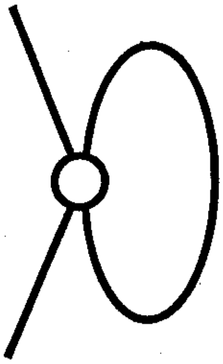
Traced Monoidal Categories



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Acknowledgments.

I would like to thank my supervisor, Ross Street, for his ideas, for being available and willing to help, for his corrections and for his encouragement. I would like to thank Elaine Vaughan, who gave me cheerful and friendly help with all the practicalities. I am also grateful to the late Doris Wallent, for providing the scholarship which I was awarded in my honours year for women in Mathematics at Macquarie.

Introduction.

This essay is based on the paper [JSV] by the same title, and presents a vast generalisation of the construction of the integers from the natural numbers. To construct the integers we take pairs (m,n) of natural numbers which we think of as $m-n$. The integers are given by the equivalence classes defined by $(m,n)\approx(m',n')$ when $m+n'=m'+n$ in $\mathbb{N}$ which we think of as $m-n=m'-n'$. Here, we construct a tortile monoidal category from a traced one, that is, we add duals. We define a new category Int$\mathcal{V}$ by taking pairs (X,U) of objects of the traced category $\mathcal{V}$, and we think of this pair as a formalisation of $X\otimes U^*$. We give definitions for the required structure, and prove that Int$\mathcal{V}$ is actually tortile. We finish with a universal property for this construction.

To prove that our constructions are the right ones, an exact correspondence between algebra and geometry is described and validated, then used to prove the categorical results. We begin by giving proofs both algebraically and pictorially, to familiarise the reader with this correspondence. We see that the pictorial reasoning is often much more enlightening and easier to use in the proofs than the algebra, hence we discontinue the algebraic proofs. However, these can be constructed from the pictorial proofs, as each step in the three dimensional reasoning corresponds precisely to an algebraic equality.

We do extend [JSV]'s paper somewhat, providing detailed algebraic proofs which were not a part of the original paper, and providing more details for all of the pictorial proofs. A new axiom is given, and proved to be equivalent to one of the original axioms, and we complete the proof of Proposition 5.1 by including the proof that the twist is natural in Int$\mathcal{V}$, which had previously been omitted.

We also add an extra initial chapter, presenting the necessary categorical definitions for the reader who is familiar only with the basic ideas of category theory.

In section 2.1 we present a 3 dimensional geometrical representation for some of the basic algebraic components of a balanced monoidal category. It has been shown [JS2] that any progressive 3 dimensional reasoning is valid in the balanced case. We extend the diagrammatic methodology in part to a traced monoidal category in that the algebraic definition of this is also presented pictorially.

Section 2.2 gives both algebraic and pictorial proofs of some useful results that follow from the axioms and establishes a new axiom, swallowing, which is proved to be equivalent to the original superposing axiom. To use the geometric reasoning in the traced case, we must be careful to use only the reasoning that is either part of the definition or has been proved to be a consequence of it. We cannot use all 3-dimensional progressive reasoning as in the balanced case. We also give a definition of a traced (trace preserving) monoidal functor.

Chapter 3 concerns tortile monoidal categories, giving the diagrammatic representation for unit and counit in these, and defining a canonical trace in terms of these. We show that this is a trace for any tortile monoidal category, and that any balanced monoidal functor between tortile monoidal categories preserves this canonical trace.

We then describe a new category Int$\mathcal{V}$ in section 4.1. We define objects, arrows, composition and identity, which we prove do give a category, and we define a functor $N:\mathcal{V}\rightarrow \mathrm{Int}\mathcal{V}$, which we prove is fully faithful.

Section 4.2 adds a tensor product to Int$\mathcal{V}$ which we prove gives it a monoidal structure.

Chapter 5 then defines a twist, a braiding, dual of objects and arrows, unit and counit, which are proved to give Int$\mathcal{V}$ a tortile monoidal structure. We then state a universal property for this construction.

Chapter I
Introductory Definitions

The aim of this chapter is to provide the reader, who is familiar with the basic ideas of category theory, with the necessary definitions to understand [JSV], and hence this essay. The reader should be familiar with notions such as category, functor, natural transformation, and isomorphism, since, in the interest of space, these will not be defined. This chapter does give definitions of (strict) monoidal (or tensor) category, (strict) monoidal functor, braiding on a monoidal category, braided monoidal category, braided monoidal functor, twist on a monoidal category, balanced monoidal category, balanced monoidal functor, dual, unit and counit, autonomous monoidal category, and tortile monoidal category. These definitions all come from [JS1].

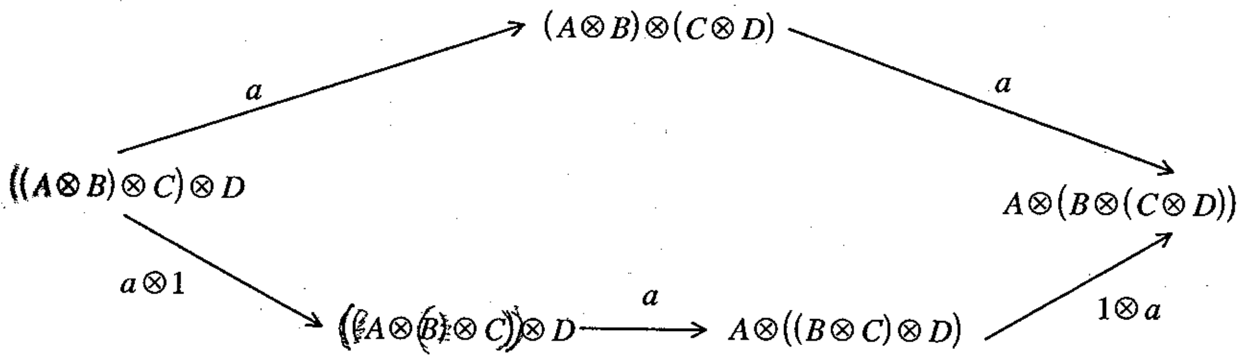
Definition 1.1 Monoidal category, monoidal functor, and strictness
A *monoidal category* (or *tensor category*) $\mathcal{V} = (\mathcal{V}, \otimes, I, a, l, r)$ consists of a category $\mathcal{V}$, a functor $\otimes: \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ (called the *tensor product*), an object I (called the *unit object*), and natural isomorphisms

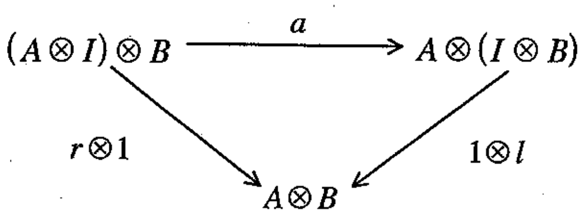
$$a = a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$$

$$l = l_A : I \otimes A \xrightarrow{\sim} A$$

$$r = r_A : A \otimes I \xrightarrow{\sim} A$$

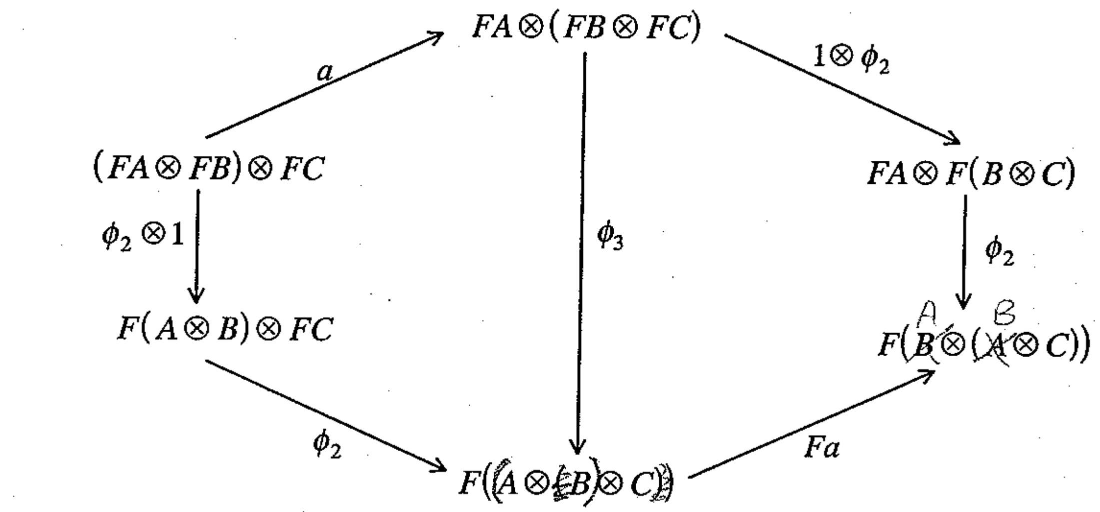
(called the *associativity*, *left unit* and *right unit constraints*, respectively) such that the following two diagrams (called the *associativity pentagon* and the *triangle for unit*) commute.

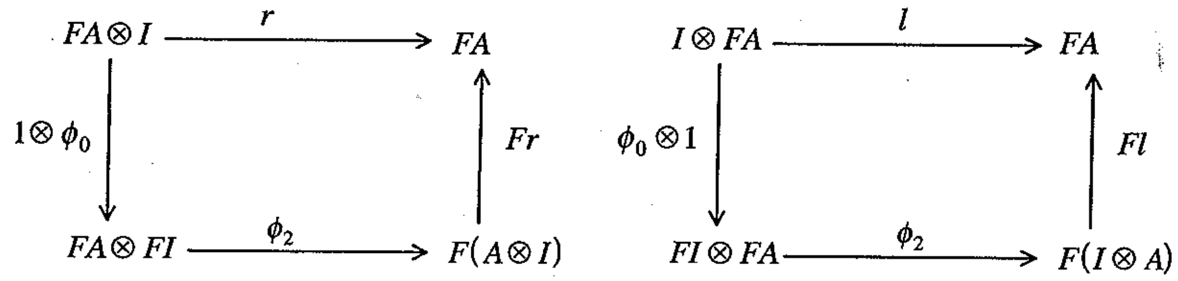




A monoidal category is called *strict* when all the constraints $a_{A,B,C}$, l_A, r_A are identities.

Suppose $\mathcal{V}$, $\mathcal{W}$ are monoidal categories. A *monoidal functor* $F = (F, \phi_2, \phi_0) : \mathcal{V} \to \mathcal{W}$ consists of a functor $F : \mathcal{V} \to \mathcal{W}$, a family of natural isomorphisms $\phi_{2,A,B} : FA \otimes FB \xrightarrow{\sim} F(A \otimes B)$ and an isomorphism $\phi_0 : I \xrightarrow{\sim} FI$ such that the following three diagrams commute:





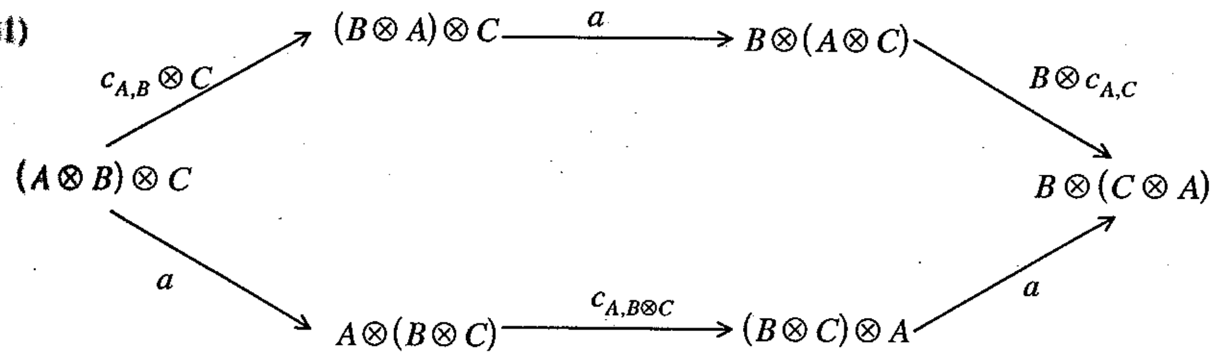
where ϕ_3 is defined by the above diagram.

The monoidal functor is called *strict* when each of the isomorphisms $\phi_{2,A,B}$, ϕ_0 is an identity.

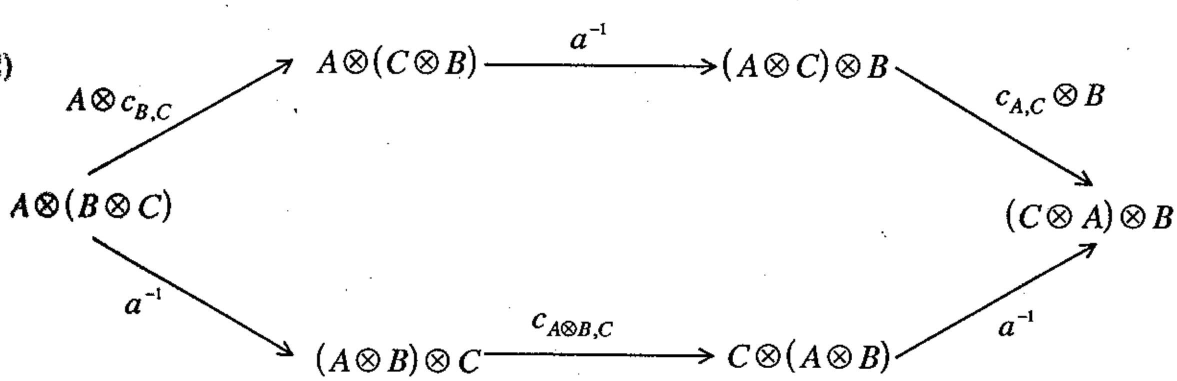
Definition 1.2 Braiding; braided monoidal category and functor.

A *braiding* for a monoidal category $\mathcal{V}$ consists of a natural family of isomorphisms $c = c_{A,B}: A\otimes B \xrightarrow{\sim} B\otimes A$ in $\mathcal{V}$ such that the two diagrams (B1) and (B2) commute.

(B1)



(B2)



A *braided monoidal category* is a monoidal category $\mathcal{V}$ with a chosen braiding c.

Suppose $\mathcal{V}, \mathcal{W}$ are braided monoidal categories. A monoidal functor $F : \mathcal{V} \to \mathcal{W}$ is said to be *braided* when the following square commutes:

$$\begin{array}{ccc} FA \otimes FB & \xrightarrow{\ \phi_2\ } & F(A \otimes B) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle Fc} \\ FB \otimes FA & \xrightarrow[\ \phi_2\ ]{} & F(B \otimes A) \end{array}$$

Definition 1.3 [JS1]: Twist; balanced monoidal category and functor

Suppose $\mathcal{V}$ is a braided monoidal category. A *(full) twist* for $\mathcal{V}$ is a natural family of isomorphisms

$$\theta = \theta_A : A \xrightarrow{\ \sim\ } A$$

such that $\theta_I = 1_I$ and the following diagram (T) commutes.

(T)

$$\begin{array}{ccc} A \otimes B & \xleftarrow{\ c_{B,A}\ } & B \otimes A \\ {\scriptstyle \theta_{A\otimes B}}\uparrow & & \uparrow{\scriptstyle \theta_B \otimes \theta_A} \\ A \otimes B & \xrightarrow[\ c_{A,B}\ ]{} & B \otimes A \end{array}$$

A monoidal category equipped with a braiding and a twist is called *balanced*.

For $\mathcal{V}, \mathcal{W}$ balanced monoidal categories, a monoidal functor $F : \mathcal{V} \to \mathcal{W}$ is called *balanced* when it is braided and preserves the twist (that is, $F\theta_A = \theta_{FA}$). We write $\mathcal{BT}en(\mathcal{V},\mathcal{W})$ for the category of balanced monoidal functors and morphisms of monoidal functors.

Definition 1.4 : Dual, unit, counit, autonomous and tortile monoidal category.

When the following adjunction triangles commute for $\eta : I \to B \otimes A$, $\varepsilon : A \otimes B \to I$, we say that the pair (η, ε) is an *adjunction between A and B*, and that A (respectively B) is *left adjoint* or *left dual* to B (respectively *right adjoint* or *right dual* to A). We write $(\eta, \varepsilon) : A \dashv B$, and call η the *unit* and ε the *counit* of the adjunction.

$$\begin{array}{ccc} A & \xrightarrow{\ 1\otimes\eta\ } & A \otimes B \otimes A \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \varepsilon\otimes 1} \\ & & A \end{array} \qquad\qquad \begin{array}{ccc} B & \xrightarrow{\ \eta\otimes 1\ } & B \otimes A \otimes B \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle 1\otimes\varepsilon} \\ & & B \end{array}$$

A monoidal category is *left (right) autonomous* when every object has a left (right) dual. It is *autonomous* when it is both left and right autonomous. In this case we write A^* for a chosen left adjoint for A. We obtain a duality functor $(\ )^{*}: \mathcal{V}^{op} \to \mathcal{V}$ by defining, for all $f: A \to B$, the arrow $f^{*}: B^{*} \to A^{*}$ to be the composite $B^{*} \xrightarrow{1\otimes\eta} B^{*} \otimes A \otimes A^{*} \xrightarrow{1\otimes f\otimes 1} B^{*} \otimes B \otimes A^{*} \xrightarrow{\varepsilon\otimes 1} A^{*}$.

A monoidal category is called *tortile* when it is autonomous and balanced, and, for all objects A, $\theta_{A^{*}} = {\theta_{A}}^{*} : A^{*} \to A^{*}$, where $(\ )^{*}: A^{op} \to A$ is the adjunction (or duality) functor.

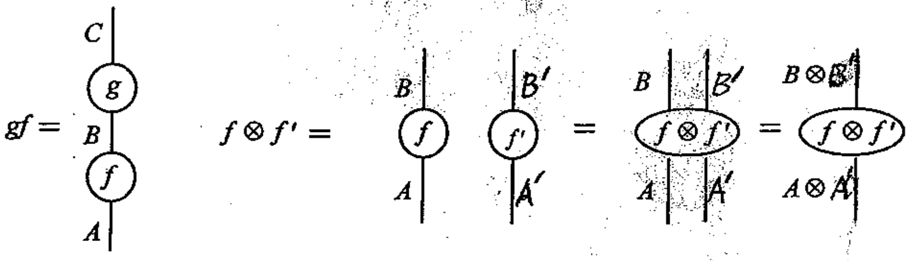
Chapter 2
Pictures and trace

The material in this chapter covers section 2 of [JSV] and includes some extra material. We look at the representation of expressions in traced monoidal categories by pictures. In section 2.1, we look at the pictures for a balanced monoidal category, and their extension to a traced monoidal category, which we define. In section 2.2 we make some slight improvements or modifications to [JSV]'s corresponding material, giving an equivalent axiom for superposing, and giving detailed proofs algebraically and pictorially for the lemmas stated in [JSV]. [JS2] rigorously shows that three dimensional reasoning on a balanced monoidal category is valid, and this chapter extends the diagrammatic methodology to a traced monoidal category, although we do not prove all three dimensional reasoning is valid in this case. We need to be careful then, in all our pictorial proofs, that the reasoning used where trace is concerned has actually been justified. However if we are careful, we no longer need the algebraic proof, just the pictorial, which is much easier to follow.

We also note that by the coherence theorem for monoidal categories [JS1], each such is equivalent to a strict one, hence for simplicity we write as if our monoidal categories were strict.

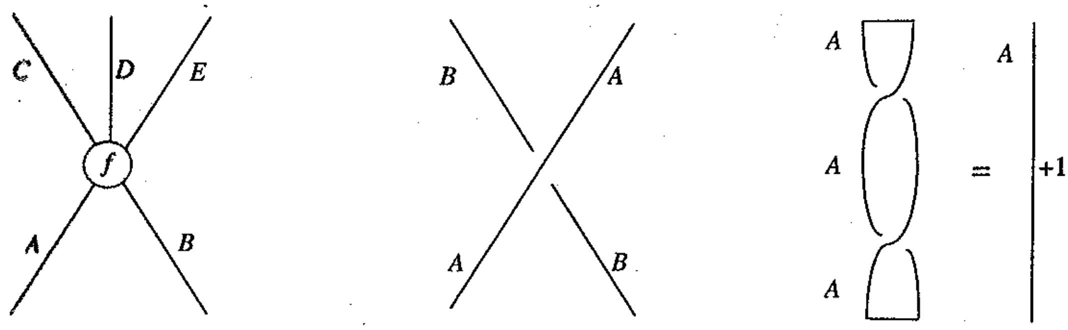
2.1 Pictures and definitions for a traced monoidal category

We start with the pictures for a balanced monoidal category. We have composition and tensoring of arrows; that is, we compose arrows $f:A \to B$, $g:B \to C$ to get $gf:A \to C$ and tensor arrows $f:A \to B, f':A' \to B'$ to get $f \otimes f':A \otimes A' \to B \otimes B'$, depicted (respectively) as follows.



We will use the pictures for $f \otimes f'$ interchangeably from now on.

We also have pictures for an arrow $f:A \otimes B \to C \otimes D \otimes E$, the braiding $c_{A,B}:A \otimes B \to B \otimes A$, and the twist $\theta_A:A \to A$, respectively, as follows.



To represent the braiding we need three dimensions (under and over crossings), and to accommodate the twist we need ribbons. However, for ease of drawing we use lines labelled with +1 for a twist on a ribbon, and labelled with -1 for a reverse twist.

Definition 2.1: A *trace* for a balanced monoidal category $\mathcal{V}$ is a natural family of functions

$$\mathrm{Tr}_{A,B} : \mathcal{V}(A \otimes U, B \otimes U) \to \mathcal{V}(A, B)$$

satisfying three axioms

vanishing:

$$\text{(V1)} \quad \mathrm{Tr}^{I}_{A,B}(f) = f, \qquad \text{(V2)} \quad \mathrm{Tr}^{U \otimes V}_{A,B}(g) = \mathrm{Tr}^{U}_{A,B}\left(\mathrm{Tr}^{V}_{A\otimes U, B \otimes U}(g)\right);$$

superposing:

$$\mathrm{Tr}^{U}_{A\otimes C, B\otimes D}\left(\left(1_B \otimes c^{-1}_{D,U}\right) \circ (f \otimes g) \circ \left(1_A \otimes c_{C,U}\right)\right) = \mathrm{Tr}^{U}_{A,B}(f) \otimes g$$
$$= \mathrm{Tr}^{U}_{A\otimes C, B\otimes D}\left(\left(1_B \otimes c_{U,D}\right) \circ (f \otimes g) \circ \left(1_A \otimes c^{-1}_{U,C}\right)\right)$$

and *yanking:*

$$\mathrm{Tr}^{U}_{U,U}\left(c_{U,U} \circ \left(\theta_U^{-1} \otimes 1_U\right)\right) = 1_U = \mathrm{Tr}^{U}_{U,U}\left({c_{U,U}}^{-1} \circ \left(\theta_U \otimes 1_U\right)\right)$$

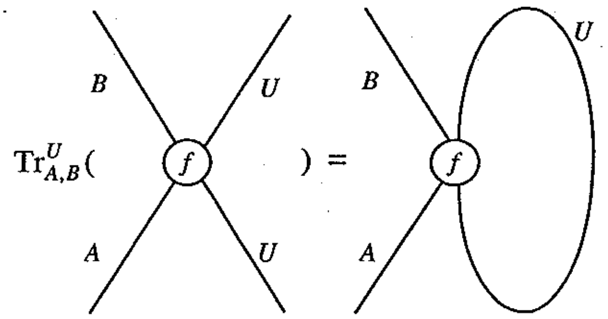
A *traced monoidal category* is a balanced monoidal category equipped with a trace.

