mathbf{D}}} & \\ K_f(\alpha) & \xrightarrow{\ \tau\ } & K_j(\alpha) & \xrightarrow{\ \lambda\ } & B^{\mathbf{D}} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle B^\alpha} \\ A^{\mathbf{C}} & \xrightarrow{\ q^{\mathbf{C}}\ } & Q^{\mathbf{C}} & \xrightarrow{\ j^{\mathbf{C}}\ } & B^{\mathbf{C}} \end{array}$$

We have to prove that the comparison j_α is an equivalence.

By the regularity assumption, it follows that it is strong epic. Indeed, q is strong epic, hence also $q^{\mathbf{C}}$, τ and σ are strong epic: by the cancellation property 3 of Theorem 4.2 also j_α is strong epic.

So, it remains to prove that j_α is monic. This is obtained by the finite monicness assumption. The procedure is as follows. First one has to prove that the property $K_q \simeq K_f$ lifts to the pair σ and τ, in the sense that $K_\sigma \simeq K_\tau$ (this is proved later) and moreover taking a new comparison $j_{\alpha\beta}$ of j_α one has again a strong epimorphism (by regularity).

Now, return to j. After a sufficient number of comparisons $j_{\alpha\beta\ldots\gamma}$ is strong epic and monic, hence it is an equivalence and going back with comparisons one has that j_α is an equivalence. Hence j is monic.

The crucial point is to prove that $K_\sigma \simeq K_\tau$. For this consider the following diagram of bipullbacks, from which one obtains $K_\tau(\beta) \simeq K_f(\alpha \times \beta)$:

$$\begin{array}{ccccc}
K_\tau(\beta) & \longrightarrow & K_j(\alpha)^{\mathbf{F}} & \longrightarrow & B^{\mathbf{D}\times\mathbf{F}} \\
\downarrow & & \downarrow{\scriptstyle K_j(\alpha)^\beta} & & \downarrow{\scriptstyle B^{\mathbf{D}\times\beta}} \\
K_f(\alpha)^{\mathbf{E}} & \xrightarrow{\tau^{\mathbf{E}}} & K_j(\alpha)^{\mathbf{E}} & \xrightarrow{\lambda^{\mathbf{E}}} & B^{\mathbf{D}\times\mathbf{E}} \\
\downarrow & & \downarrow{\scriptstyle \pi^{\mathbf{E}}} & & \downarrow{\scriptstyle B^{\alpha\times\mathbf{E}}} \\
A^{\mathbf{C}\times\mathbf{E}} & \xrightarrow{q^{\mathbf{C}\times\mathbf{E}}} & Q^{\mathbf{C}\times\mathbf{E}} & \xrightarrow{j^{\mathbf{C}\times\mathbf{E}}} & B^{\mathbf{C}\times\mathbf{E}}
\end{array}$$

Analogously, one obtains $K_\sigma(\beta) \simeq K_q(\alpha \times \beta)$ by inspection of the following diagram of bipullbacks:

$$\begin{array}{ccc}
K_\sigma(\beta) & \longrightarrow & Q^{\mathbf{D}\times\mathbf{F}} \\
\downarrow & & \downarrow{\scriptstyle Q^{\mathbf{D}\times\beta}} \\
K_q(\alpha)^{\mathbf{E}} & \xrightarrow{\sigma^{\mathbf{E}}} & Q^{\mathbf{D}\times\mathbf{E}} \\
\simeq\downarrow & & \downarrow{\scriptstyle j_\alpha^{\mathbf{E}}} \\
K_f(\alpha)^{\mathbf{E}} & \xrightarrow{\tau^{\mathbf{E}}} & K_j(\alpha)^{\mathbf{E}} \\
\downarrow & & \downarrow{\scriptstyle \pi^{\mathbf{E}}} \\
A^{\mathbf{C}\times\mathbf{E}} & \xrightarrow{q^{\mathbf{C}\times\mathbf{E}}} & Q^{\mathbf{C}\times\mathbf{E}}
\end{array}$$

Now $K_\sigma(\beta) \simeq K_q(\alpha \times \beta) \simeq K_f(\alpha \times \beta) \simeq K_\tau(\beta)$ for any weight β and the proof is completed. □

8. $\mathcal{W}$-exactness

In this section we consider the case in which the natural transformation $K\epsilon : KQK \to K$ is an equivalence in $Hom(\mathbf{2}, \mathcal{X})$.

First observe that a calculation involving only the triangular isomorphisms of the $Q \dashv\!\!\| K$ adjointness provides:

Lemma 8.1. *$K\epsilon : K \to KQK$ is an equivalence in $Hom(\mathbf{2}, \mathcal{X})$ if and only if $Q\eta : Q \to QKQ$ is an equivalence in $Hom(\mathcal{W}^{op}, \mathcal{X})$.*

Proof. For any f and any G:

$$KQK_f \xrightarrow{K\epsilon_f} K_f \xrightarrow{\eta_{K_f}} KQK_f$$

is isomorphic to I_{KQK_f}, if and only if

$$QKQ_G \xrightarrow{\epsilon_{Q_G}} Q_G \xrightarrow{Q\eta_G} QKQ_G$$

is isomorphic to I_{QKQ_G}.

Naturality with respect to f and G follows from naturality of the adjointness.

□

Now the problem is to study when the biadjointess $Q \dashv\!\!\| K$ restricts to an equivalence between the subcategory of functors of the type $K_f : \mathcal{W}^{op} \to \mathcal{X}$ and that of functors of the type $Q_G : \mathbf{2} \to \mathcal{X}$.

As in the one dimensional case, kernels can be described by congruences, in the following sense.