For $\mathcal{V}$, $\mathcal{W}$ traced monoidal categories, we say a monoidal functor $F : \mathcal{V} \to \mathcal{W}$ is *traced* when it is balanced and preserves trace in the following sense:

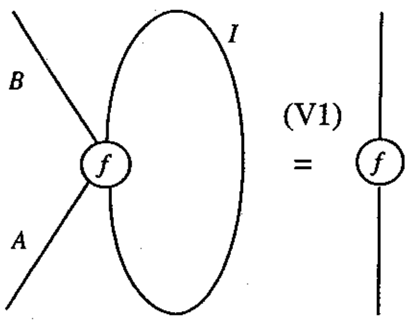
$$\mathrm{Tr}^{FU}_{FA,FB}\left(FA \otimes FU \xrightarrow{\phi_{2,A,U}} F(A \otimes U) \xrightarrow{Ff} F(B \otimes U) \xrightarrow{{\phi_{2,B,U}}^{-1}} FB \otimes FU \right)$$
$$= F\left(\mathrm{Tr}^{U}_{A,B}(f)\right) : FA \to FB .$$

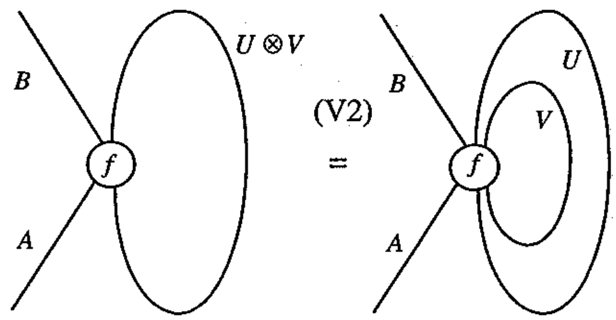
For a traced monoidal category $\mathcal{V}$, we have the following pictures:

The *trace* of $f : A \otimes U \to B \otimes U$; that is, $\mathrm{Tr}^{U}_{A,B}(f)$:

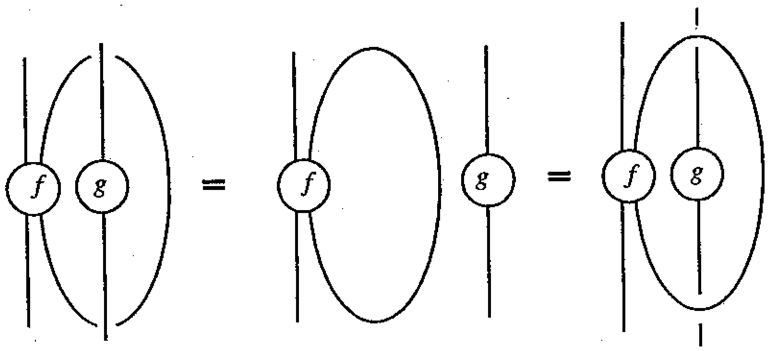


vanishing:

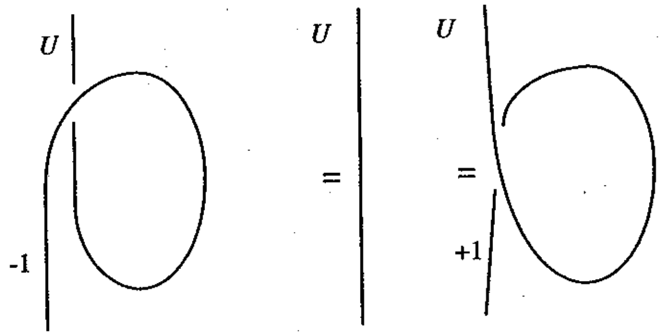




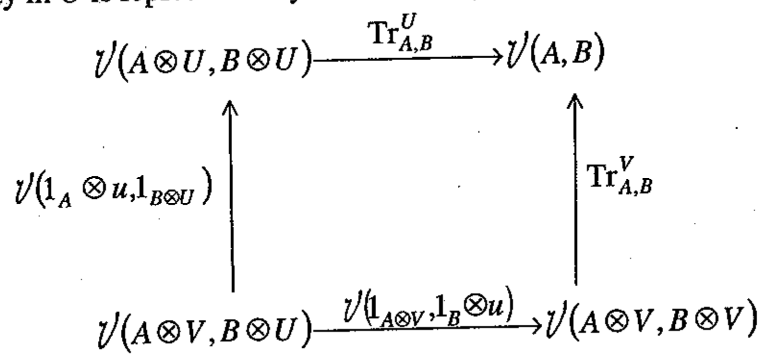
superposing



and *yanking*



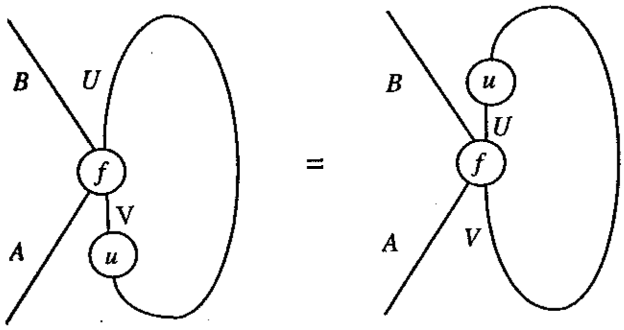
Naturality of $\mathrm{Tr}^{U}_{A,B}$ gives diagrams for di-naturality in U and naturality in A, B. Di-naturality in U is represented by the following commutative square, where $u\colon U \to V$



We call this *sliding*; that is, for $f : A\otimes V \to B\otimes U$, it means

$$\mathrm{Tr}^{U}_{A,B}\big(f\circ(1_A\otimes u)\big)=\mathrm{Tr}^{V}_{A,B}\big((1_B\otimes u)\circ f\big);$$

or diagrammatically

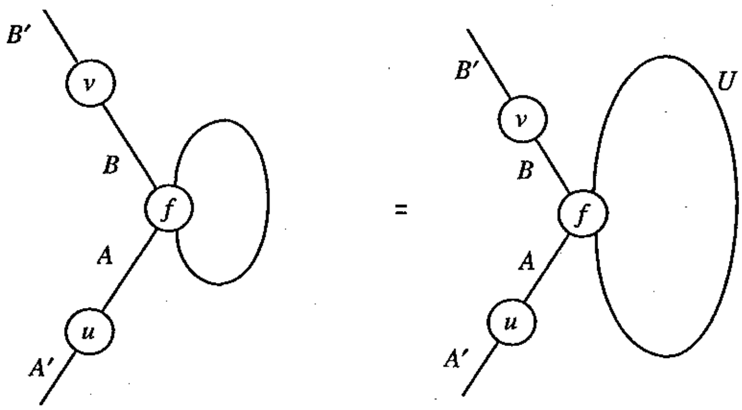


Naturality in A,B is represented by the following commutative square, where $u: A' \to A$, $v: B \to B'$.

$$\begin{array}{ccc} \mathcal{V}(A\otimes U, B\otimes U) & \xrightarrow{\mathrm{Tr}^U_{A,B}} & \mathcal{V}(A,B) \\ \mathcal{V}(u\otimes 1_U, v\otimes 1_U)\Big\downarrow & & \Big\downarrow \mathcal{V}(u,v) \\ \mathcal{V}(A'\otimes U, B'\otimes U) & \xrightarrow{\mathrm{Tr}^U_{A',B'}} & \mathcal{V}(A',B') \end{array}$$

We call this *tightening*; that is, for $f: A\otimes U \to B\otimes U$, it means

$$v\circ \mathrm{Tr}^U_{A,B}(f)\circ u = \mathrm{Tr}^V_{A',B'}((v\otimes 1_U)\circ f\circ (u\otimes 1_U))$$

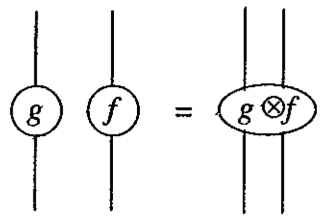


Remark 2.2: In addition to [JSV], we observe that if we replace twists with their reverse and crossings $c_{A,B}: A\otimes B \to B\otimes A$ with inverse crossings $c_{B,A}{}^{-1}: A\otimes B \to B\otimes A$, we get another balanced monoidal category with the same trace as the original since the axioms for trace either do not involve crossings or twists, or are symmetric. This means that any results that we prove containing c,θ are also valid if we replace $c_{A,B}$ with $c_{B,A}{}^{-1}$ and replace θ with θ^{-1}.

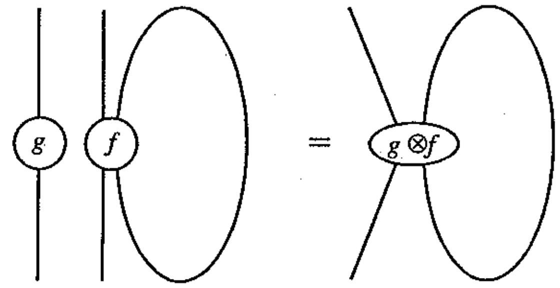
2.2 Consequences of the trace axioms

We now look at making the definition of trace slightly easier to use. Our first modification is to show the equivalence of superposing and *swallowing*. Then we look at sliding, and find that we can weaken it to *sliding of crossings* which imply all other slidings in the presence of the axioms. We also present a few properties that we will use in later proofs. To start with, the proofs will be done both algebraically and pictorially. Looking at the pictorial proof gives the motivation for the steps taken in the algebraic proof. Once we have set up the pictures, we will no longer need to do the algebraic proofs, but putting a few of them in shows why it really is easier to use pictorial proofs. Properties of balanced categories are used in the proofs, labelled explicitly to start with, and later labelled with just *balance*.

Before we start proving things we need to look at a subtlety in the pictures. In a balanced category we have

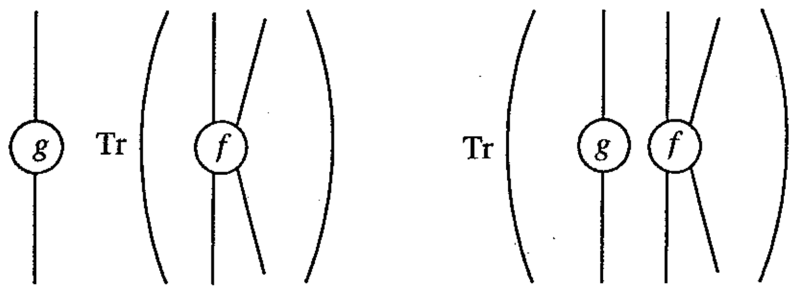


Using this reasoning in a *traced* monoidal category, we obtain $g \otimes \mathrm{Tr}(f) = \mathrm{Tr}(g \otimes f)$; pictorially



We call this swallowing, and prove it is equivalent to superposing, hence it is true in a traced monoidal category. This means we can use the above picture to represent both.

However, in the proof that superposing implies swallowing, we cannot assume swallowing, but using the picture for $\mathrm{Tr}(g \otimes f)$ reduces much of the pictorial proof to an algebraic proof within the [illegible]. Instead, we rather awkwardly use the following pictures to distinguish $g \otimes \mathrm{Tr}(f)$ and $\mathrm{Tr}(g \otimes f)$, respectively:



Then we return to the pictures for a balanced category until we again have $\mathrm{Tr}(g \otimes f)$ or $g \otimes \mathrm{Tr}(f)$ in the algebra. In other proofs where swallowing has been used, we will just use the picture for $g \otimes \mathrm{Tr}(f)$ as this is more enlightening.

Proposition 2.3: *Swallowing ⇔ superposing*

That is, $\{ \mathrm{Tr}^U_{C\otimes A,D\otimes B}(g\otimes f) = g\otimes Tr^U_{A,B}(f) \} \Leftrightarrow$

$$\{ \mathrm{Tr}^U_{A\otimes C,B\otimes D}\left(\left(1_B\otimes c^{-1}_{D,U}\right)\circ\left(f\otimes g\right)\circ\left(1_A\otimes c_{C,U}\right)\right) = \mathrm{Tr}^U_{A,B}(f)\otimes g$$

$$= \mathrm{Tr}^U_{A\otimes C,B\otimes D}\left(\left(1_B\otimes c_{U,D}\right)\circ\left(f\otimes g\right)\circ\left(1_A\otimes c^{-1}_{U,C}\right)\right)\}$$

Proof: Assume superposing.

Then $g\otimes Tr^U_{A,B}(f) \overset{naturality}{=} c_{D,B}{}^{-1}\circ\left(\mathrm{Tr}^U_{A,B}(f)\otimes g\right)\circ c_{C,A}$ (12)

$\overset{superposing}{=} c_{D,B}{}^{-1}\circ \mathrm{Tr}^U_{A\otimes C,B\otimes D}\left(\left(1_B\otimes c^{-1}_{D,U}\right)\circ\left(f\otimes g\right)\circ\left(1_A\otimes c_{C,U}\right)\right)\circ c_{C,A}$ (13)

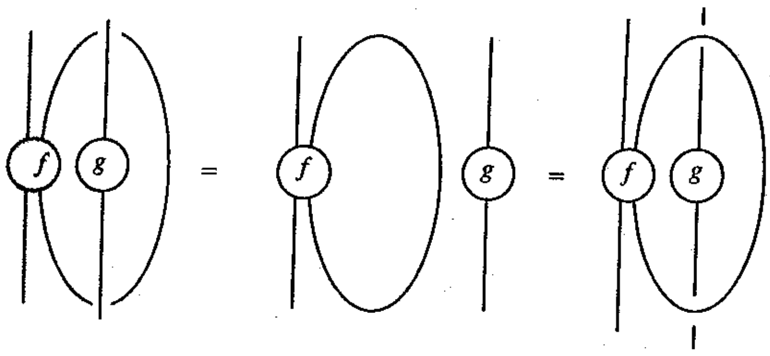
$\overset{[illegible]}{=} \mathrm{Tr}^U_{C\otimes A,D\otimes B}\left(\left(c_{D,B}{}^{-1}\otimes 1_U\right)\circ\left(1_B\otimes c^{-1}_{D,U}\right)\circ\left(f\otimes g\right)\circ\left(1_A\otimes c_{C,U}\right)\circ\left(c_{C,A}\otimes 1_U\right)\right)$ (14)

$\overset{[illegible]}{=} \mathrm{Tr}^U_{C\otimes A,D\otimes B}\left(c_{D,B\otimes U}{}^{-1}\circ\left(f\otimes g\right)\circ c_{C,A\otimes U}\right)$ (15)

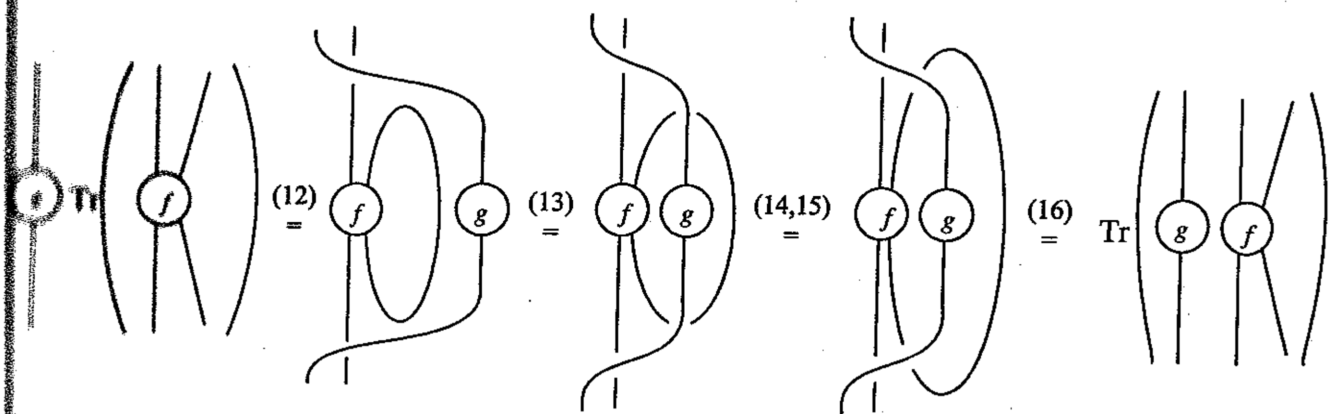
$\overset{naturality}{=} \mathrm{Tr}^U_{C\otimes A,D\otimes B}(g\otimes f)$, and so we have swallowing. (16)

We also give the proof pictorially, and since we are trying to prove swallowing, we use the alternative pictures.

Assume superposing:



Then



Now assume swallowing, that is, $\mathrm{Tr}^{U}_{C\otimes A,D\otimes B}(g\otimes f)=g\otimes Tr^{U}_{A,B}(f)$. (1)

Then $\mathrm{Tr}^{U}_{A\otimes C,B\otimes D}\left(\left(1_B\otimes c^{-1}_{D,U}\right)\circ(f\otimes g)\circ\left(1_A\otimes c_{C,U}\right)\right)$

$$= \mathrm{Tr}^{U}_{A\otimes C,B\otimes D}\left(\left(c_{D,B}\otimes 1_U\right)\circ c^{-1}_{D,B\otimes U}\circ(f\otimes g)\circ c_{C,A\otimes U}\circ\left(c^{-1}_{C,A}\otimes 1_U\right)\right) \tag{2}$$

$$= c_{D,B}\circ \mathrm{Tr}^{U}_{C\otimes A,D\otimes B}\left(c^{-1}_{D,B\otimes U}\circ(f\otimes g)\circ c_{C,A\otimes U}\right)\circ c^{-1}_{C,A} \tag{3}$$

$$= c_{D,B}\circ \mathrm{Tr}^{U}_{C\otimes A,D\otimes B}(g\otimes f)\circ c^{-1}_{C,A} \tag{4}$$

$$= c_{D,B}\circ\left(g\otimes \mathrm{Tr}^{U}_{A,B}(f)\right)\circ c^{-1}_{C,A} \tag{5}$$

$$= \mathrm{Tr}^{U}_{A,B}(f)\otimes g \tag{6}$$

$$= c^{-1}_{B,D}\circ\left(g\otimes \mathrm{Tr}^{U}_{A,B}(f)\right)\circ c_{A,C} \tag{7}$$

$$= c^{-1}_{B,D}\circ \mathrm{Tr}^{U}_{C\otimes A,D\otimes B}(g\otimes f)\circ c_{A,C} \tag{8}$$

$$= c^{-1}_{B,D}\circ \mathrm{Tr}^{U}_{C\otimes A,D\otimes B}\left(c_{B\otimes U,D}\circ(f\otimes g)\circ c^{-1}_{A\otimes U,C}\right)\circ c_{A,C} \tag{9}$$

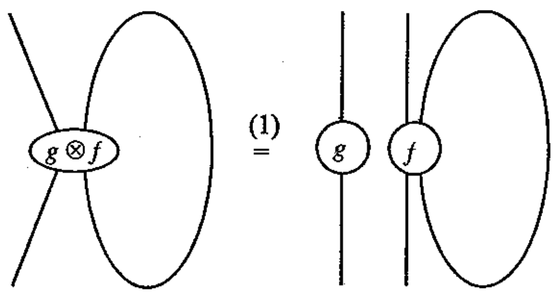
$$= \mathrm{Tr}^{U}_{A\otimes C,B\otimes D}\left(\left(c^{-1}_{B,D}\otimes 1_U\right)\circ c_{B\otimes U,D}\circ(f\otimes g)\circ c^{-1}_{A\otimes U,C}\circ\left(c_{A,C}\otimes 1_U\right)\right) \tag{10}$$

$$= \mathrm{Tr}^{U}_{A\otimes C,B\otimes D}\left(\left(1_B\otimes c_{U,D}\right)\circ(f\otimes g)\circ\left(1_A\otimes c^{-1}_{U,C}\right)\right). \tag{11}$$

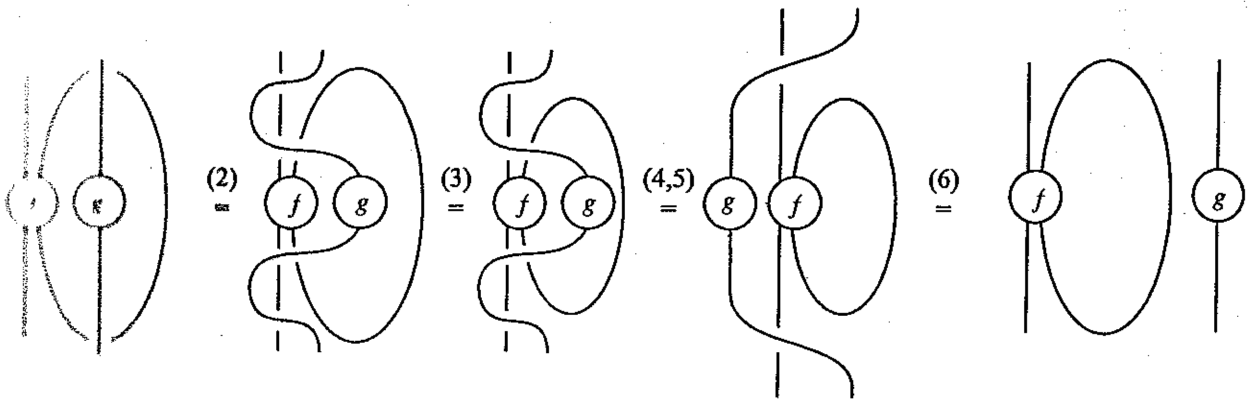
So we have superposing from(2), (6) and (11).

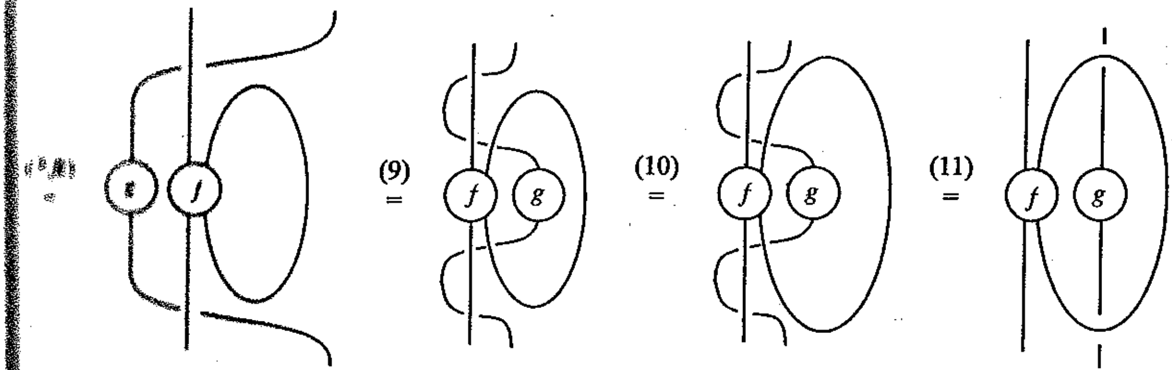
We give the proof pictorially, and since we assume swallowing, we do not need to use the alternative form of picture. The proof is as follows:

Assume



Then





We now can use swallowing and superposing interchangeably.

Remark 2.4: $\theta_U = \mathrm{Tr}^U_{U,U}\left(c_{U,U}\right)$ and $\theta_U{}^{-1} = \mathrm{Tr}^U_{U,U}\left(c_{U,U}{}^{-1}\right)$.

Proof: $1_U = \mathrm{Tr}^U_{U,U}\left(c_{U,U} \circ \left(\theta_U{}^{-1} \otimes 1_U\right)\right)$ yanking

$\theta_U = \mathrm{Tr}^U_{U,U}\left(c_{U,U} \circ \left(\theta_U{}^{-1} \otimes 1_U\right)\right) \circ \theta_U$

$= \mathrm{Tr}^U_{U,U}\left(c_{U,U}\right) \circ \theta_U{}^{-1} \circ \theta_U$ tightening

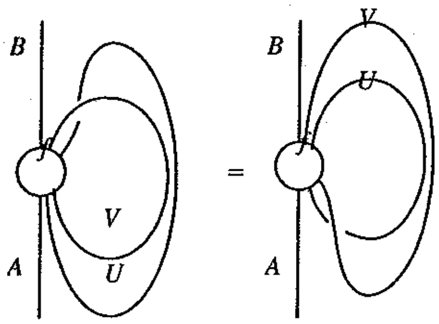
$= \mathrm{Tr}^U_{U,U}\left(c_{U,U}\right)$

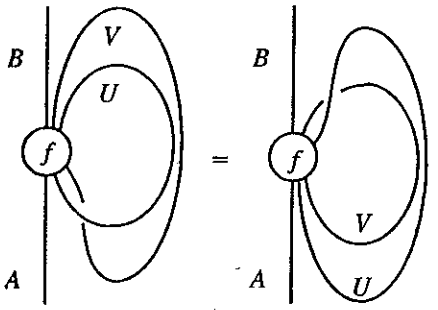
Similarly, $\theta_U{}^{-1} = \mathrm{Tr}^U_{U,U}\left(c_{U,U}{}^{-1}\right)$.

Lemma 2.5: *Sliding of crossings implies all other slidings in the presence of the other axioms.*

Proof: First we will prove that sliding of crossings implies sliding of twists, and then that these two imply all slidings in the presence of the other axioms. Actually, we only use the axioms yanking, tightening, and superposing/swallowing, as well as some observations about a balanced category.

Assume sliding of crossings; that is, for $f : A \otimes U \otimes V \to B \otimes V \otimes U$,





Algebraically, the first of these can be written as either

$$\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^V_{A\otimes U,B\otimes U}\left(\left(1_B \otimes c_{V,U}\right)\circ f\right)\right)=\mathrm{Tr}^V_{A,B}\left(\mathrm{Tr}^U_{A\otimes V,B\otimes V}\left(f\circ\left(1_A\otimes c_{V,U}\right)\right)\right)$$

or

$$\mathrm{Tr}^{U\otimes V}_{A,B}\left(\left(1_B \otimes c_{V,U}\right)\circ f\right)=\mathrm{Tr}^{V\otimes U}_{A,B}\left(f\circ\left(1_A\circ c_{V,U}\right)\right).$$

We note that, in the presence of axiom (V2), these are equivalent, and we use the first of these in the diagrams. Similarly, we express the second diagram algebraically as

$$\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^V_{A\otimes U,B\otimes U}\left(\left(1_B \otimes c_{U,V}{}^{-1}\right)\circ f\right)\right)=\mathrm{Tr}^V_{A,B}\left(\mathrm{Tr}^U_{A\otimes V,B\otimes V}\left(f\circ\left(1_A\otimes c_{U,V}{}^{-1}\right)\right)\right).$$

Now for $f : A\otimes U \to B\otimes U$,

$$\mathrm{Tr}^U_{A,B}\left(\left(1_B \otimes \theta_U\right)\circ f\right)$$

$$= \mathrm{Tr}^U_{A,B}\left(\left(1_B \otimes \mathrm{Tr}^U_{U,U}\left(c_{U,U}\right)\right)\circ f\right) \quad (1)$$

$$= \mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(1_B \otimes c_{U,U}\right)\circ f\right) \quad (2)$$

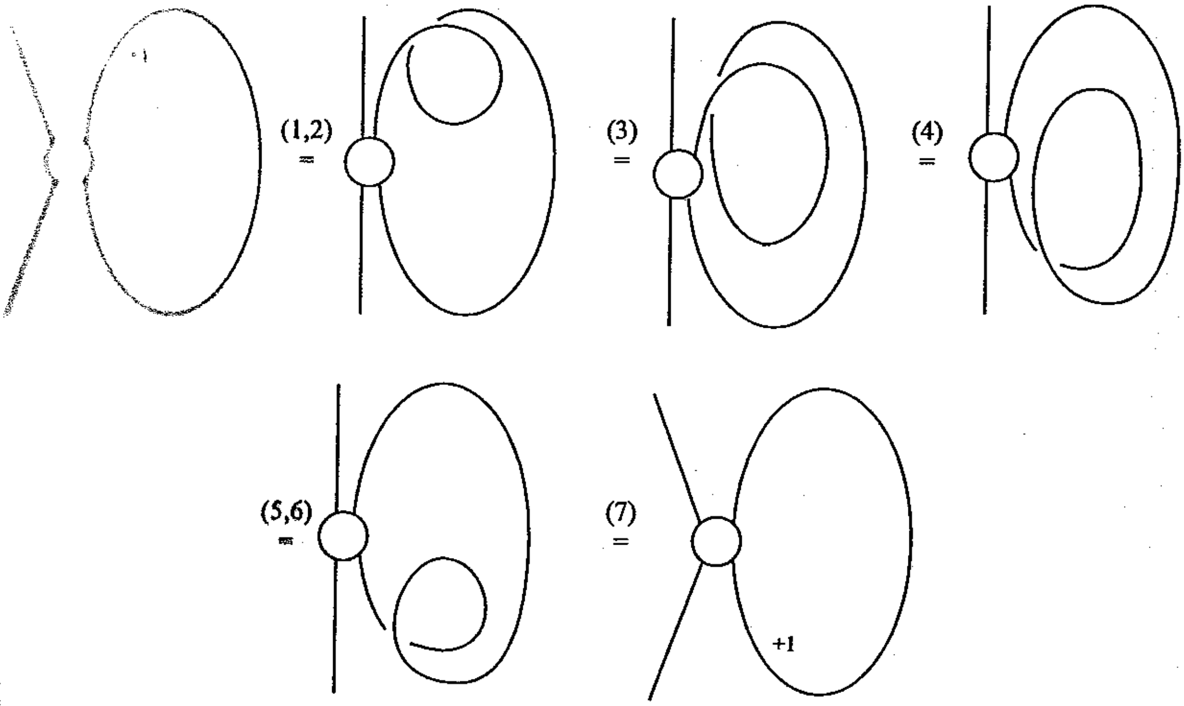
$$= \mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{A\otimes U,B\otimes U}\left(1_B \otimes c_{U,U}\right)\circ\left(f\otimes 1_U\right)\right) \quad (3)$$

$$= \mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{A\otimes U,B\otimes U}\left(f\otimes 1_U\right)\circ\left(1_A\otimes c_{U,U}\right)\right) \quad (4)$$

$$= \mathrm{Tr}^U_{A,B}\left(f\circ \mathrm{Tr}^U_{A\otimes U,A\otimes U}\left(1_A \otimes c_{U,U}\right)\right) \quad (5)$$

$$= \mathrm{Tr}^U_{A,B}\left(f\circ \left(1_A \otimes \mathrm{Tr}^U_{A\otimes U,A\otimes U}\left(c_{U,U}\right)\right)\right) \quad (6)$$

$$= \mathrm{Tr}^U_{A,B}\left(f\circ\left(1_A\otimes\theta_U\right)\right) \quad (7)$$



Similarly, $\mathrm{Tr}^U_{A,B}\left(\left(1_B \otimes \theta_U^{-1}\right)\circ f\right) = \mathrm{Tr}^U_{A,B}\left(f \circ \left(1_A \otimes \theta_U^{-1}\right)\right)$

We now wish to see that sliding of crossings and twists imply all slidings in the presence of the other axioms.

Let $f : A \otimes V \to B \otimes U$, $u : U \to V$,

$$\mathrm{Tr}^V_{A,B}\left(\left(1_B \otimes u\right)\circ f\right) = \mathrm{Tr}^V_{A,B}\left(\left(1_B \otimes u\right)\circ\left(1_B \otimes 1_U\right)\circ f\right) \tag{8}$$

$$= \mathrm{Tr}^V_{A,B}\left(\left(1_B \otimes u\right)\circ\left(1_B \otimes \mathrm{Tr}^U_{U,U}\left(c_{U,U}\circ\left(\theta_U^{-1}\otimes 1_U\right)\right)\right)\circ f\right) \tag{9}$$

$$= \mathrm{Tr}^V_{A,B}\left(\left(1_B \otimes \left(u \circ \mathrm{Tr}^U_{U,U}\left(c_{U,U}\circ\left(\theta_U^{-1}\otimes 1_U\right)\right)\right)\right)\circ f\right)$$

$$= \mathrm{Tr}^V_{A,B}\left(\left(1_B \otimes \left(\mathrm{Tr}^U_{U,V}\left(\left(u\otimes 1_U\right)\circ c_{U,U}\circ\left(\theta_U^{-1}\otimes 1_U\right)\right)\right)\right)\circ f\right) \tag{10}$$

$$= \mathrm{Tr}^V_{A,B}\left(\mathrm{Tr}^U_{A\otimes V,B\otimes V}\left(1_B \otimes \left[\left(u\otimes 1_U\right)\circ c_{U,U}\circ\left(\theta_U^{-1}\otimes 1_U\right)\right]\right)\circ f\right) \tag{11}$$

$$= \mathrm{Tr}^V_{A,B}\left(\mathrm{Tr}^U_{A\otimes V,B\otimes V}\left(1_B \otimes \left[\left(u\otimes 1_U\right)\circ c_{U,U}\circ\left(\theta_U^{-1}\otimes 1_U\right)\right]\circ\left(f\otimes 1_U\right)\right)\right) \tag{12}$$

$$= \mathrm{Tr}^V_{A,B}\left(\mathrm{Tr}^U_{A\otimes V,B\otimes V}\left(1_B \otimes \left[c_{U,V}\circ\left(1_U\otimes u\right)\circ\left(\theta_U^{-1}\otimes 1_U\right)\right]\circ\left(f\otimes 1_U\right)\right)\right) \tag{13}$$

$$= \mathrm{Tr}^V_{A,B}\left(\mathrm{Tr}^U_{A\otimes V,B\otimes V}\left(\left(1_B \otimes c_{U,V}\right)\circ\left(1_B\otimes\left[\left(1_U\otimes u\right)\circ\left(\theta_U^{-1}\otimes 1_U\right)\right]\right)\circ\left(f\otimes 1_U\right)\right)\right)$$

$$= \mathrm{Tr}^V_{A,B}\left(\mathrm{Tr}^U_{A\otimes U,B\otimes V}\left(\left(1_B \otimes\left[\left(1_U\otimes u\right)\circ\left(\theta_U^{-1}\otimes 1_U\right)\right]\right)\circ\left(f\otimes 1_U\right)\circ\left(1_A\otimes c_{U,V}\right)\right)\right) \tag{14}$$

$$= \mathrm{Tr}^V_{A,B}\left(\mathrm{Tr}^U_{A\otimes U,B\otimes V}\left(\left(1_B \otimes \theta_U^{-1}\otimes 1_V\right)\circ\left(f\otimes u\right)\circ\left(1_A\otimes c_{U,V}\right)\right)\right) \tag{15}$$

$$= \mathrm{Tr}^V_{A,B}\left(\mathrm{Tr}^U_{A\otimes U,B\otimes V}\left(\left(1_B \otimes \theta_U^{-1}\otimes 1_V\right)\circ\left(f\otimes 1_V\right)\circ\left(1_A\otimes 1_V\otimes u\right)\circ\left(1_A\otimes c_{U,V}\right)\right)\right) \tag{16}$$

$$= \mathrm{Tr}^{U}_{A,B}\left(\mathrm{Tr}^{V}_{A\otimes U,B\otimes U}\left(\left(1_B\otimes\theta_U{}^{-1}\otimes 1_V\right)\circ\left(f\otimes 1_V\right)\circ\left(1_A\otimes c_{V,V}\right)\circ\left(1_A\otimes u\otimes 1_V\right)\right)\right) \tag{17}$$

$$= \mathrm{Tr}^{U}_{A,B}\left(\left(1_B\otimes\theta_U{}^{-1}\right)\circ \mathrm{Tr}^{V}_{A\otimes V,B\otimes U}\left(\left(f\otimes 1_V\right)\circ\left(1_A\otimes c_{V,V}\right)\right)\circ\left(1_A\otimes u\right)\right) \tag{18}$$

$$= \mathrm{Tr}^{U}_{A,B}\left(\left(1_B\otimes\theta_U{}^{-1}\right)\circ f\circ \mathrm{Tr}^{V}_{A\otimes V,A\otimes V}\left(1_A\otimes c_{V,V}\right)\circ\left(1_A\otimes u\right)\right) \tag{19}$$

$$= \mathrm{Tr}^{U}_{A,B}\left(\left(1_B\otimes\theta_U{}^{-1}\right)\circ f\circ\left(1_A\otimes \mathrm{Tr}^{V}_{V,V}\left(c_{V,V}\right)\right)\circ\left(1_A\otimes u\right)\right) \tag{20}$$

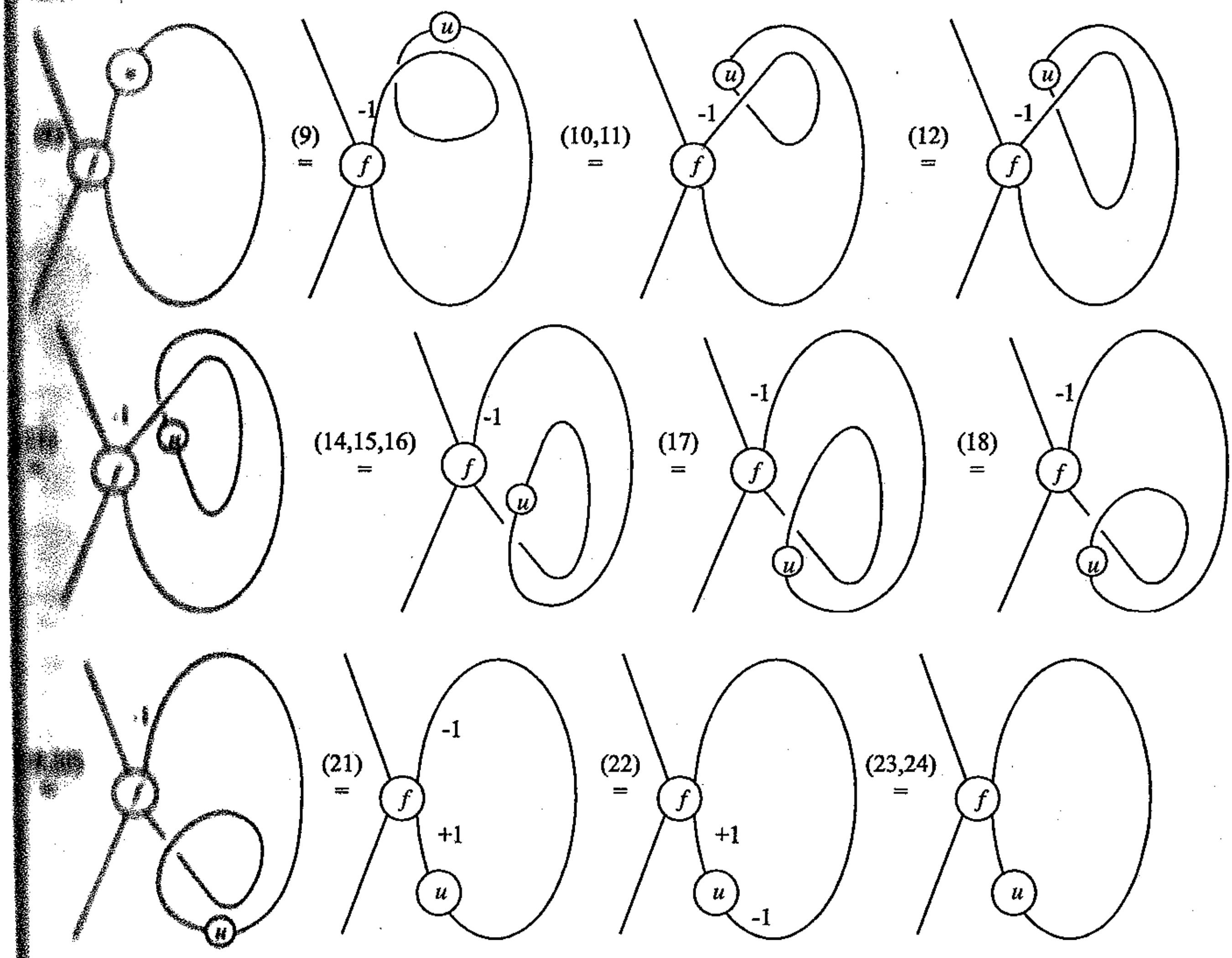
$$= \mathrm{Tr}^{U}_{A,B}\left(\left(1_B\otimes\theta_U{}^{-1}\right)\circ f\circ\left(1_A\otimes\theta_U\right)\circ\left(1_A\otimes u\right)\right) \tag{21}$$

$$= \mathrm{Tr}^{U}_{A,B}\left(f\circ\left(1_A\otimes\theta_U\right)\circ\left(1_A\otimes u\right)\circ\left(1_A\otimes\theta_U{}^{-1}\right)\right) \tag{22}$$

$$= \mathrm{Tr}^{U}_{A,B}\left(f\circ\left(1_A\otimes\left(\theta_U\circ u\circ\theta_U{}^{-1}\right)\right)\right) \tag{23}$$

$$= \mathrm{Tr}^{U}_{A,B}\left(f\circ\left(1_A\otimes u\right)\right) \tag{24}$$

Pictorially



Lemma 2.5 (*Flipping*) The trace Tr^U of $f: A\otimes U \to B\otimes U$ is equal to the trace of the composite

$$U\otimes A \xrightarrow{\ c\ } A\otimes U \xrightarrow{\ f\ } B\otimes U \xrightarrow{\ c^{-1}\ } U\otimes B \xrightarrow{\ c^{-1}\ } B\otimes U.$$

Proof.

$$\mathrm{Tr}^U_{A,B}\left(c_{B,U}^{-1}\circ c_{U,B}^{-1}\circ f\circ c_{U,A}\circ c_{A,U}\right) \tag{1}$$

$$=\mathrm{Tr}^U_{A,B}\left(c_{B,U}^{-1}\circ c_{U,B}^{-1}\circ\left((1_B\circ 1_B)\otimes\left(\theta_U\circ\theta_U^{-1}\right)\right)\circ f\circ c_{U,A}\circ c_{A,U}\right)$$

$$=\mathrm{Tr}^U_{A,B}\left(c_{B,U}^{-1}\circ c_{U,B}^{-1}\circ\left(1_B\otimes\left(\theta_U\circ \mathrm{Tr}^U_{U,U}\left(c_{U,U}^{-1}\right)\right)\right)\circ f\circ c_{U,A}\circ c_{A,U}\right) \tag{2}$$

$$=\mathrm{Tr}^U_{A,B}\left((1_B\otimes\theta_U)\circ c_{B,U}^{-1}\circ c_{U,B}^{-1}\circ\left(1_B\otimes \mathrm{Tr}^U_{U,U}\left(c_{U,U}^{-1}\right)\right)\circ f\circ c_{U,A}\circ c_{A,U}\right) \tag{3}$$

$$=\mathrm{Tr}^U_{A,B}\left((1_B\otimes\theta_U)\circ c_{B,U}^{-1}\circ c_{U,B}^{-1}\circ \mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(1_B\otimes c_{U,U}^{-1}\right)\circ f\circ c_{U,A}\circ c_{A,U}\right) \tag{4}$$

$$=\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(\left(\left((1_B\otimes\theta_U)\circ c_{B,U}^{-1}\circ c_{U,B}^{-1}\right)\otimes 1_U\right)\circ\left(1_B\otimes c_{U,U}^{-1}\right)\circ\left(\left(f\circ c_{U,A}\circ c_{A,U}\right)\otimes 1_U\right)\right)\right) \tag{5}$$

$$=\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(\left(\left((1_B\otimes\theta_U)\circ c_{B,U}^{-1}\right)\otimes 1_U\right)\circ c_{U,B\otimes U}^{-1}\circ\left(\left(f\circ c_{U,A}\circ c_{A,U}\right)\otimes 1_U\right)\right)\right) \tag{6}$$

$$=\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(\left(\left((1_B\otimes\theta_U)\circ c_{B,U}^{-1}\right)\otimes 1_U\right)\circ\left(1_U\otimes\left(f\circ c_{U,A}\circ c_{A,U}\right)\right)\circ c_{U,A\otimes U}^{-1}\right)\right) \tag{7}$$

$$=\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(\left(\left((1_B\otimes\theta_U)\circ c_{B,U}^{-1}\right)\otimes 1_U\right)\circ\left(1_U\otimes\left(f\circ c_{U,A}\circ c_{A,U}\right)\right)\circ\left(c_{U,A}^{-1}\otimes 1_U\right)\circ\left(1_A\otimes c_{U,U}^{-1}\right)\right)\right) \tag{8}$$

$$=\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(\left(1_B\otimes c_{U,U}^{-1}\right)\circ\left(\left((1_B\otimes\theta_U)\circ c_{B,U}^{-1}\right)\otimes 1_U\right)\circ\left(1_U\otimes\left(f\circ c_{U,A}\circ c_{A,U}\right)\right)\circ\left(c_{U,A}^{-1}\otimes 1_U\right)\right)\right) \tag{9}$$

$$=\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(\left(1_B\otimes c_{U,U}^{-1}\right)\circ\left(c_{B,U}^{-1}\otimes 1_U\right)\circ\left(\theta_U\otimes 1_B\otimes 1_U\right)\circ\left(1_U\otimes\left(f\circ c\circ c\right)\right)\circ\left(c_{U,A}^{-1}\otimes 1_U\right)\right)\right) \tag{10}$$

$$=\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(c_{B\otimes U,U}^{-1}\circ\left(\theta_U\otimes 1_B\otimes 1_U\right)\circ\left(1_U\otimes\left(f\circ c_{U,A}\circ c_{A,U}\right)\right)\circ\left(c_{U,A}^{-1}\otimes 1_U\right)\right)\right) \tag{11}$$

$$=\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(\left(\left(f\circ c_{U,A}\circ c_{A,U}\right)\otimes\theta_U\right)\circ c_{A\otimes U,U}^{-1}\circ\left(c_{U,A}^{-1}\otimes 1_U\right)\right)\right) \tag{12}$$

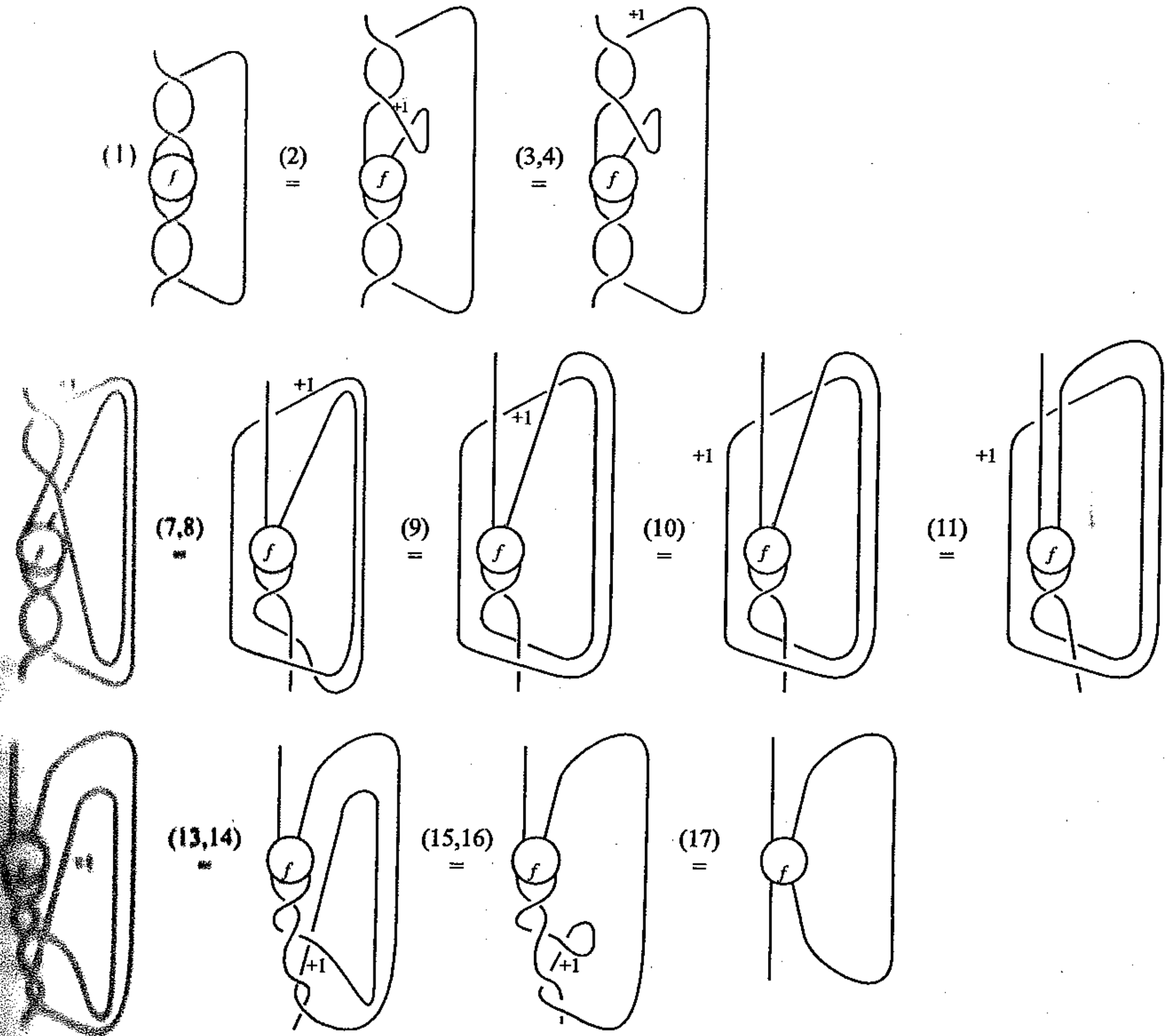
$$=\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(\left(\left(f\circ c_{U,A}\circ c_{A,U}\right)\otimes\theta_U\right)\circ\left(1_A\otimes c_{U,U}^{-1}\right)\circ\left(\left(c_{A,U}^{-1}\circ c_{U,A}^{-1}\right)\otimes 1_U\right)\right)\right) \tag{13}$$

$$=\mathrm{Tr}^U_{A,B}\left(\mathrm{Tr}^U_{B\otimes U,B\otimes U}\left(\left(\left(f\circ c_{U,A}\circ c_{A,U}\right)\otimes 1_U\right)\circ\left(1_A\otimes\left(c_{U,U}^{-1}\circ\left(\theta_U\otimes 1_U\right)\right)\right)\circ\left(\left(c_{A,U}^{-1}\circ c_{U,A}^{-1}\right)\otimes 1_U\right)\right)\right) \tag{14}$$