Definition 8.2

A functor $G : \mathcal{W}^{op} \to \mathcal{X}$ is said to be a **precongruence** *in $\mathcal{X}$ (relative to the weights $\mathcal{W}$) if $G(\mathbf{2}(0, -)) \simeq Q_G(0)$.*

It is said to be a **congruence** *if moreover, for any object X in $\mathcal{X}$ there exists an arrow $g_X : \mathbf{2} \to Cat$ such that $\mathcal{X}(X, G) \simeq K_{g_X}$:*

$$\begin{array}{ccc} \mathcal{W}^{op} & & \\ {\scriptstyle G}\big\downarrow & \searrow {\scriptstyle K_{g_X}} & \\ \mathcal{X} & \xrightarrow{\mathcal{X}(X,-)} & Cat \end{array}$$

In other words, G is a congruence in $\mathcal{X}$ when, for any X the functor $\mathcal{X}(X, G) : \mathcal{W}^{op} \to Cat$ is a kernel in Cat.

Observe that, for any $f : \mathbf{2} \to \mathcal{X}$, the kernel K_f is a precongruence by the Yoneda equivalences of 6.1. It is moreover a congruence because the construction K is preserved by representables:

$$\mathcal{X}(X, K_f) \simeq K_{\mathcal{X}(X,f)}$$

For any arrow $f : \mathbf{2} \to \mathcal{X}$, its kernel K_f is called the **congruence associated** with f.

Definition 8.3

Say that a bicategory $\mathcal{X}$ is **exact** *(in the sense of $\mathcal{W}$) when it satisfies $\mathcal{W}$-regularity (Def. 7.1) and moreover every congruence G is equivalent to the kernel of its quotient: $\eta_G : G \simeq KQ_G$.*

An immediate consequence is now that, in an exact bicategory: $KQK_f \simeq K_f$ for any arrow f and hence also $QKQ_G \simeq Q_G$ is true for any G (Lemma 8.1).

Here is the result that extends directly to bicategories, provided a family of weights $\mathcal{W}$ is chosen, the known result on exact categories.

Theorem 8.4

In an exact bicategory, if the finite monicness assumption holds:

1) the canonical factorization constitutes a factorization structure,

2) every strong epimorphism is the quotient for the congruence associated to it.

Proof. *i)* For any arrow f in $\mathcal{X}$, by exactness $KQK_f \simeq K_f$. Theorem 7.3 applies and the second arrow of the canonical factorization is monic.

ii) If q is a strong epimorphism, consider its canonical factorization $q \simeq jp$. Here p is a strong epimorphism and j is monic. But j is also a strong epimorphism by the cancellation law 3 of Theorem 4.2. Hence j is an equivalence and $q \simeq QK_q$.

□

9. Further examples

Consider again the notions of monomorphism given in the examples of 3.6.

The weights:

$$
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\mathbf{2}(0,-)} & \mathbf{1} \\
\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\
2 & \longrightarrow & \mathbf{2} \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
3 & \longrightarrow & \mathbf{3}
\end{array}
$$

provide in Cat the factorization via an essentially surjective on objects functor followed by a fully-faithful one. The same factorization restricts to the category $Cat_{\times}$ of categories with finite products and products preserving functors i.e. to the category of algebraic theories.

In [13] Zawadowski shows that the above weights provide a factorization structure in the bicategory LFP of locally finitely presentable categories: any functor in LFP is factored into a weakly surjective followed by a fully faithful functor ($F : \mathcal{A} \to \mathcal{B}$ in LFP is *weakly surjective* if the smallest full subcategory of $\mathcal{B}$, containing the image of F and closed under limits, filtered colimits and isomorphisms is $\mathcal{B}$ itself).

Again in [13], a lex morphism $I : \mathbf{C} \to \mathbf{D}$ is called *strongly conservative* if composition with I is a functor I^* (in LFP) which is weakly surjective. Moreover, I is said to be a *false quotient* if I^* is fully-faithful. Hence, with the terminology of [13] and by means of the Gabriel-Ulmer duality between Lex and LFP, one has that the above weights provide the "false quotient - strongly conservative" factorization in *Lex* (the *coregular* factorization).

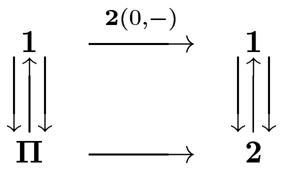
The papers [9] and [13] give more examples of the coregular factorization: it coincides with the "quotient - conservative" factorization in the bicategories Ex^{op} (Barr exact categories, [1]), in $Pretop^{op}$ and in $BPretop^{op}$ (pretopoi and boolean pretopoi respectively).

In *Lex*, the "quotient - conservative" factorization (recall that *Lex* is a *conservational bicategory* in the sense of [5]) can be obtained by the weights:

$$
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\mathbf{2}(0,-)} & \mathbf{1} \\
\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\
\mathbf{2} & \longrightarrow & \tilde{\mathbf{2}}
\end{array}
$$

The same factorization restricts to the category *Ladj* of left exact categories and left adjoints as arrows, considered by Day [6], where it becomes the factorization into a reflection followed by a conservative left adjoint. Moreover, it restricts to the bicategory Top^{op} of elementary topoi and inverse images of geometric morphisms as arrows: in this case it becomes the "surjection - inclusion" (inverse images) factorization.

Yet another notion of monomorphism is considered in 3.6 by means of the weights:



where Π is the category $\cdot \rightrightarrows \cdot$ with two non trivial parallel arrows. It is easy to see that these weights provide a factorization structure in *Cat* and in some of its subcategories (such as *Lex*, for instance).

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