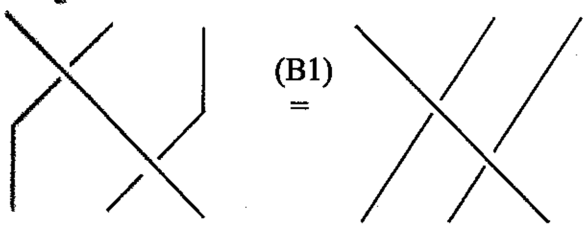
$$=\mathrm{Tr}^U_{A,B}\left(f\circ c_{U,A}\circ c_{A,U}\circ \mathrm{Tr}^U_{A\otimes U,A\otimes U}\left(1_A\otimes c_{U,U}^{-1}\right)\circ\left(1\ \otimes\theta_U\right)\circ c_{A,U}^{-1}\circ c_{U,A}^{-1}\right) \tag{15}$$

$$=\mathrm{Tr}^U_{A,B}\left(f\circ c_{U,A}\circ c_{A,U}\circ\left(1_A\otimes \mathrm{Tr}^U_{U,U}\left(c_{U,U}^{-1}\right)\right)\circ\left(1_A\otimes\theta_U\right)\circ c_{A,U}^{-1}\circ c_{U,A}^{-1}\right) \tag{16}$$

$$=\mathrm{Tr}^U_{A,B}(f). \tag{17}$$



… following balanced result:

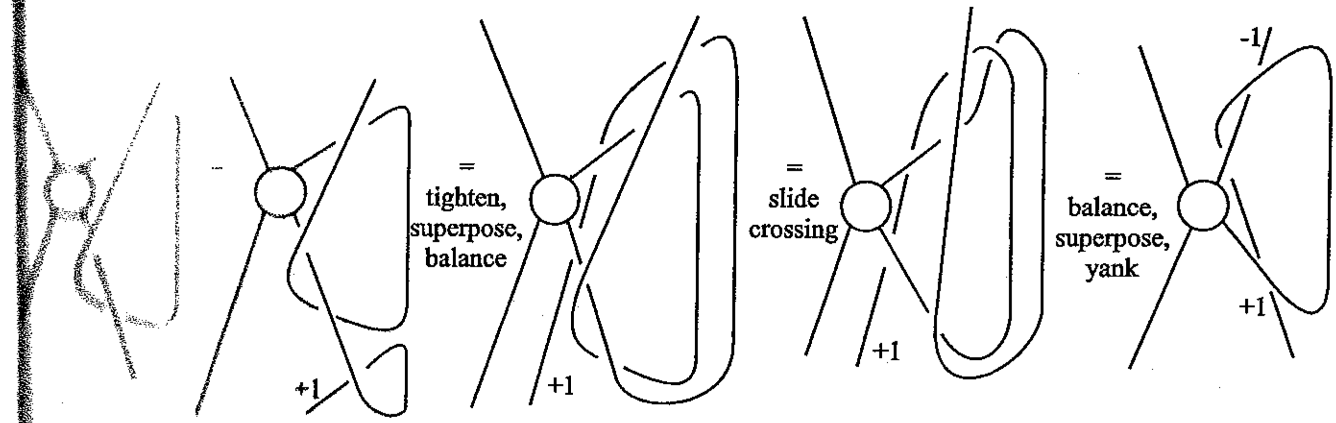


… *(Trace swapping) For all $f : A\otimes B \to C\otimes D$, the following composites have the same* …

$$\cdots \longrightarrow A\otimes D\otimes B \xrightarrow{1\otimes c} A\otimes B\otimes D \xrightarrow{f\otimes 1} C\otimes D\otimes D \xrightarrow{1\otimes c^{-1}} C\otimes D\otimes D$$

$$A\otimes B\otimes B \longrightarrow A\otimes B\otimes B \xrightarrow{1\otimes c^{-1}} A\otimes B\otimes B \xrightarrow{f\otimes 1} C\otimes D\otimes B$$
$$\longrightarrow C\otimes B\otimes D \xrightarrow{1\otimes c} C\otimes D\otimes B \xrightarrow{1\otimes\theta^{-1}\otimes 1} C\otimes D\otimes B$$

Pictorially:



Lemma 2.7: *Suppose $F : \mathcal{V} \to \mathcal{W}$ is a fully faithful balanced monoidal functor with $\mathcal{W}$ traced monoidal. Then there exists a unique trace on $\mathcal{V}$ for which F is a traced monoidal functor. (This is called the trace on $\mathcal{V}$ induced from $\mathcal{W}$ along F.)*

Proof:

Suppose $F : \mathcal{V} \to \mathcal{W}$ is a fully faithful balanced monoidal functor with $\mathcal{W}$ traced monoidal.
Given $f : A \otimes U \to B \otimes U$ in $\mathcal{V}$, let

$$g = \mathrm{Tr}^{FU}_{FA,FB}\left(FA \otimes FU \xrightarrow{\phi_2} F(A \otimes U) \xrightarrow{Ff} F(B \otimes U) \xrightarrow{\phi_2^{-1}} FB \otimes FU \right) \text{ in } \mathcal{W}.$$

Then, since F is fully faithful, there exists a unique $h_f : A \to B$ in $\mathcal{V}$, with $Fh_f = g$.
We define $\mathrm{Tr}^{U}_{A,B}(f) = h_f$.
We now wish to prove that h_f satisfies the axioms for trace. To see that $h_1 = h_2$ in $\mathcal{V}$, since F is fully faithful, it is enough to see that $Fh_1 = Fh_2$ in $\mathcal{W}$.

Vanishing: for any $f : A \otimes I \to B \otimes I$ in $\mathcal{V}$, we must show that $\mathrm{Tr}^{I}_{A,B}(f) = f$, so it is enough to prove $F\left(\mathrm{Tr}^{I}_{A,B}(f)\right) = Ff$. Starting on the left:

$$F\left(\mathrm{Tr}^{I}_{A,B}(f)\right) = \mathrm{Tr}^{FI}_{FA,FB}\left(FA \otimes FI \xrightarrow{\phi_2} F(A \otimes I) \xrightarrow{Ff} F(B \otimes I) \xrightarrow{\phi_2^{-1}} FB \otimes FI \right)$$

$$FA \otimes FI \xrightarrow{1 \otimes \phi_0^{-1}} FA \otimes I, \quad FA \otimes FI \xrightarrow{1} FA \otimes FI, \quad FA \otimes I \xrightarrow{1 \otimes \phi_0} FA \otimes FI$$

$$= \mathrm{Tr}^{I}_{FA,FB}\left(FA \otimes FI \longrightarrow FA \otimes I F(A \otimes I) \xrightarrow{Ff} F(B \otimes I) \xrightarrow{\phi_2^{-1}} FB \otimes FI \xrightarrow{1 \otimes \phi_0^{-1}} FB \otimes I \right)$$

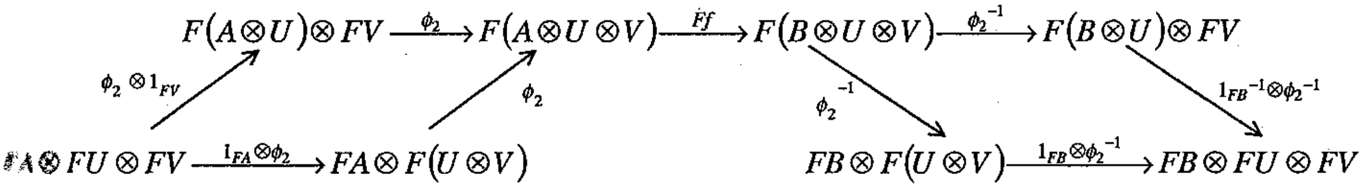
$$= \mathrm{Tr}^{I}_{FA,FB}\left(F(A \otimes I) \xrightarrow{Ff} F(B \otimes I) \right) \text{ as required.}$$

For: it is enough to prove that

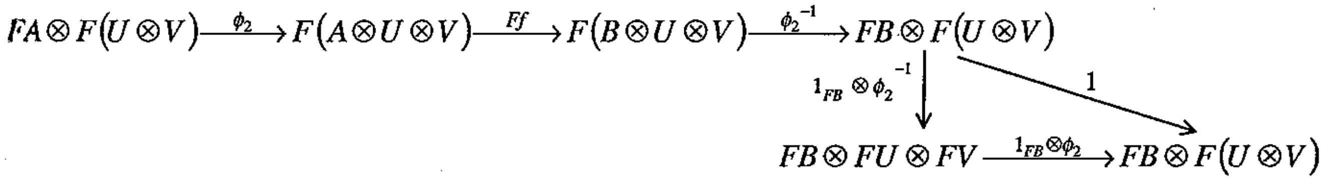
$$\mathrm{Tr}^{F(U \otimes V)}_{FA,FB}\left(FA \otimes F(U \otimes V) \xrightarrow{\phi_2} F(A \otimes U \otimes V) \xrightarrow{Ff} F(B \otimes U \otimes V) \xrightarrow{\phi_2^{-1}} FB \otimes F(U \otimes V) \right)$$

$$= \mathrm{Tr}^{FU}_{FA,FB}\left(FA \otimes FU \xrightarrow{\phi_2} F(A \otimes U) \xrightarrow{F\left(\mathrm{Tr}^{V}_{A \otimes U, B \otimes U}(f)\right)} F(B \otimes U) \xrightarrow{\phi_2^{-1}} FB \otimes FU \right)$$

The last of these is equal under tightening to the value of $\mathrm{Tr}_{FA,FB}^{FU} \circ \mathrm{Tr}_{FA\otimes FU,FB\otimes FU}^{FV}$ at the composite



which equals, under (V2) and sliding, the value of $\mathrm{Tr}_{FA,FB}^{F(U\otimes V)}$ at the composite

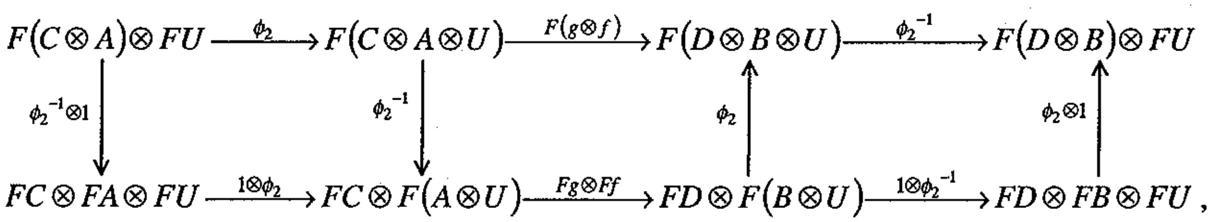


as required.

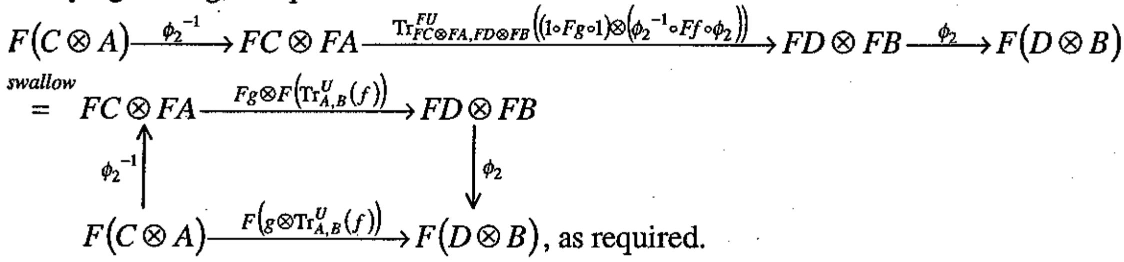
Swallowing: We need to prove for $f : A\otimes U \to B\otimes U, g : C \to D$ that

$$F\left(g\otimes \mathrm{Tr}_{A,B}^{U}(f)\right) = F\left(\mathrm{Tr}_{C\otimes A,D\otimes B}^{U}(g\otimes f)\right)$$

The right hand side is equal to the value of $\mathrm{Tr}_{F(C\otimes A),F(D\otimes B)}^{FU}$ at the composite



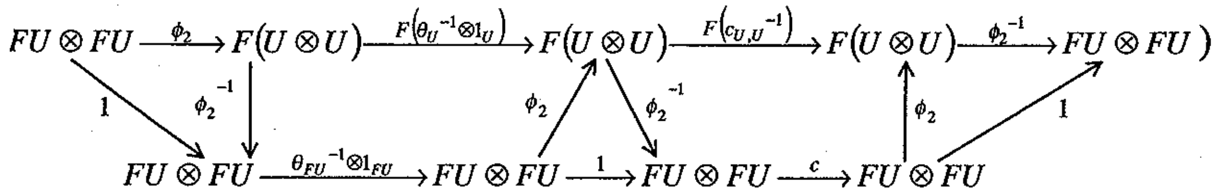
which, by tightening, is equal to



, as required.

Yanking: $F\left(\mathrm{Tr}_{U,U}^{U}\left(c_{U,U} \circ \left(\theta_U^{-1} \otimes 1_U\right)\right)\right) = F1_U = F\left(\mathrm{Tr}_{U,U}^{U}\left(c_{U,U}^{-1} \circ \left(\theta_U \otimes 1_U\right)\right)\right)$

The left hand side is equal to the value of $\mathrm{Tr}_{FU,FU}^{FU}$ at the composite



and this is the identity under yanking in $\mathcal{W}$.

Yanking in the other direction gives the value of $\mathrm{Tr}^{FU}_{FU,FU}$ at the composite

$$
\begin{array}{ccccccc}
FU\otimes FU & \xrightarrow{\theta_{FU}\otimes 1_{FU}} & FU\otimes FU & \xrightarrow{1} & FU\otimes FU & \xrightarrow{c^{-1}} & FU\otimes FU\,) \\
\Big\downarrow{\scriptstyle \phi_2} & & {\scriptstyle \phi_2^{-1}}\searrow & & \nearrow{\scriptstyle \phi_2} & & \Big\uparrow{\scriptstyle \phi_2^{-1}} \\
F(U\otimes U) & \xrightarrow{F(\theta_U^{-1}\otimes 1_U)} & & F(U\otimes U) & & \xrightarrow{F(c_{U,U}^{-1})} & F(U\otimes U),
\end{array}
$$

as required.

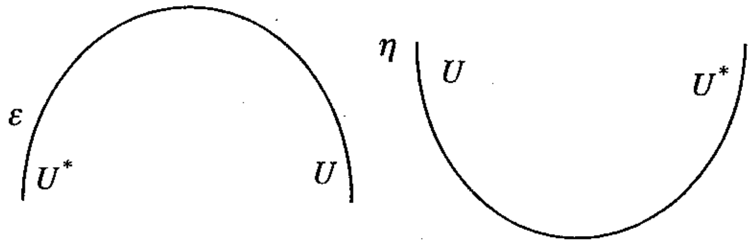
We also need to see that trace is natural. The definition amounts to the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{V}(A\otimes U, B\otimes U) & \xrightarrow{\mathrm{Tr}^U} & \mathcal{V}(A,B) \\
\Big\downarrow{\scriptstyle F} & & \\
\mathcal{W}(F(A\otimes U), F(B\otimes U)) & & {\scriptstyle F}\Big\downarrow\;\Big\uparrow{\scriptstyle F^{-1}} \\
\Big\downarrow{\scriptstyle \mathcal{W}(\phi_2,\phi_2^{-1})} & & \\
\mathcal{W}(FA\otimes FU, FB\otimes FU) & \xrightarrow{\mathrm{Tr}^{FU}} & \mathcal{W}(FA,FB)
\end{array}
$$

which shows that trace is natural because all the factors are.

Chapter 3
Canonical Trace In A Tortile Monoidal Category

In a tortile monoidal category $\mathcal{W}$, each object has both a left and right dual. Let U be an object in it, then U^* is its left dual, and the corresponding counit is $\varepsilon : U \otimes U^* \to I$ and the unit is $\eta : I \to U \otimes U^*$, with diagrams:

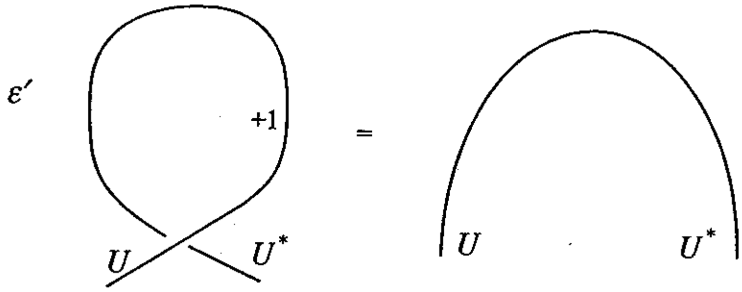


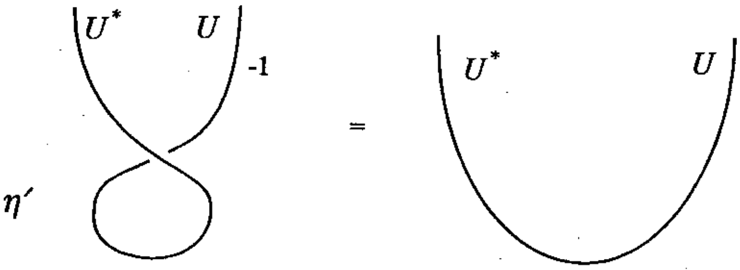
As justified in [JS3], the appropriate counit and unit for U^* as a right dual of U are

$$\varepsilon' : U \otimes U^* \xrightarrow{c_{U,U}^{-1}} U^* \otimes U \xrightarrow{1_{U^*} \otimes \theta_U} U^* \otimes U \xrightarrow{\varepsilon} I$$

$$\eta' : I \xrightarrow{\eta} U \otimes U^* \xrightarrow{c_{U^*,U}^{-1}} U^* \otimes U \xrightarrow{1_{U^*} \otimes \theta_U^{-1}} U^* \otimes U$$

which are illustrated by the following diagrams:





Proposition 3.1: *In any tortile monoidal category, a trace, called the canonical trace, is defined by the following formula*

$$\mathrm{Tr}_{A,B}^{U}(f) = A \xrightarrow{1 \otimes \eta} A \otimes U \otimes U^* \xrightarrow{f \otimes 1} B \otimes U \otimes U^* \xrightarrow{1 \otimes \varepsilon'} B$$

Furthermore, every balanced monoidal functor between tortile monoidal categories is traced with respect to the canonical trace.

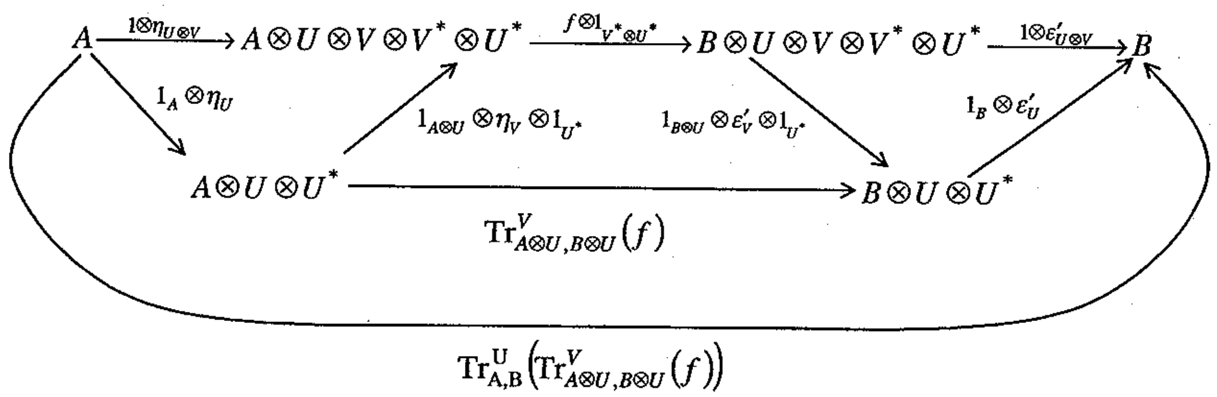
Proof: We do not use three dimensional reasoning in the tortile monoidal category, as this has not been rigorously proved as in the balanced case. Instead, we will use algebra.

We need to see that the trace as defined above does satisfy the axioms for trace.

(V1) $\mathrm{Tr}^{I}_{A,B}(f) = A \xrightarrow{1\otimes\eta_I} A\otimes I\otimes I \xrightarrow{f\otimes 1} B\otimes I\otimes I \xrightarrow{1\otimes\varepsilon'_I} B$

$= A \xrightarrow{1\otimes 1} A\otimes I\otimes I \xrightarrow{f\otimes 1} B\otimes I\otimes I \xrightarrow{1\otimes 1} B = f$

(V2) $\mathrm{Tr}^{U\otimes V}_{A,B}(f) = \mathrm{Tr}^{U}_{A,B}\left(\mathrm{Tr}^{V}_{A\otimes U,B\otimes U}(f)\right)$ is shown by the following commutative diagram.



Swallowing: $\mathrm{Tr}^{U}_{C\otimes A,D\otimes B}(g\otimes f) =$

$C\otimes A \xrightarrow{1_C\otimes 1_A\otimes\eta_U} C\otimes A\otimes U\otimes U^* \xrightarrow{g\otimes f\otimes 1_{U^*}} D\otimes B\otimes U\otimes U^* \xrightarrow{1_D\otimes 1_B\otimes\varepsilon'_U} D\otimes B$

$= (1_C\circ g\circ 1_D)\otimes\left((1_A\otimes\eta_U)\circ(f\otimes 1_{U^*})\circ(1_B\otimes\varepsilon'_U)\right)$

$= g\otimes \mathrm{Tr}^{U}_{A,B}(f)$

Yanking:

$U \xrightarrow{1_U\otimes\eta_U} U\otimes U\otimes U^* \xrightarrow{\left(c_{U,U}\circ(\theta_U^{-1}\otimes 1_U)\right)\otimes 1_{U^*}} U\otimes U\otimes U^* \xrightarrow{1_U\otimes\varepsilon'_U} U$

$= U \xrightarrow{1_U\otimes\eta_U} U\otimes U\otimes U^* \xrightarrow{\theta_U^{-1}\otimes 1_U\otimes 1_{U^*}} U\otimes U\otimes U^* \xrightarrow{c_{U,U}\otimes 1_{U^*}} U\otimes U\otimes U^*$

$\xrightarrow{1_U\otimes c_{U,U^*}} U\otimes U^*\otimes U \xrightarrow{1_U\otimes 1_{U^*}\otimes\theta_U} U\otimes U^*\otimes U \xrightarrow{1_U\otimes\varepsilon} U$

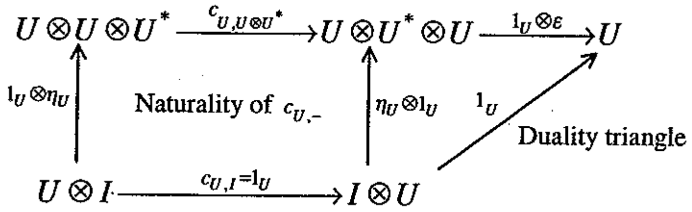
$= U \xrightarrow{1_U\otimes\eta_U} U\otimes U\otimes U^* \xrightarrow{\theta_U^{-1}\otimes 1_U\otimes 1_{U^*}} U\otimes U\otimes U^*$

$\xrightarrow{c_{U,U\otimes U^*}} U\otimes U^*\otimes U \xrightarrow{1_U\otimes 1_{U^*}\otimes\theta_U} U\otimes U^*\otimes U \xrightarrow{1_U\otimes\varepsilon} U$

$= U \xrightarrow{1_U\otimes\eta_U} U\otimes U\otimes U^* \xrightarrow{c_{U,U\otimes U^*}} U\otimes U^*\otimes U$

$\xrightarrow{1_U\otimes 1_{U^*}\otimes\theta_U^{-1}} U\otimes U^*\otimes U \xrightarrow{1_U\otimes 1_{U^*}\otimes\theta_U} U\otimes U^*\otimes U \xrightarrow{1_U\otimes\varepsilon} U$

$=$

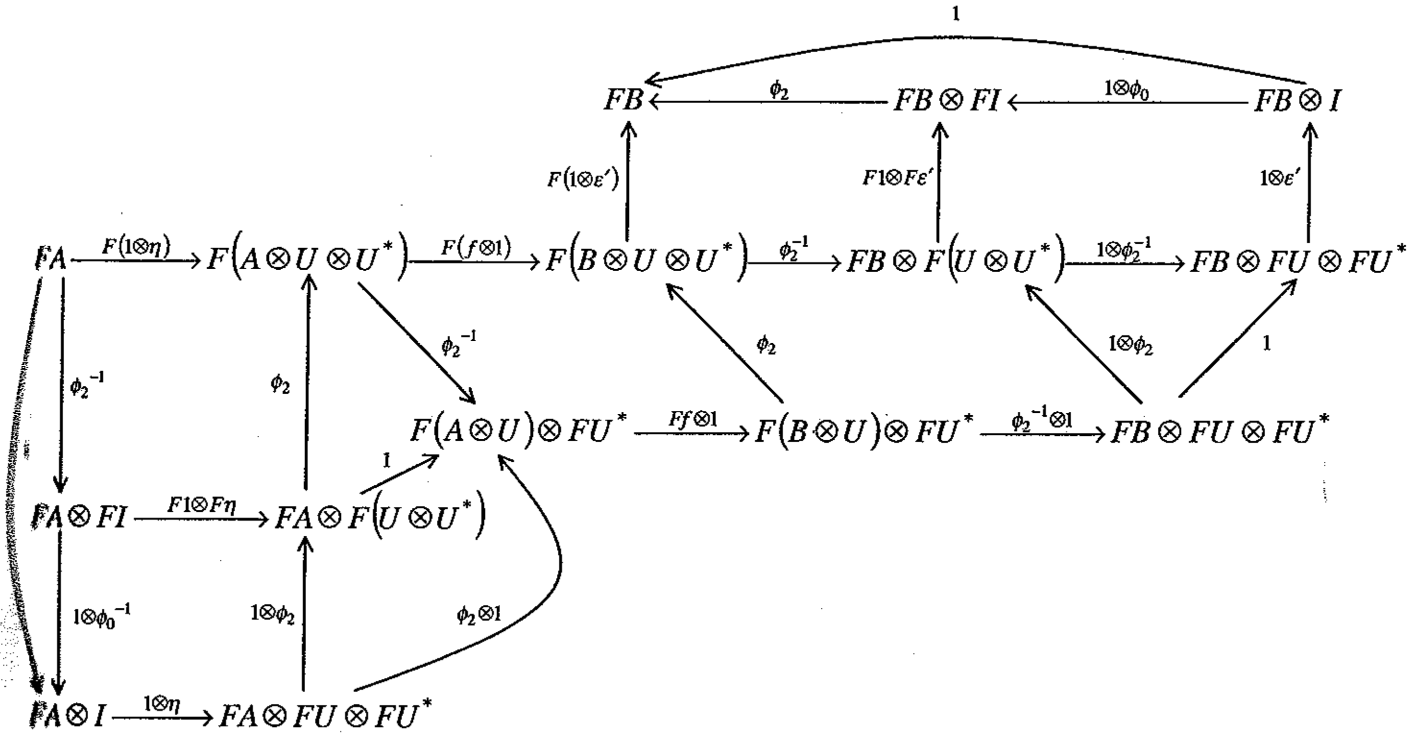


The last sentence of the proposition is true because balanced monoidal functors preserve duals. Let $\mathcal{V}, \mathcal{W}$ be tortile monoidal categories, and $F:\mathcal{V}\to\mathcal{W}$ be a balanced monoidal functor between them. Then

$$F\left(\mathrm{Tr}^{U}_{A,B}(f)\right) = F\left(A \xrightarrow{1\otimes\eta} A\otimes U\otimes U^* \xrightarrow{f\otimes 1} B\otimes U\otimes U^* \xrightarrow{1\otimes\varepsilon'} B\right)$$

This is equal to the upper composite in the following diagram. We show, using commutativity of the diagrams in Definition 1.1, functoriality of tensor, and the definition of $F\eta$, that this is equal to the lower composite in the following diagram. This is the following algebraic equality, which means F preserves trace.

$$F\left(\mathrm{Tr}^{U}_{A,B}(f)\right)=(1\otimes\varepsilon')\circ\left({\phi_2}^{-1}\otimes 1\right)\circ(Ff\otimes 1)\circ(\phi_2\otimes 1)\circ(1\otimes\eta)=\mathrm{Tr}^{FU}_{FA,FB}\left({\phi_2}^{-1}\circ Ff\circ\phi_2\right),$$



Chapter 4
Making a tortile monoidal category from a traced one

In this chapter we start with a traced monoidal category $\mathcal{V}$, and define a new category Int$\mathcal{V}$. We add a tensor product functor, a braiding, and a twist, enriching Int$\mathcal{V}$ with a tortile monoidal structure. This construction is a vast generalisation of the construction of the integers from the natural numbers. We finish by stating without proof a unviersal property for this construction.

4.1 The category IntV

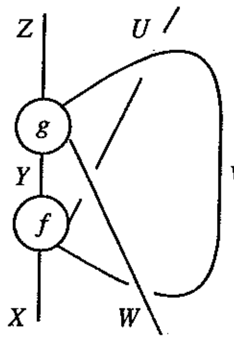
We let $\mathcal{V}$ be any traced monoidal category, which we can assume to be strict by the coherence theorem for monoidal categories [JS1]. To define a category we need to define objects, arrows, source and target, composition of arrows, and identity arrows. We define the objects of Int$\mathcal{V}$ as pairs (X,U) of objects X, U in $\mathcal{V}$, and arrows $f:(X,U)\to(Y,V)$ in Int$\mathcal{V}$ as arrows $f: X\otimes V\to Y\otimes U$ in $\mathcal{V}$. The composite of $f:(X,U)\to(Y,V)$ and $g:(Y,V)\to(Z,W)$ is defined to be the value of the trace function

$$\mathrm{Tr}^{V}_{X\otimes W,Z\otimes U}:\mathcal{V}(X\otimes W\otimes V, Z\otimes U\otimes V)\to\mathcal{V}(X\otimes W, Z\otimes U)$$

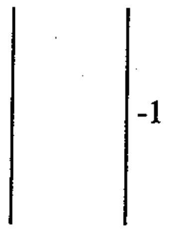
at the composite

$$X\otimes W\otimes V\xrightarrow{1\otimes c}X\otimes V\otimes W\xrightarrow{f\otimes 1}Y\otimes U\otimes W\xrightarrow{1\otimes c^{-1}}Y\otimes W\otimes U$$
$$\xrightarrow{g\otimes 1}Z\otimes V\otimes U\xrightarrow{1\otimes c}Z\otimes U\otimes V$$

which in pictures is



The identity of the object (X,U) is defined to be $1_X\otimes\theta_U^{-1}: X\otimes U\to X\otimes U$



Proposition 4.1: *The above data do define a category* Int$\mathcal{V}$ *and a fully faithful functor* $N:\mathcal{V}\to$ Int$\mathcal{V}$ *is defined by* $N(X)=(X,I)$, $N(f)=f$.
To prove this we need to show that

I) Int$\mathcal{V}$ is a category:
   (a) Associativity, that is $h\circ(g\circ f)=(h\circ g)\circ f$, whenever the arrows are thus composable.
   (b) $1_{(Y,V)}\circ f=f=f\circ 1_{(X,U)}$ for $f:(X,U)\to(Y,V)$.

2) That N is a fully faithful functor:

(a) $s(N(f)) = N(s(f))$ and $t(N(f)) = N(t(f))$ where s is the source map and t is the target map

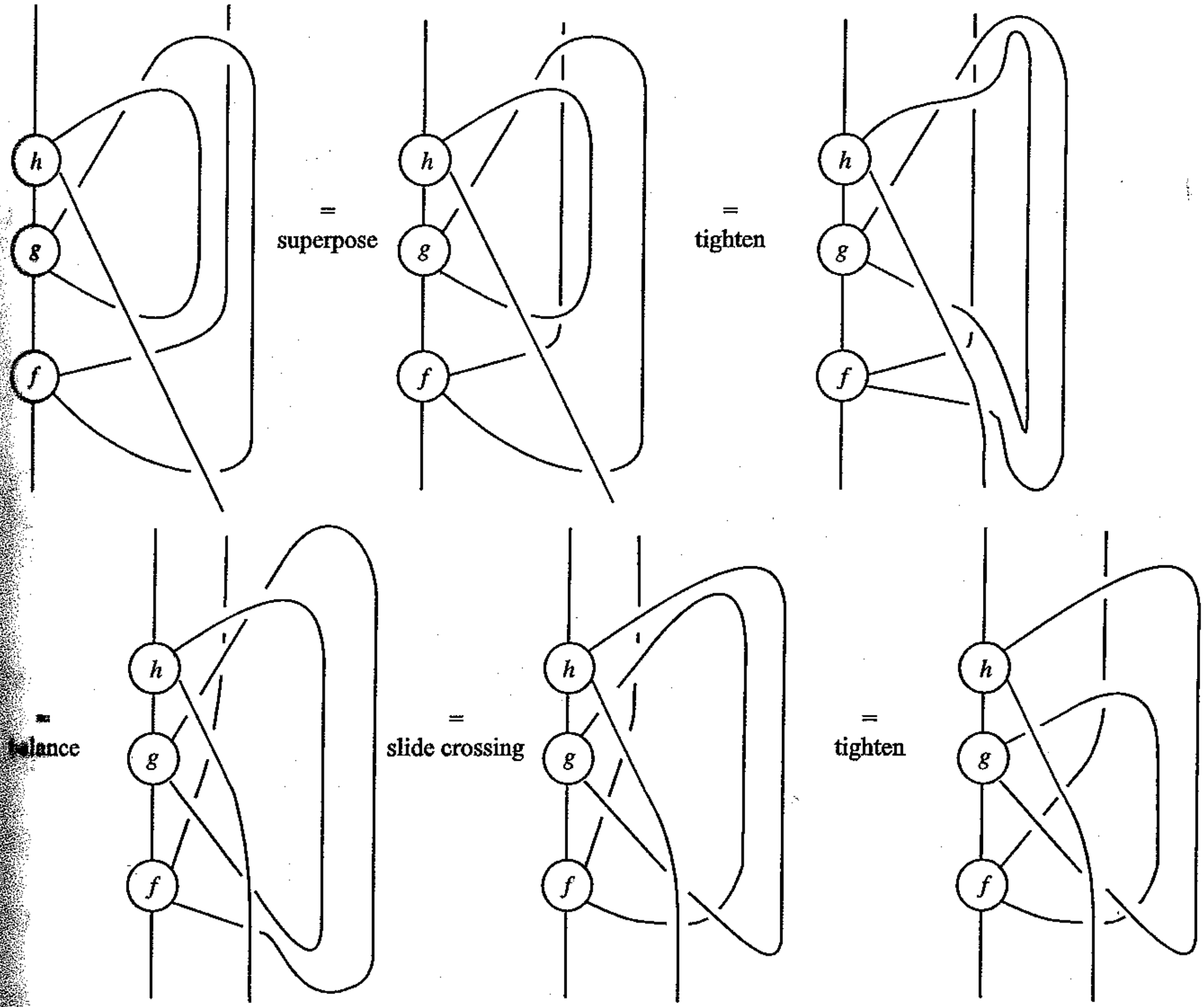
(b) $N(g \circ f) = Ng \circ Nf$

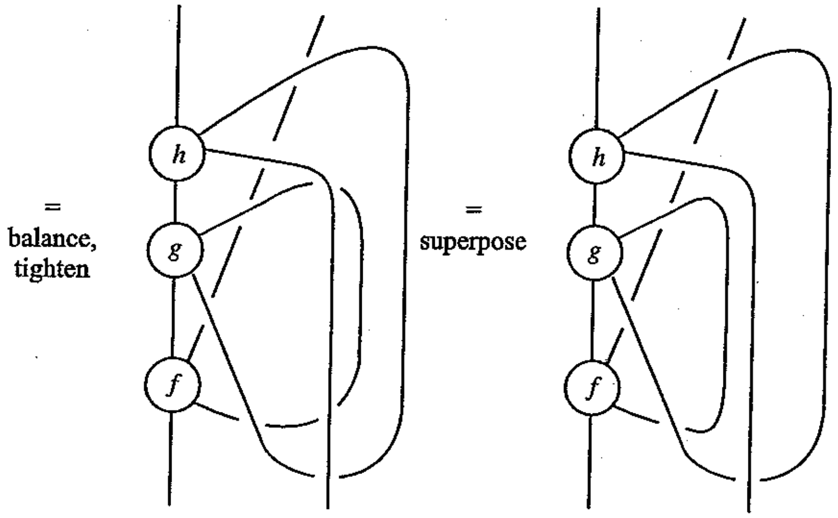
(c) $N(1_X) = 1_{N(X)}$

(d) N is full

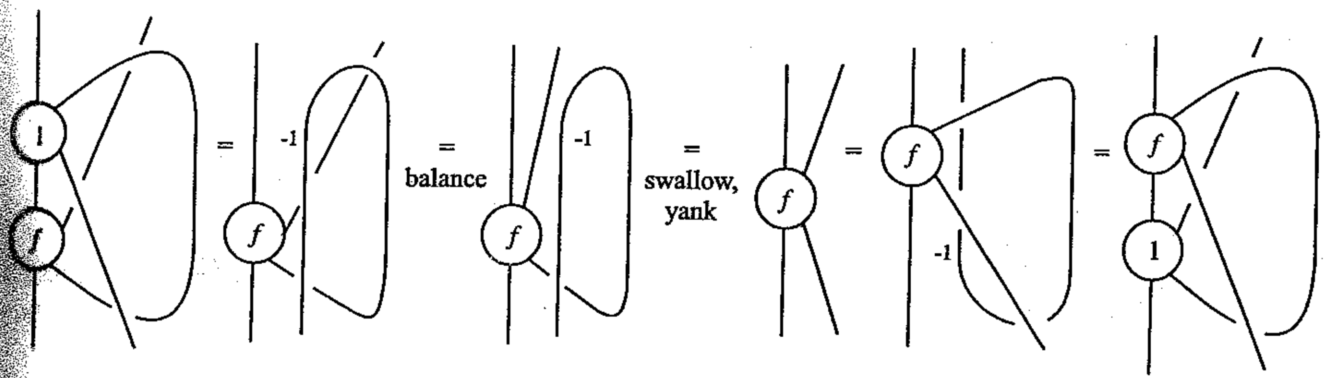
(e) N is faithful

Proof: 1a) Associativity is proved diagrammatically as follows





1b) We again give a pictorial proof



a) For $f : X \to Y$

$$s(N(f)) = (X, I) = N(X) = N(s(f))$$

and $$t(N(f)) = (Y, I) = N(Y) = N(t(f)).$$

b) $N(g \circ f) = g \circ f = Ng \circ Nf$ since $N(f)=f$.

c) $N(1_X) = 1_X = 1_X \otimes \theta_I = 1_{(X,I)} = 1_{N(X)}$ since $\theta_I = 1_I$.

d) We need to prove that $\forall X, X' \in \mathcal{V}$, if $g : (X, I) \to (X', I)$, then $\exists f \in \mathcal{V}$ with $N(f) = g$.
Now, for $g : (X, I) \to (X', I)$ in Int$\mathcal{V}$, we have $g : X \otimes I \to X' \otimes I$ in $\mathcal{V}$, since $\mathcal{V}$ is strict, $g : X \to X'$, and $N(g) = g$.

e) We need to prove that, for any $X, X' \in \mathcal{V}$, and any $f_1 : X \to X', f_2 : X \to X'$, we have $N(f_1) = N(f_2) \Rightarrow f_1 = f_2$, which is clearly true, since by definition $N(f_1) = f_1, N(f_2) = f_2$.

4.2 Adding a tensor product to make $\mathrm{Int}\mathcal{V}$ monoidal

We define a tensor product functor $\otimes' : \mathrm{Int}\mathcal{V} \times \mathrm{Int}\mathcal{V} \to \mathrm{Int}\mathcal{V}$ as follows.

On objects: $(X,U) \otimes' (X',U') = (X \otimes' X', U' \otimes U)$

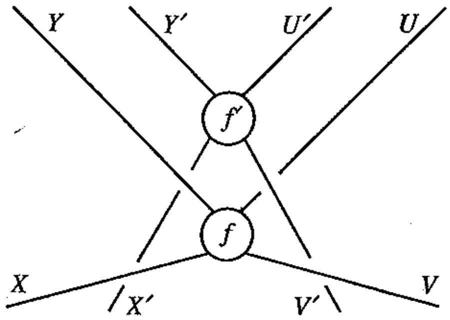
On arrows: For $f:(X,U) \to (Y,V), f':(X',U') \to (Y',V')$ in $\mathrm{Int}\mathcal{V}$,

$f \otimes' f : (X \otimes X', U' \otimes U) \to (Y \otimes Y', V' \otimes V)$

is defined to be the following composite:

$$X \otimes X' \otimes V' \otimes V \xrightarrow{c \otimes c^{-1}} X' \otimes X \otimes V \otimes V' \xrightarrow{1 \otimes f \otimes 1} X' \otimes Y \otimes U \otimes V'$$
$$\xrightarrow{c^{-1} \otimes c^{-1}} Y \otimes X' \otimes V' \otimes U \xrightarrow{1 \otimes f' \otimes 1} Y \otimes Y' \otimes U' \otimes U$$

with diagram



Proposition 4.2: *The above tensor product enriches* $\mathrm{Int}\mathcal{V}$ *with the structure of a monoidal category, and the functor* $N : \mathcal{V} \to \mathrm{Int}\mathcal{V}$ *is then monoidal.*

We have shown that $\mathrm{Int}\mathcal{V}$ is a category in Proposition 4.1, and we have, from the definition of $\otimes'$ on objects, that $f \otimes' f' : (X,U) \otimes' (X',U') \to (Y,V) \otimes' (Y',V')$.

It remains to prove

1) That (I,I) is the unit object for $\mathrm{Int}\mathcal{V} \times \mathrm{Int}\mathcal{V}$

2) That $\otimes'$ is a functor; that is,

(a) $1_{(X,U)} \otimes' 1_{(X',U')} = 1_{(X,U) \otimes' (X',U')}$

(b) $\otimes'((g,g') \circ (f,f')) = (g \otimes' g') \circ (f \otimes' f')$;

that is, $\quad (g \circ f) \otimes' (g' \circ f') = (g \otimes' g') \circ (f \otimes' f')$.

3) That we have natural isomorphisms for associativity, left and right unit constraint and that the associativity pentagon and triangle for unit commute

4) That we have a natural family of isomorphisms $\phi_{2,A,B} : N(X) \otimes' N(X') \xrightarrow{\sim} N(X \otimes X')$ and an isomorphism $\phi_0 : I \xrightarrow{\sim} N(I)$ such that the three required diagrams commute.

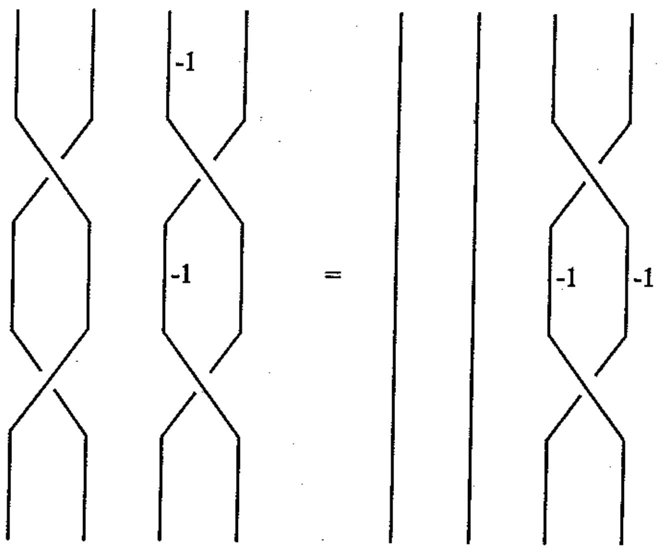
Proof:

1) $(X,U) \otimes' (I,I) = (X \otimes I, I \otimes U) = (X,U) = (I \otimes X, U \otimes I) = (I,I) \otimes (X,U)$, hence (I,I) is the identity object.

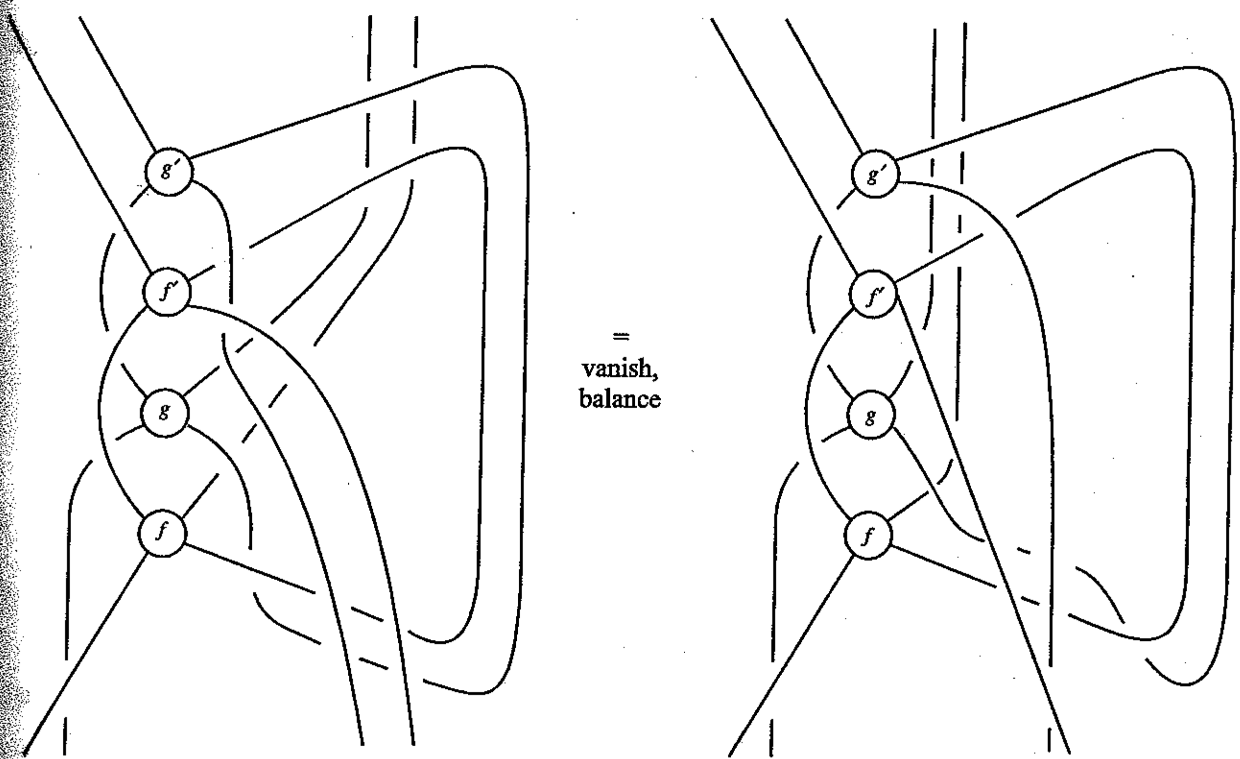
2a) $1_{(X,U)} \otimes' 1_{(X',U')}$

$$= \left(1_X \otimes 1_{(X',U')} \otimes 1_U\right) \circ \left(c_{X,X'}{}^{-1} \otimes c_{U',U}{}^{-1}\right) \circ \left(1_{X'} \otimes 1_{(X,U)} \otimes 1_{U'}\right) \circ \left(c_{X,X'} \otimes c_{U,U'}{}^{-1}\right)$$
$$= \left(1_X \otimes 1_{X'} \otimes \theta_{U'}{}^{-1} \otimes 1_U\right) \circ \left(c_{X,X'}{}^{-1} \otimes c_{U',U}{}^{-1}\right) \circ \left(1_{X'} \otimes 1_X \otimes \theta_U{}^{-1} \otimes 1_{U'}\right) \circ \left(c_{X,X'} \otimes c_{U,U'}{}^{-1}\right)$$
$$= \left(\left(1_X \otimes 1_{X'}\right) \circ c_{X,X'}{}^{-1} \circ \left(1_{X'} \otimes 1_X\right) \circ c_{X,X'}\right) \otimes \left(\left(\theta_{U'}{}^{-1} \otimes 1_U\right) \circ c_{U',U}{}^{-1} \circ \left(\theta_U{}^{-1} \otimes 1_{U'}\right) \circ c_{U,U'}{}^{-1}\right)$$
$$= 1_X \otimes 1_{X'} \otimes \left(c_{U',U}{}^{-1} \circ \left(1_U \otimes \theta_{U'}{}^{-1}\right) \circ \left(\theta_U{}^{-1} \otimes 1_{U'}\right) \circ c_{U,U'}{}^{-1}\right)$$
$$= 1_{X \otimes X'} \otimes \left(c_{U',U}{}^{-1} \circ \left(\theta_U{}^{-1} \otimes \theta_{U'}{}^{-1}\right) \circ c_{U,U'}{}^{-1}\right)$$
$$= 1_{X \otimes X'} \otimes \theta_{U' \otimes U}{}^{-1} = 1_{X \otimes X', U' \otimes U} = 1_{(X,U) \otimes' (X',U')}$$

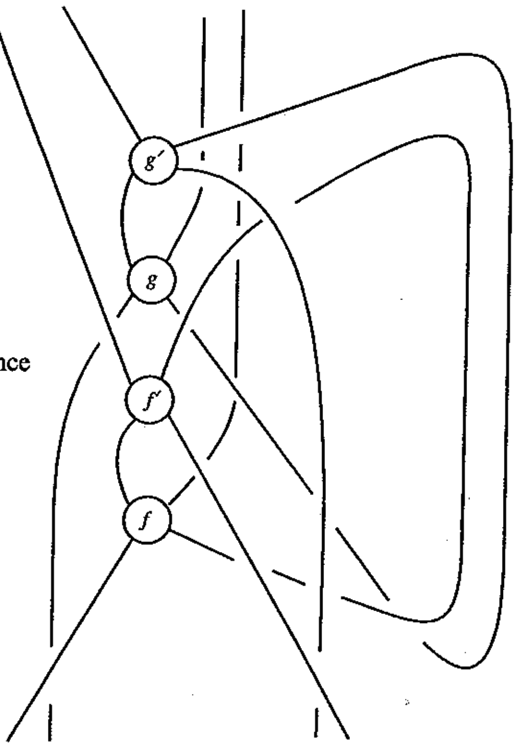
Pictorially,



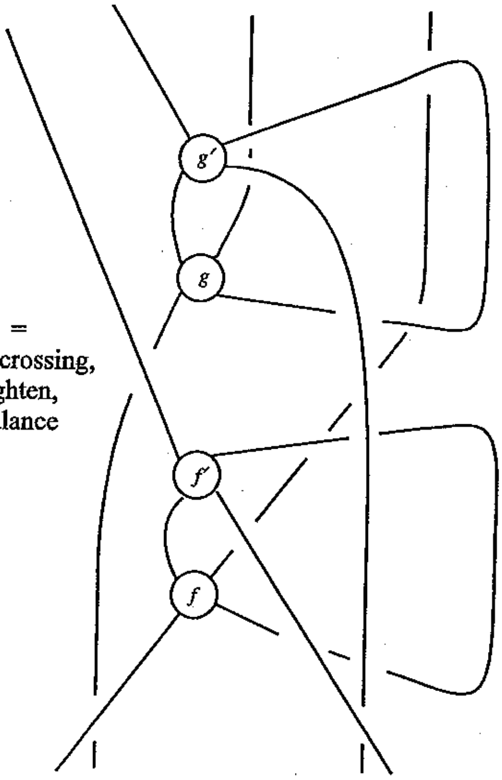
2b) We prove this diagrammatically as follows:



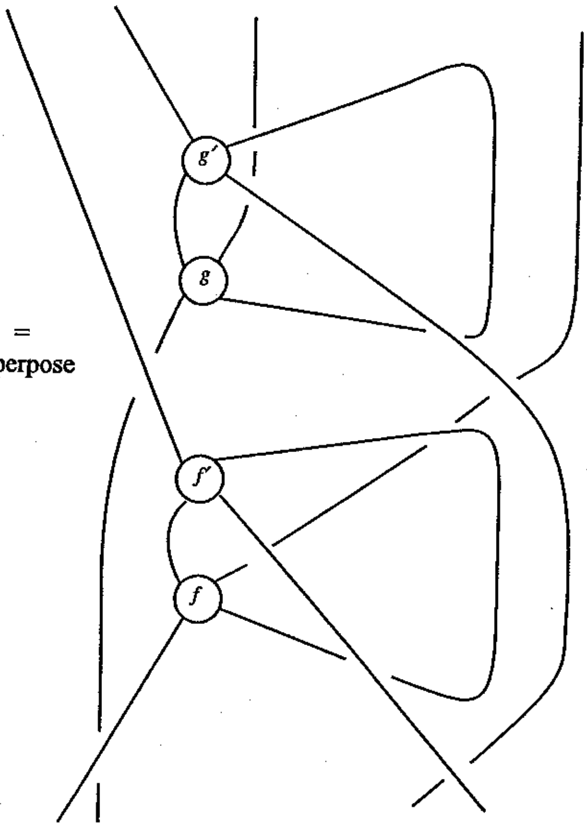
=
balance



=
slide crossing,
tighten,
balance



=
superpose



3) We need a natural family of isomorphisms

$$a_{(X,U),(Y,V),(Z,W)}:\big((X,U)\otimes'(Y,V)\big)\otimes'(Z,W)\to(X,U)\otimes'\big((Y,V)\otimes'(Z,W)\big)$$

Using the definition of $\otimes'$, we have

$$\big((X,U)\otimes'(Y,V)\big)\otimes'(Z,W)=(X\otimes Y,V\otimes U)\otimes'(Z,W)$$

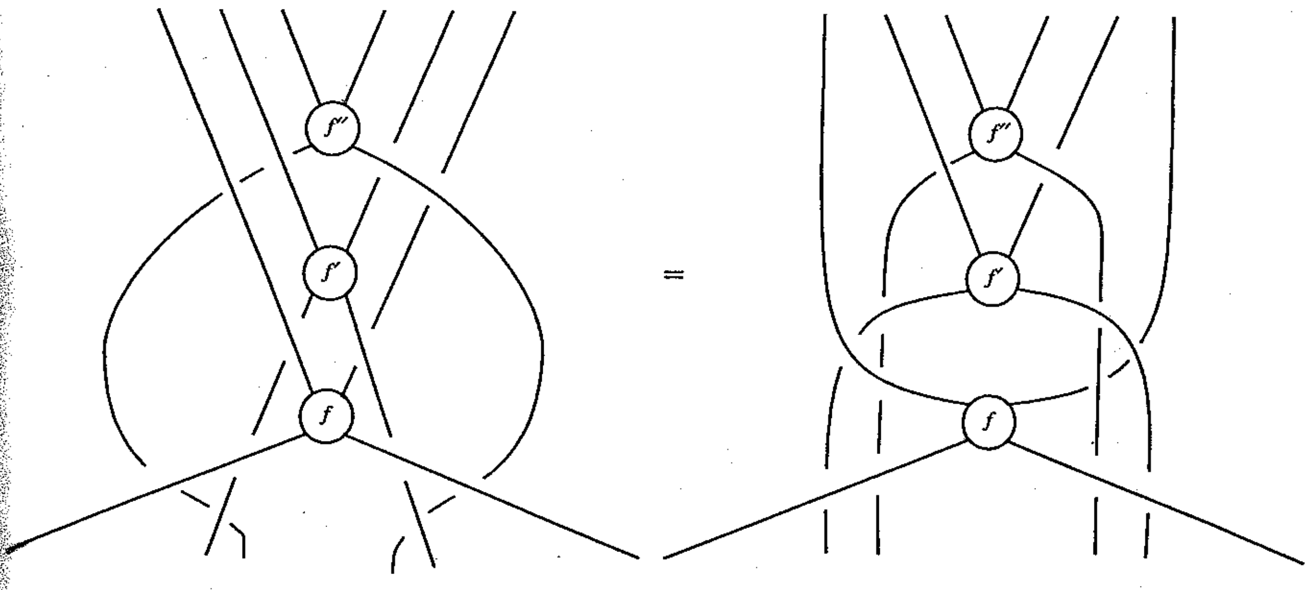
$$=(X\otimes Y\otimes Z, W\otimes V\otimes U)=(X,U)\otimes'(Y\otimes Z, W\otimes V)=(X,U)\otimes'((Y,V)\otimes'(Z,W))$$

showing that $\otimes'$ is associative on objects, so we take a to be the identity, making the associativity pentagon commute.

Naturality of a requires the following square to commute:

$$\begin{array}{ccc}((X,U)\otimes'(Y,V))\otimes'(Z,W) & \xrightarrow{a} & (X,U)\otimes'((Y,V)\otimes'(Z,W)) \\ \downarrow{\scriptstyle (f\otimes' f')\otimes' f''} & & \downarrow{\scriptstyle f\otimes'(f'\otimes' f'')} \\ ((X,U)\otimes'(Y,V))\otimes'(Z,W) & \xrightarrow{a} & (X,U)\otimes'((Y,V)\otimes'(Z,W))\end{array}$$

for any $f:(X,U)\to(X',U')$, $f':(Y,V)\to(Y',V')$, $f'':(Z,W)\to(Z',W')$. Since a is the identity, we need to prove that $f\otimes'(f'\otimes' f'')=(f\otimes' f')\otimes' f''$, which is the following purely balanced diagrammatic observation.



We can take $l_{X,U}=1_{X,U}$, since $l_{(X,U)}:(I,I)\otimes'(X,U)\to(X,U)$, and $(I,I)\otimes'(X,U)=(X,U)$. Similarly, $r_{X,U}:(X,U)\to(X,U)$ and we need $l=r$ to satisfy the triangle for unit, so we have $l=r=1$ The naturality conditions for l and r are then satisfied.

We need a natural family of isomorphisms

$$\phi_{2,X,X'}:N(X)\otimes' N(X')\to N(X\otimes X'),$$

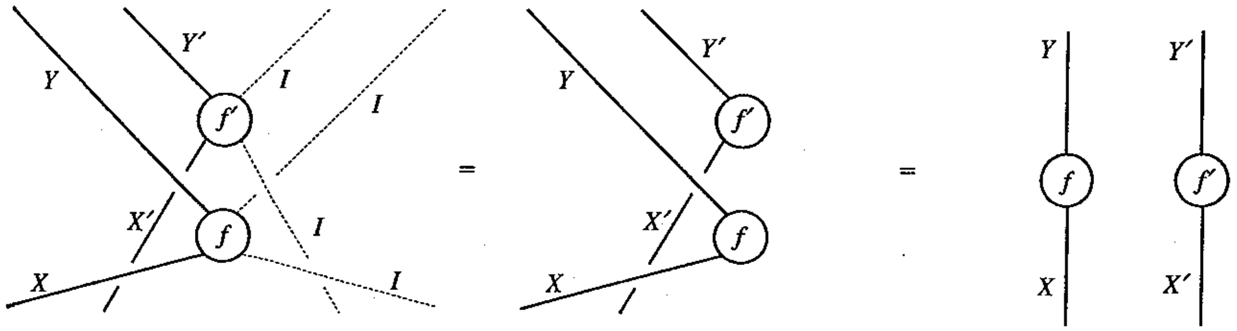
and we can take these to be the identity, since

$$N(X)\otimes' N(X')=(X,I)\otimes'(X',I)=(X\otimes X', I\otimes I)=N(X\otimes X'),$$

Naturality of ϕ_2 requires commutativity of the following square:

$$\begin{array}{ccc} N(X)\otimes' N(X') & \xrightarrow{\phi_2} & N(X\otimes X') \\ {\scriptstyle N(f)\otimes' N(f')}\downarrow & & \downarrow{\scriptstyle N(f\otimes f')} \\ N(Y)\otimes' N(Y') & \xrightarrow{\phi_2} & N(Y\otimes Y') \end{array}$$

which, with ϕ_2 the identity, requires $Nf \otimes' Nf' = N(f \otimes f') = f \otimes f'$. The easiest way to see this is pictorially as follows, with the identity objects drawn as dotted lines.



We also need an isomorphism $\phi_0 : (I,I) \to N(I)$; that is $\phi_0 : (I,I) \to (I,I)$, so we take it to be the identity, making all the required diagrams commute.

Chapter 5
The tortile Structure on Int$\mathcal{V}$

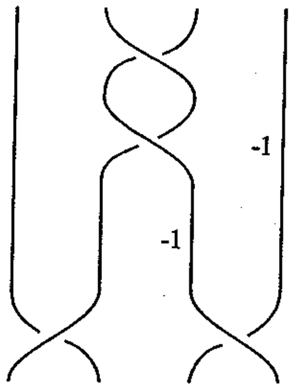
For each pair $(X,U),(X',U')$ of objects in Int$\mathcal{V}$, let

$$c_{(X,U),(X',U')}:(X,U)\otimes'(X',U')\rightarrow(X',U')\otimes'(X,U)$$

be the arrow in Int$\mathcal{V}$ given by the composite in $\mathcal{V}$ of the following four arrows.

$$X\otimes X'\otimes U\otimes U'\xrightarrow{c_{X,X'}\otimes c_{U',U}{}^{-1}}X'\otimes X\otimes U'\otimes U\xrightarrow{1\otimes1\otimes\theta_{U'}{}^{-1}\otimes1}X'\otimes X\otimes U'\otimes U$$
$$\xrightarrow{1\otimes c_{U',X}{}^{-1}\otimes\theta_U{}^{-1}}X'\otimes U'\otimes X\otimes U\xrightarrow{1\otimes c_{X,U'}\otimes1}X'\otimes X\otimes U'\otimes U.$$

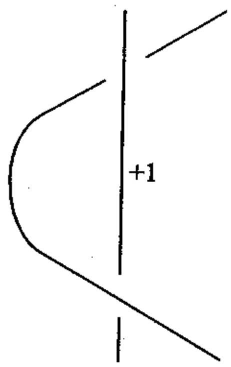
Diagrammatically



For each object (X,U) of Int$\mathcal{V}$, let $\theta_{(X,U)}:(X,U)\rightarrow(X,U)$ be the arrow in Int$\mathcal{V}$ given by the composite in $\mathcal{V}$ of three arrows,

$$X\otimes U\xrightarrow{c_{U,X}{}^{-1}}U\otimes X\xrightarrow{1\otimes\theta_X}U\otimes X\xrightarrow{c_{X,U}{}^{-1}}X\otimes U$$

represented diagrammatically as



The dual $(X,U)^*$ of (X,U) is (U,X). The counit is the arrow $\varepsilon:(U,X)\otimes'(X,U)\rightarrow(I,I)$ in Int$\mathcal{V}$ given by the arrow $1_U\otimes\theta_X:U\otimes X\rightarrow U\otimes X$ in $\mathcal{V}$. The unit $\eta:(I,I)\rightarrow(X,U)\otimes'(U,X)$ is given by the arrow $1_X\otimes\theta_U{}^{-1}:X\otimes U\rightarrow X\otimes U$ in $\mathcal{V}$. The dual $f^*:(Y,V)^*\rightarrow(X,U)^*$ of an arrow $f:(X,U)\rightarrow(Y,V)$ in Int$\mathcal{V}$ is the following composite in $\mathcal{V}$:

$$V\otimes X\xrightarrow{\theta_V\otimes1}V\otimes X\xrightarrow{c}X\otimes V\xrightarrow{f}Y\otimes U\xrightarrow{c^{-1}}U\otimes Y\xrightarrow{1\otimes\theta_Y{}^{-1}}U\otimes Y$$

Proposition 5.1: *The arrows* $c_{(X,U),(X',U')}$, $\theta_{(X,U)}$, ε *and* η *enrich the monoidal category* $\mathrm{Int}\mathcal{V}$ *with a tortile structure, and the monoidal functor* $N:\mathcal{V}\to\mathrm{Int}\mathcal{V}$ *(fully faithful by Proposition 4.1) is then traced.*

We must prove the following

1. **Braiding**

a) That $c_{(X,U),(X',U')}$ is a natural family of isomorphisms

b) That B1 commutes

c) That B2 commutes

2. **Twist**

a) $\theta_{(X,U)}$ is natural

b) $\theta_{(I,I)}=1_I$

c) Diagram T commutes

3. **Adjoints**

a) That dual objects, unit, counit as described are actually these, that is, the duality triangles commute.

b) $\theta_{(X,U)^*}=\theta_{(X,U)}^{\;*}$

4. ***N* is traced**

a) *N* is balanced

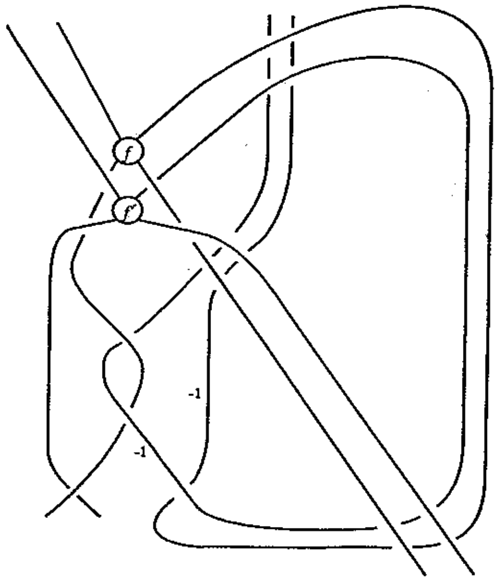
b) *N* preserves trace.

Proof:

1a) Naturality of the braiding is expressed by commutativity of the following square:

$$\begin{array}{ccc}
(X,U)\otimes'(X',U') & \xrightarrow{\;c\;} & (X',U')\otimes'(X,U) \\
{\scriptstyle f\otimes' f'}\downarrow & & \downarrow{\scriptstyle f'\otimes' f} \\
(Y,V)\otimes'(Y',V') & \xrightarrow{\;c\;} & (Y',V')\otimes'(Y,V)
\end{array}$$

We prove this diagrammatically as follows.



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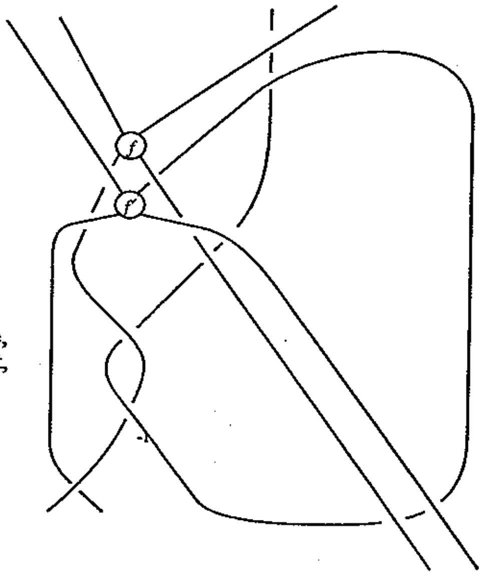

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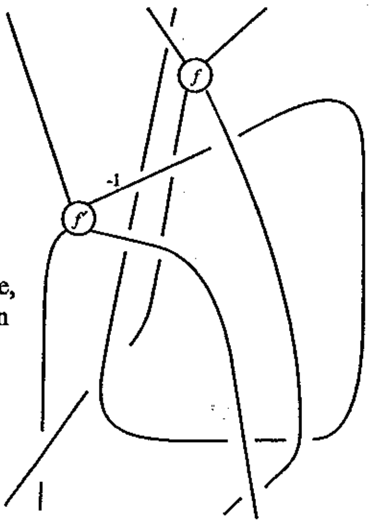

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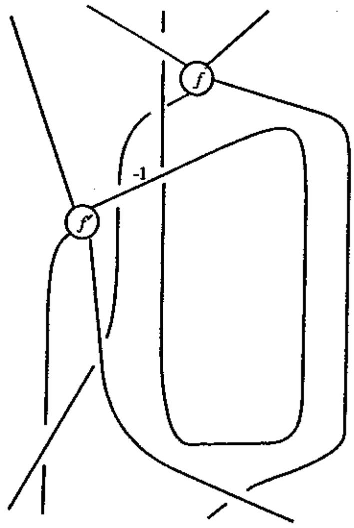

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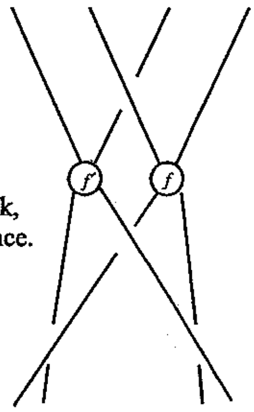

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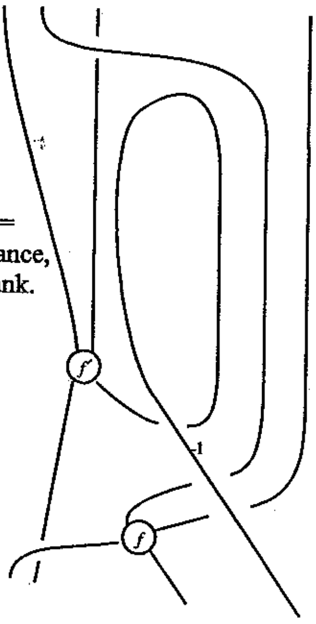

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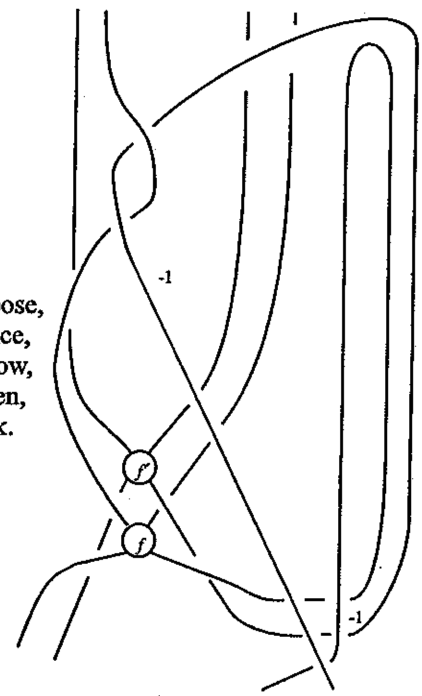

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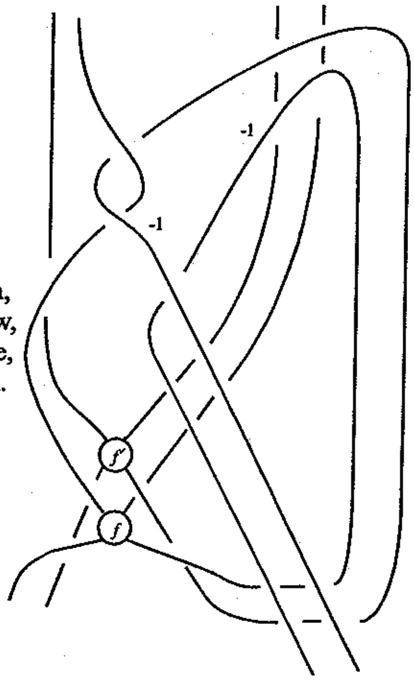
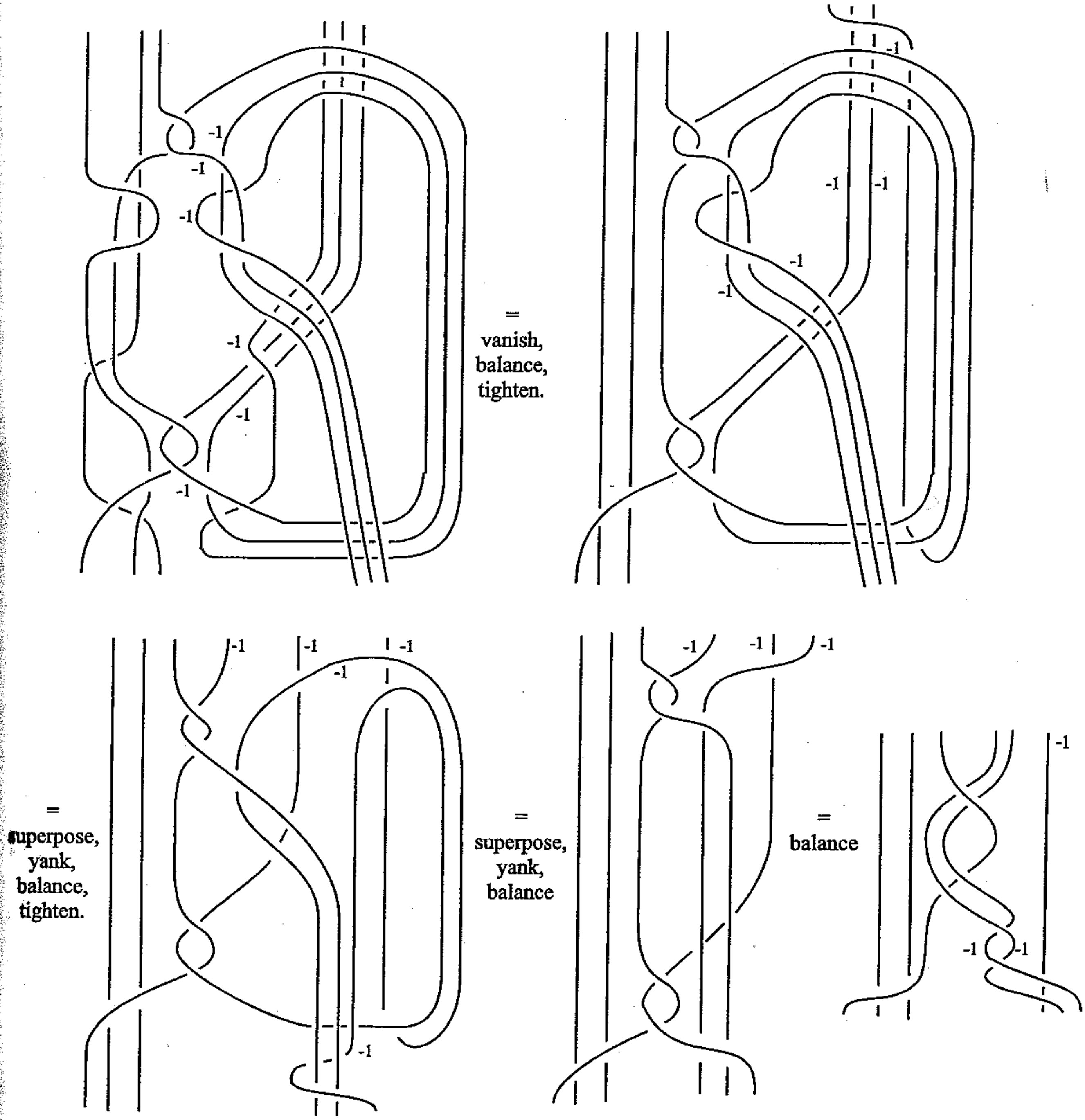


1b) We have seen that the associativity isomorphism is actually an identity, so diagram (B1) reduces to the following triangle, which we prove is commutative using the following diagrams.

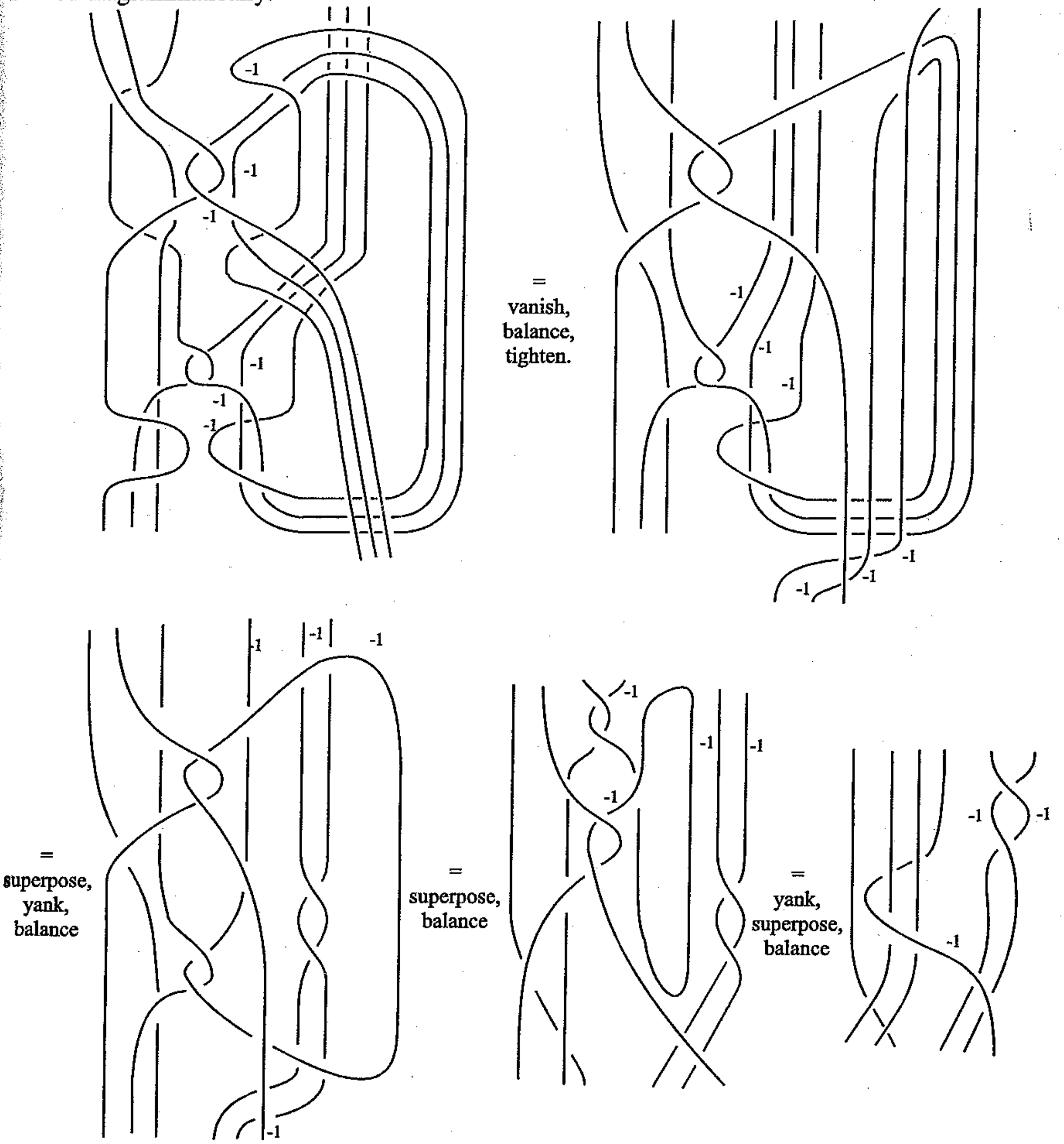
$$\begin{array}{ccc} (X,U)\otimes'(Y,V)\otimes'(Z,W) & \xrightarrow{c_{(X,U),(Y,V)\otimes'(Z,W)}} & (Y,V)\otimes'(Z,W)\otimes'(X,U) \\ & {\scriptstyle c\otimes'1} \searrow \quad \nearrow {\scriptstyle 1\otimes' c} & \\ & (Y,V)\otimes'(X,U)\otimes'(Z,W) & \end{array}$$



1c) Similarly, commutativity of (B2) reduces to commutativity of the following triangle,

$$\begin{array}{ccc} (X,U)\otimes'(Y,V)\otimes'(Z,W) & \xrightarrow{c_{(X,U)\otimes'(Y,V),(Z,W)}} & (Z,W)\otimes'(X,U)\otimes'(Y,V) \\ & {\scriptstyle 1\otimes' c}\searrow \quad \nearrow{\scriptstyle c\otimes' 1} & \\ & (X,U)\otimes'(Z,W)\otimes'(Y,V) & \end{array}$$

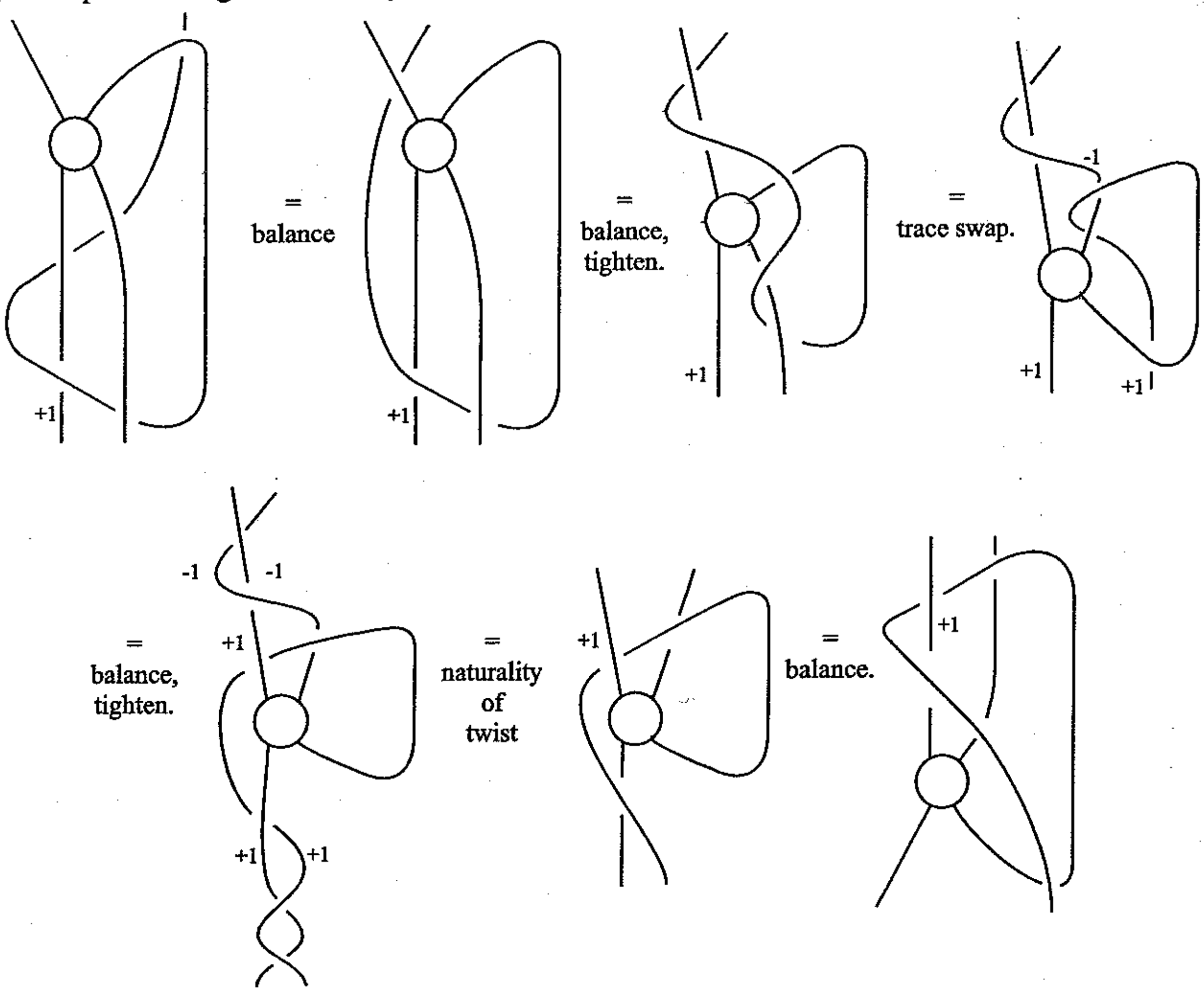
Proved diagrammatically:



2a) We wish to see that θ is natural (this was omitted from [JSV]); that is, for any $f:(X,U)\rightarrow(Y,V)$, the following diagram commutes.

$$\begin{array}{ccc}(X,U) & \xrightarrow{\theta(X,U)} & (X,U)\\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f}\\ (Y,V) & \xrightarrow{\theta(Y,V)} & (Y,V)\end{array}$$

This is proved diagrammatically as follows:



2b) By definition, $\theta_{I,I}=c_{I,I}\circ(1\otimes\theta_I)\circ c_{I,I}{}^{-1}$, and we have that $\theta_I=1$ since $\mathcal{V}$ is balanced.
From (B1), with A=B=C=I, and a = 1, we have

$(c_{I,I}\otimes 1)\circ(1\otimes c_{I,I})=c_{I,I\otimes I}$

$\Rightarrow\quad c_{I,I}\circ c_{I,I}=c_{I,I}$

$\Rightarrow\quad c_{I,I}\circ c_{I,I}\circ c_{I,I}{}^{-1}=c_{I,I}\circ c_{I,I}{}^{-1}$

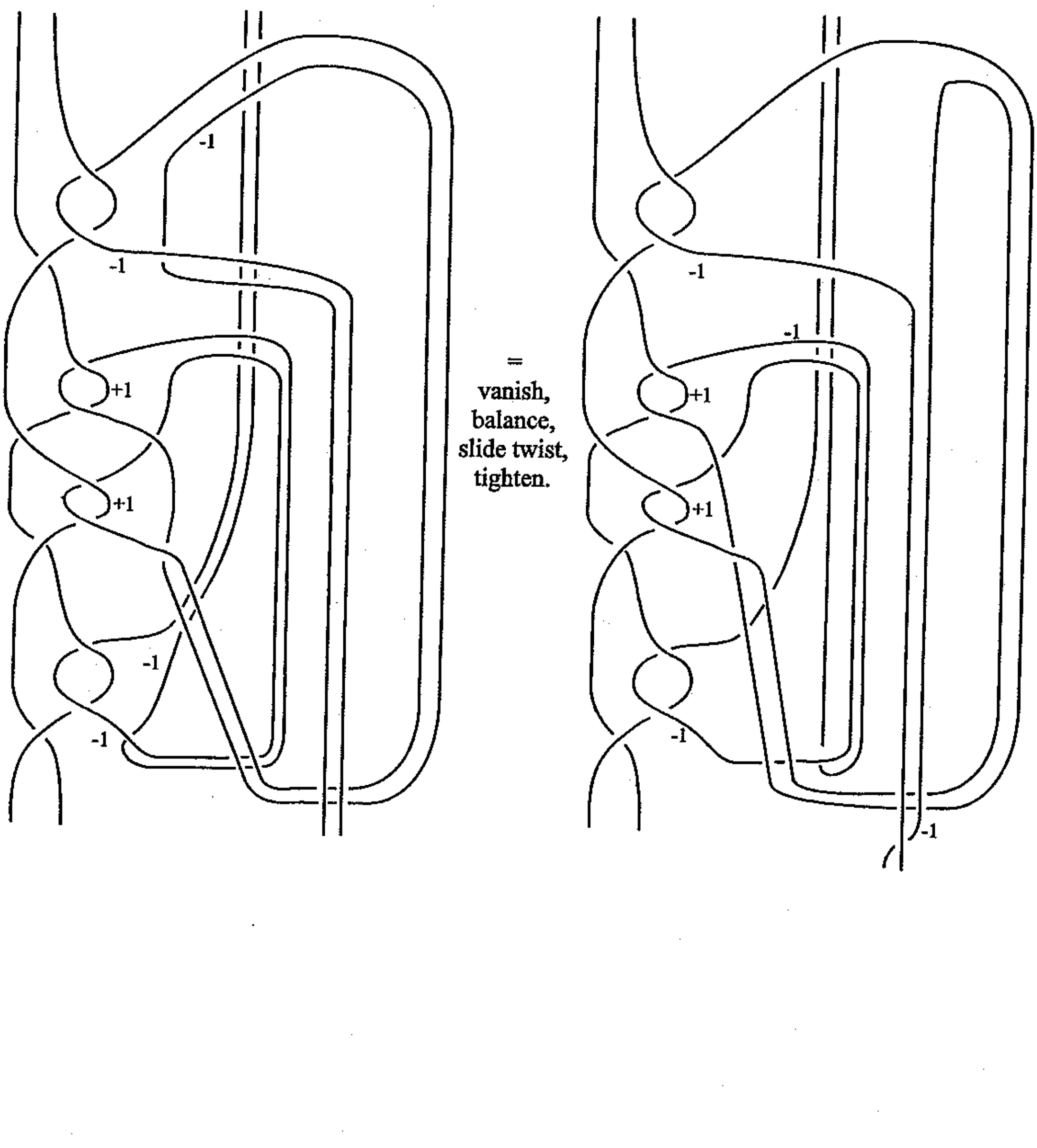
$\Rightarrow\quad c_{I,I}=1_{I,I}$

Hence $\theta_{I,I}=1_{I,I}$ as required.

2c) We need the following diagram, (T), to commute:

$$\begin{array}{ccc} (X,U)\otimes'(X',U') & \xrightarrow{c} & (X',U')\otimes'(X,U) \\ \theta\downarrow & & \downarrow\theta\otimes'\theta \\ (X,U)\otimes'(X',U') & \xleftarrow{c} & (X',U')\otimes'(X,U) \end{array}$$

We prove this diagrammatically as follows:



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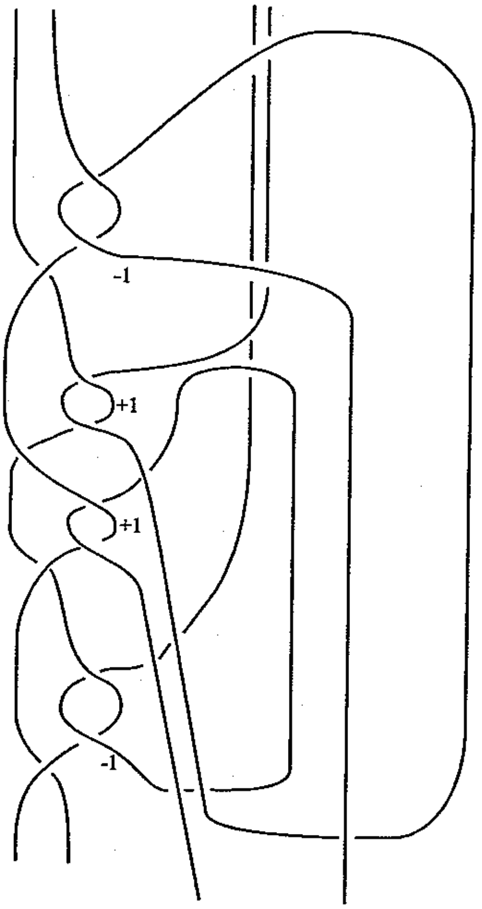

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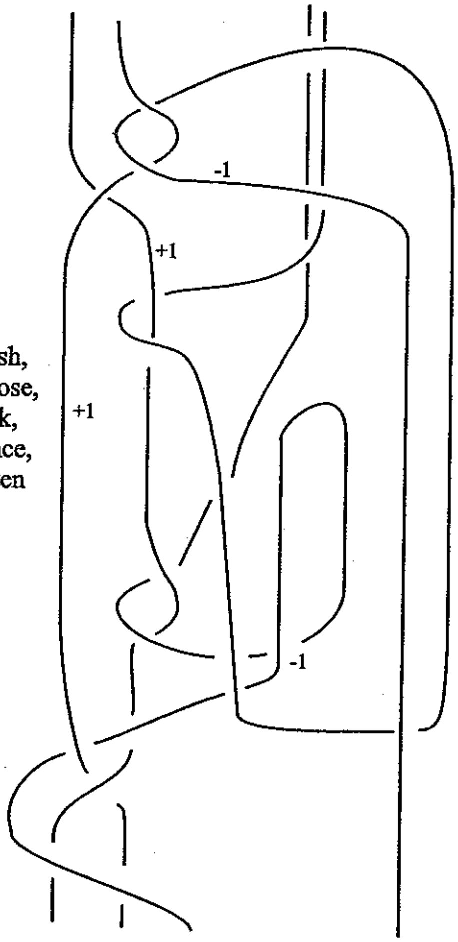

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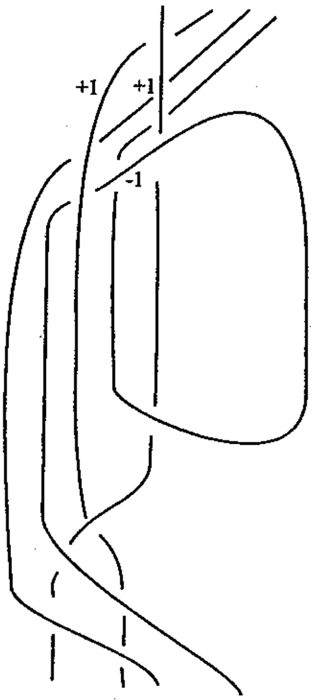

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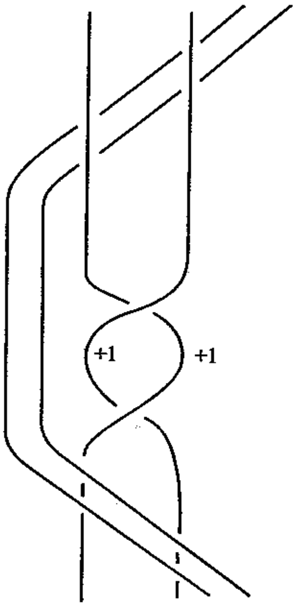


3a) We need to see that the duality triangles commute.
The first is

$$\begin{array}{ccc} (U,X) & \xrightarrow{1\otimes'\eta} & (U,X)\otimes'(X,U)\otimes'(U,X) \\ & {\searrow}^{1} & \downarrow{\scriptstyle \varepsilon\otimes' 1} \\ & & (U,X) \end{array}$$

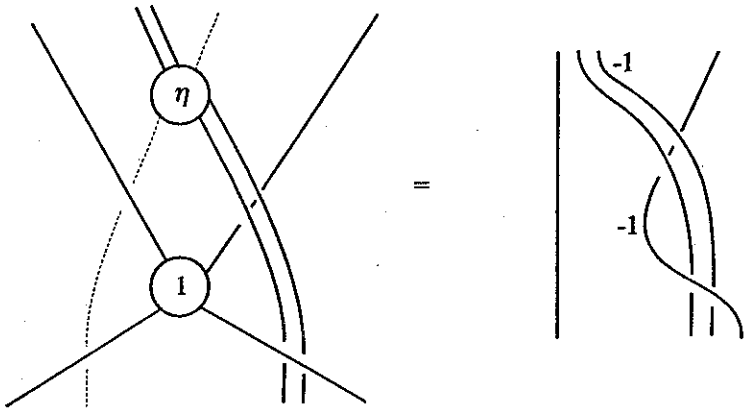
Drawing the correct diagram is somewhat subtle, so some of the reasoning will be presented.
We have $1:(U,X)\to(U,X)$ and $\eta:(I,I)\to(X\otimes U,X\otimes U)$, hence

$$1\otimes'\eta:(U\otimes I,I\otimes X)\to(U\otimes X\otimes U,X\otimes U\otimes X);$$

that is

$$1\otimes'\eta:((U\otimes I)\otimes(X\otimes U\otimes X))\to((I\otimes X)\otimes(U\otimes X\otimes U))$$

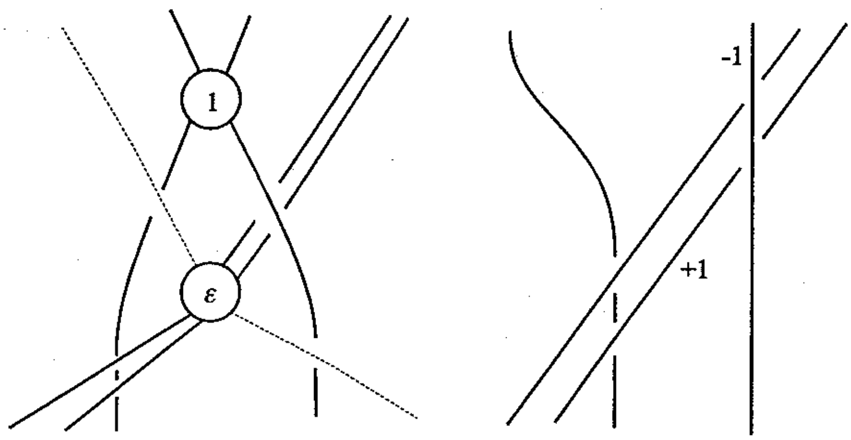
so we have the following diagram, with identity objects as dotted lines



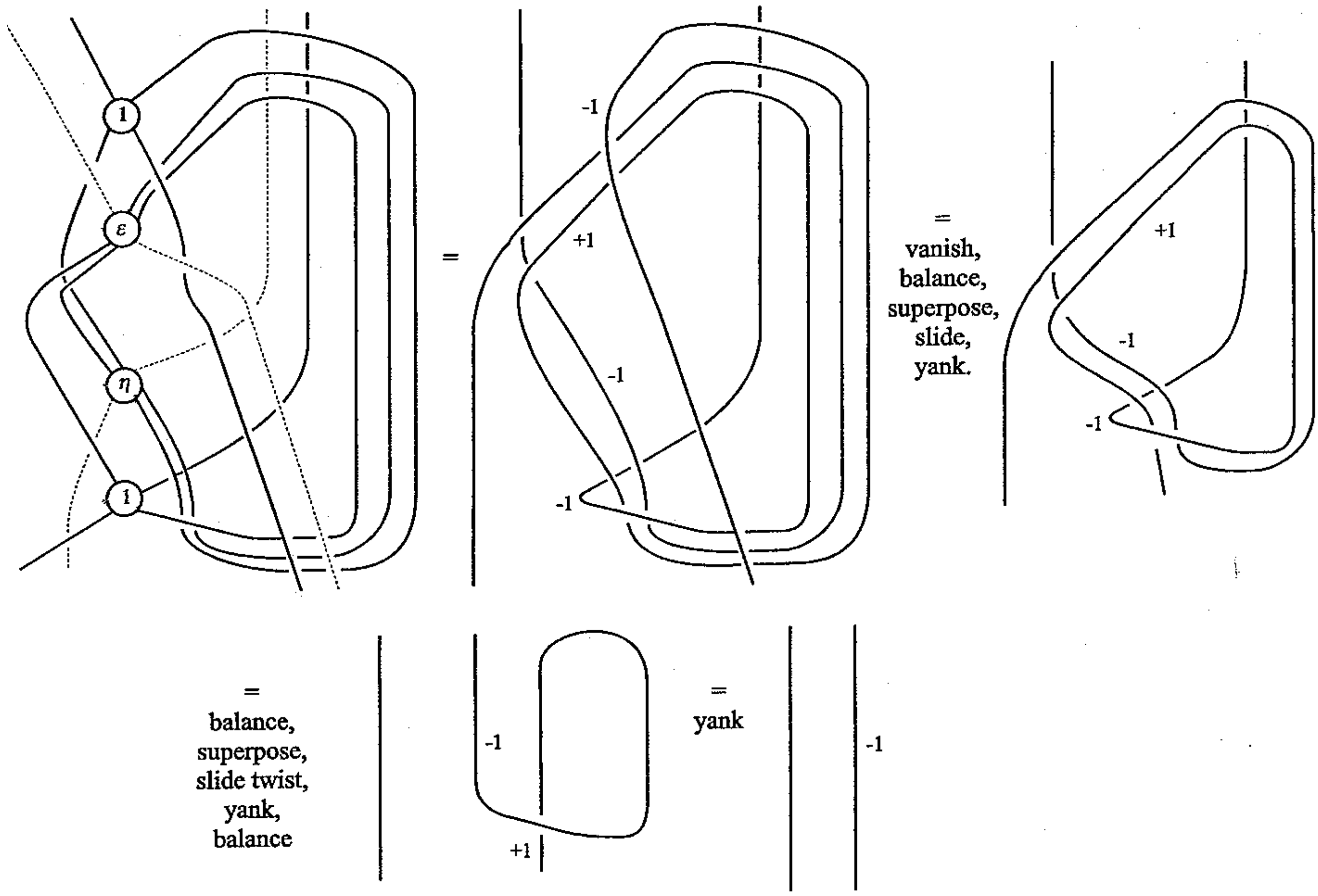
Similarly,

$$\varepsilon\otimes'1:((U\otimes X\otimes U)\otimes(X))\to((X\otimes U\otimes X)\otimes(U)),$$

pictorially



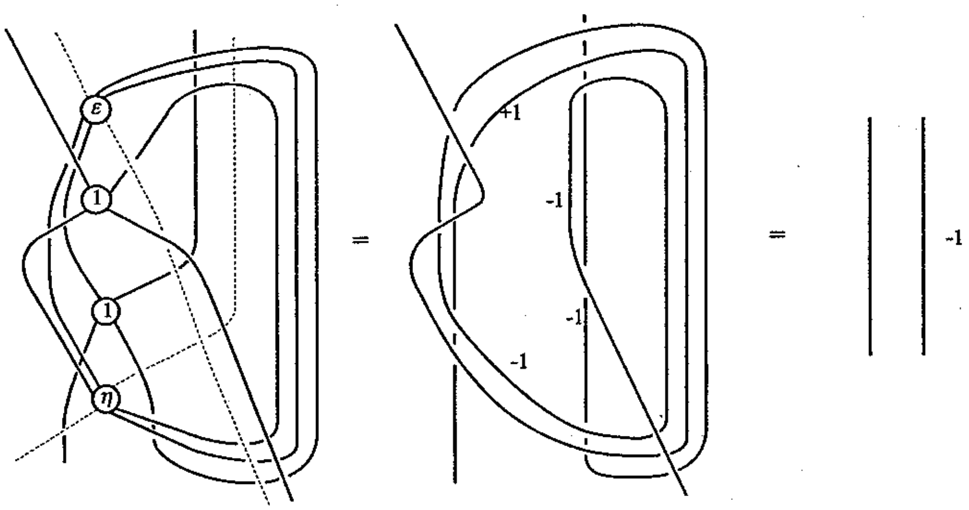
So composing gives



The second triangle is

$$\begin{array}{ccc} (X,U) & \xrightarrow{\eta\otimes'1} & (X,U)\otimes'(U,X)\otimes'(X,U) \\ & \searrow^{1} & \downarrow{\scriptstyle 1\otimes'\varepsilon} \\ & & (X,U) \end{array}$$

which is proved diagrammatically as follows:

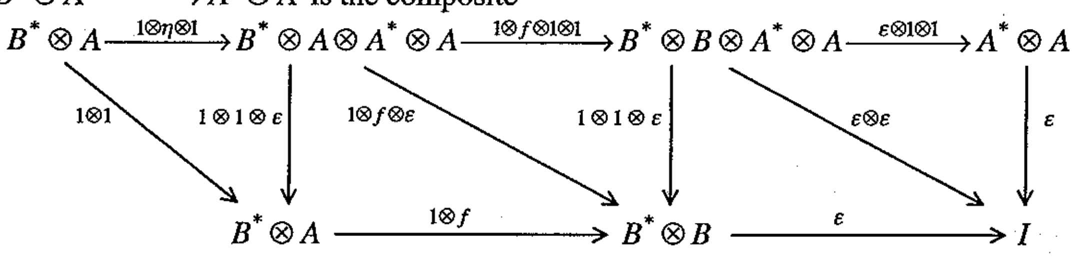


3b) To prove this, we first verify the formula for the dual of a map, and then apply it to θ.

For $A \xrightarrow{f} B$, $B^* \xrightarrow{f^*} A^*$ is defined to be the composite

$$B^* \xrightarrow{1\otimes\eta} B^* \otimes A \otimes A^* \xrightarrow{1\otimes f\otimes 1} B^* \otimes B \otimes A^* \xrightarrow{\varepsilon\otimes 1} A^*,$$

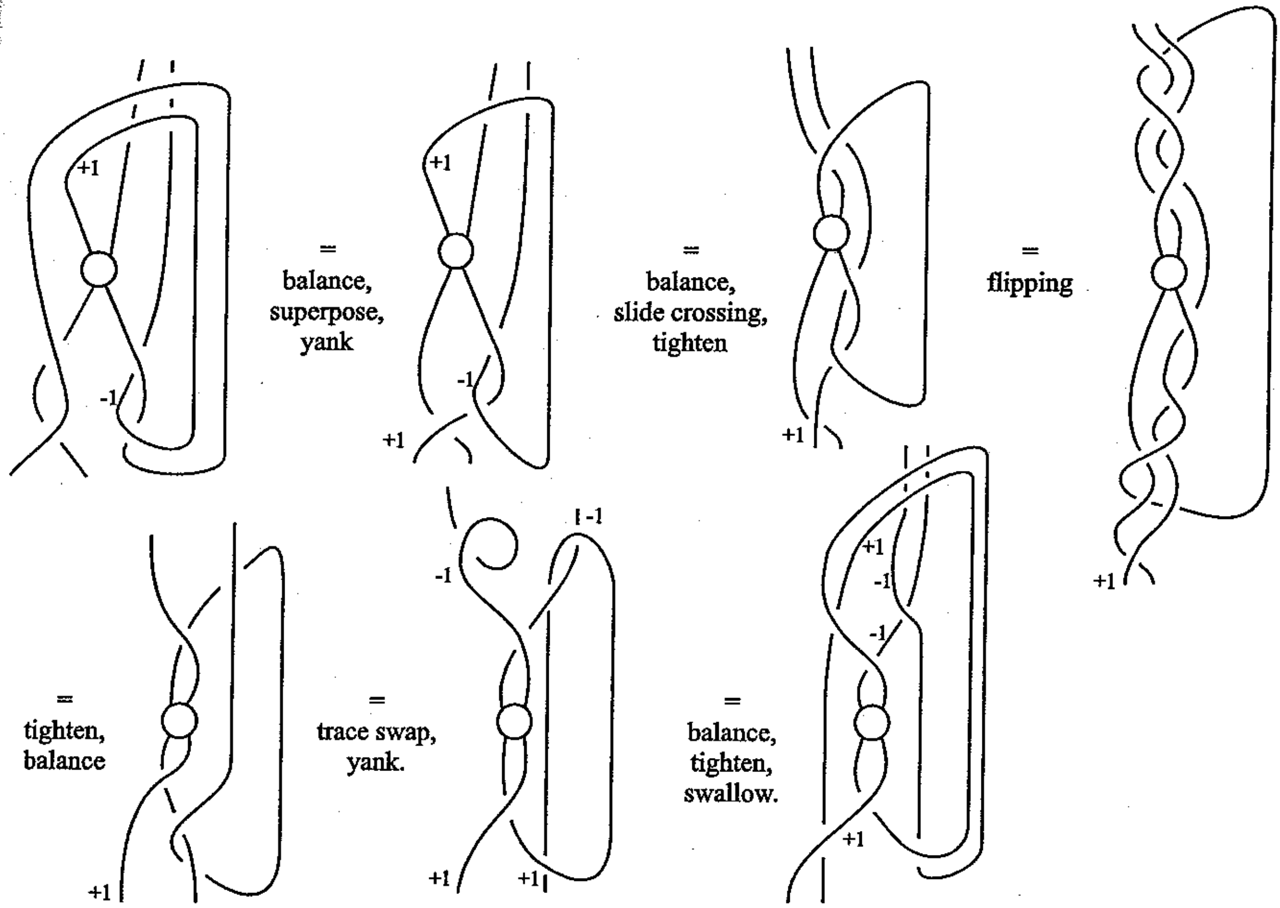
hence $B^* \otimes A \xrightarrow{f^*\otimes 1} A^* \otimes A$ is the composite



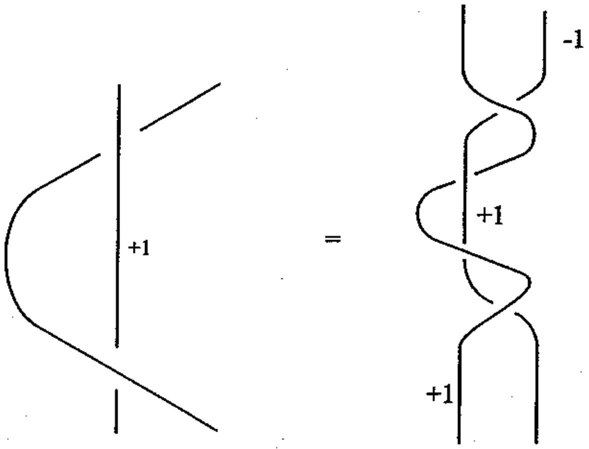
and by applying the duality triangle and functoriality of $\otimes$ as described by the commuting diagram above, we have that $\varepsilon \circ (1\otimes f) = \varepsilon \circ (f^* \otimes 1)$. Actually, it is easy to show that each of the following diagrams uniquely determines f^*.

$$\begin{array}{ccc} I & \xrightarrow{\eta_A} & A\otimes A^* \\ {\scriptstyle \eta_B}\downarrow & & \downarrow{\scriptstyle f\otimes 1} \\ B\otimes B^* & \xrightarrow{1\otimes f^*} & B\otimes A^* \end{array} \qquad \begin{array}{ccc} B^*\otimes A & \xrightarrow{f^*\otimes 1} & A^*\otimes A \\ {\scriptstyle 1\otimes f}\downarrow & & \downarrow{\scriptstyle \varepsilon_A} \\ B^*\otimes B & \xrightarrow{\varepsilon_B} & I \end{array}$$

Hence to see that what we have defined as f^* is actually the dual of *f*, we must show that $\varepsilon \circ (1\otimes f) = \varepsilon \circ (f^* \otimes 1)$, which is proved in the following diagrams.



Now we wish to see that $\theta_{(U,X)} = \theta_{(X,U)}^{*}$, which is the following purely balanced diagrammatic observation:



4a) We need to see that N is balanced. We have from Proposition 4.2 that N is monoidal. Since ϕ_2 and ϕ_0 are identities, for N to be braided we need $N(c) = c$, which we have from the definition of N.

We need to see that N preserves the twist, that is $\theta_X = N(\theta_X) = \theta_{N(X)} = \theta_{(X,I)}$. By definition, $\theta_{(X,I)} = c_{(X,I)} \circ (1_I \otimes \theta_X) \circ c_{I,X}$, and from (B1) and (B2), the braiding diagrams, it is easy to see that $c_{(X,I)} = 1_X = c_{I,X}$, hence $\theta_{(X,I)} = \theta_X$, so N is balanced.

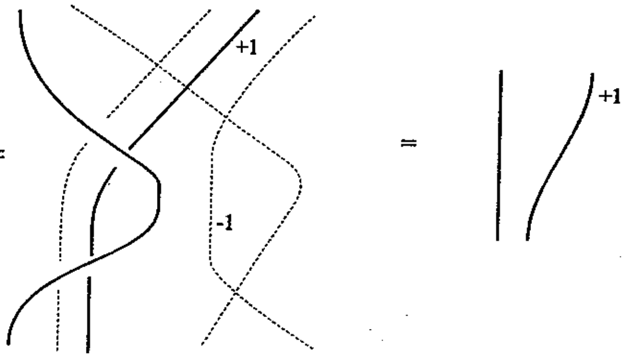
4b) We need to see that N preserves trace, that is, for all arrows $f : A \otimes U \to B \otimes U$ in $\mathcal{V}$, the canonical trace of $f : (A \otimes U, I) \to (B \otimes U, I)$ in Int$\mathcal{V}$ is $N(\mathrm{Tr}^U(f))$. That is, we must see that the following composite in Int$\mathcal{V}$ is equal to $\mathrm{Tr}^U(f) : (A,I) \to (B,I)$.

$$(A,I) \xrightarrow{1\otimes'\eta} (A,I) \otimes' (U,I) \otimes' (I,U) \xrightarrow{f\otimes'1} (B,I) \otimes' (U,I) \otimes' (I,U)$$
$$\xrightarrow{1\otimes'c} (B,I) \otimes' (I,U) \otimes' (U,I) \xrightarrow{1\otimes'1\otimes'\theta} (B,I) \otimes' (I,U) \otimes' (U,I) \xrightarrow{1\otimes'\varepsilon} (B,I)$$

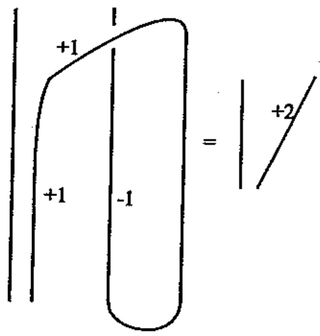
Two diagrammatic proofs are given. The first presents the diagrams for each factor in the same order as in the equation above, and then composes them in pairs, from top to bottom, simplifying at each step. The second proof presents all the factors composed from top to bottom in one diagram, which is then simplified.

Again we draw the identity object I as a dotted line. We need to draw it initially to keep track of how to tensor and compose things, but since crossings with I and twists on I are just identities, we can remove I from our diagrams.

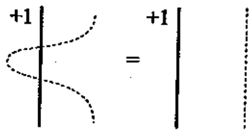
$1\otimes' \varepsilon =$



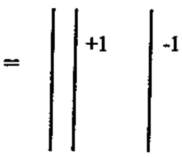
Composing with $1\otimes' 1\otimes' \theta$ gives



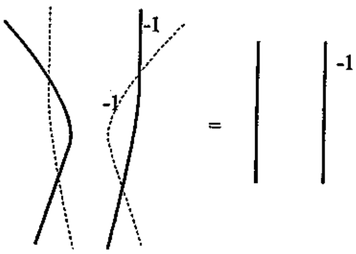
$\theta =$



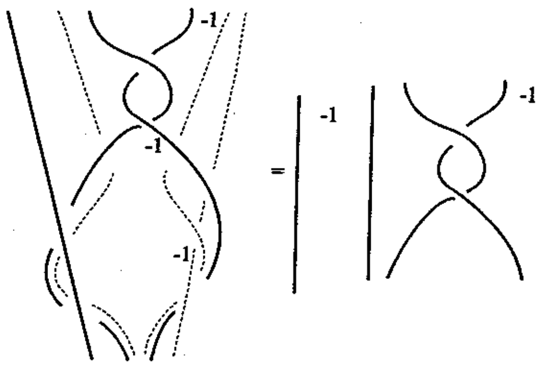
$1\otimes' 1\otimes' \theta =$



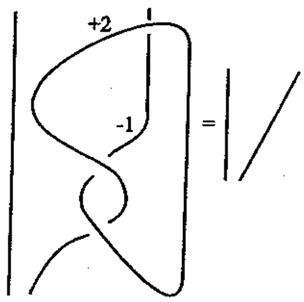
$1\otimes 1 =$



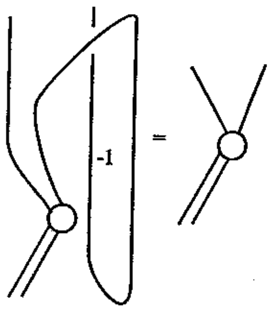
$1\otimes' c =$



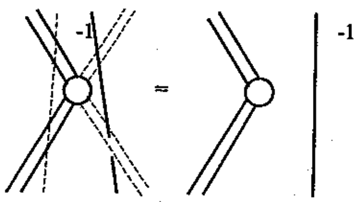
Composing the above with $1\otimes' c$ gives



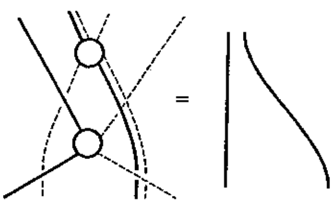
Composing the above with $f\otimes' 1$ gives



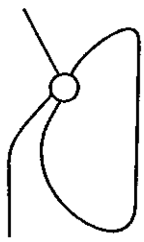
$f\otimes' 1 =$



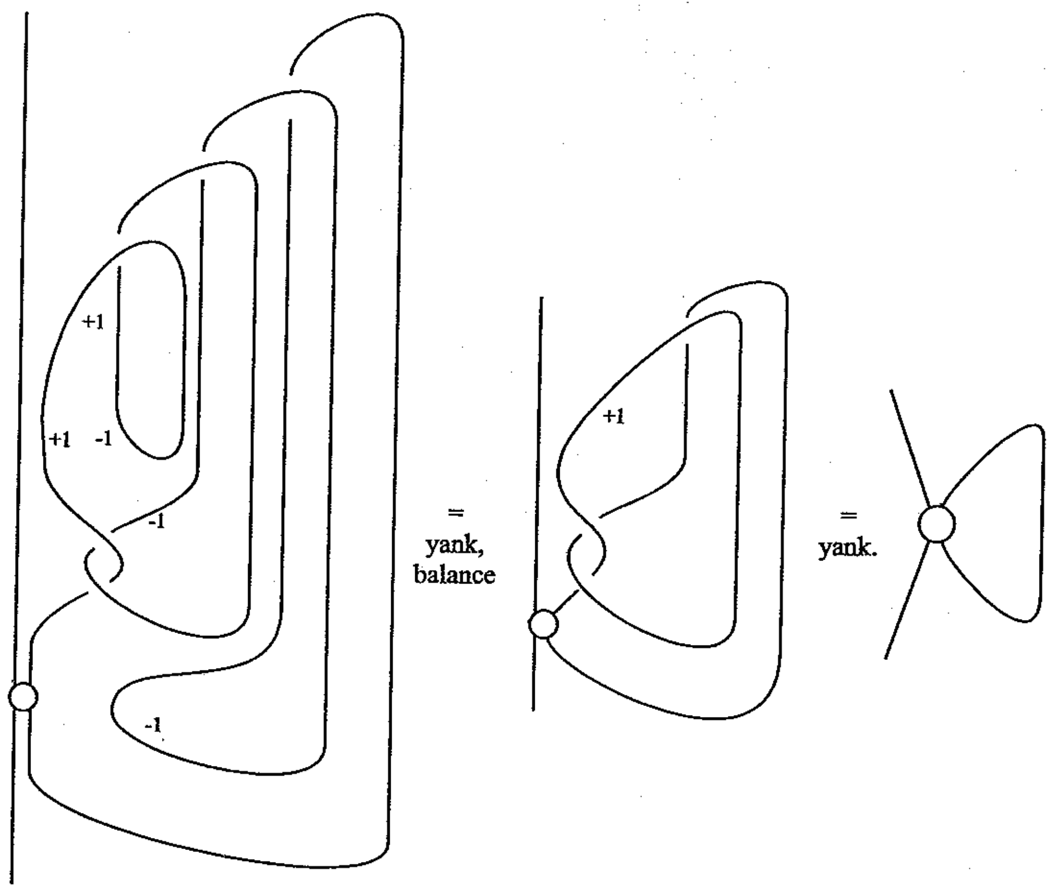
$1\otimes' \eta =$



Composing the above with $1\otimes' \eta$ gives

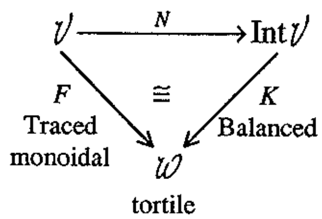


The second diagrammatic proof is as follows.



We now give a universal property for the construction.

Proposition 5.2: *Suppose $\mathcal{V}$ is a traced monoidal category and $\mathcal{W}$ is a tortile monoidal category. Then, for all traced monoidal functors $F : \mathcal{V} \to \mathcal{W}$, there exists a balanced monoidal functor $K : \mathrm{Int}\,\mathcal{V} \to \mathcal{W}$ which is unique up to monoidal natural isomorphism with the property $KN \cong F$.*



This result is stated without proof, however we give the definition of *K*, and the proof can be found in [JSV].

On the object (X,U) of Int $\mathcal{V}$, put $K(X,U) = FX \otimes (FU)^*$.

For $f : (X,U) \to (Y,V)$, define

$$K(f) : K(X,U) \to K(Y,V) = (1 \otimes \varepsilon') \circ (1 \otimes c^{-1}) \circ (Ff \otimes 1 \otimes 1) \circ (1 \otimes \eta \otimes 1).$$

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[JS3] A. JOYAL & R. STREET, The geometry of tensor calculus II. *In progress.*

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