Branches of higher dimensional algebra

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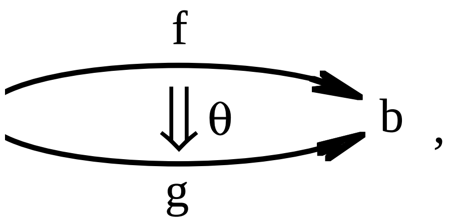
ABSTRACT

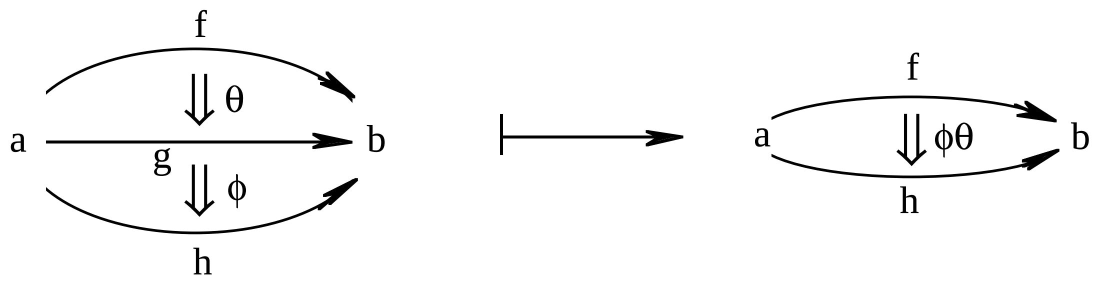
The talk will survey recent advances in the study of higher dimensional categorical structures involving the higher operads of Michael Batanin defined in terms of plane trees.

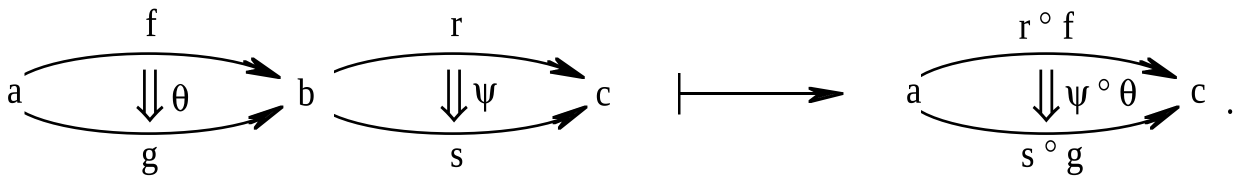
2-categories

A *2-category* A consists of objects a, b, c, . . . , arrows $f : a \longrightarrow b$, and 2-arrows $\theta : f \Rightarrow g : a \longrightarrow b$ displayed thus



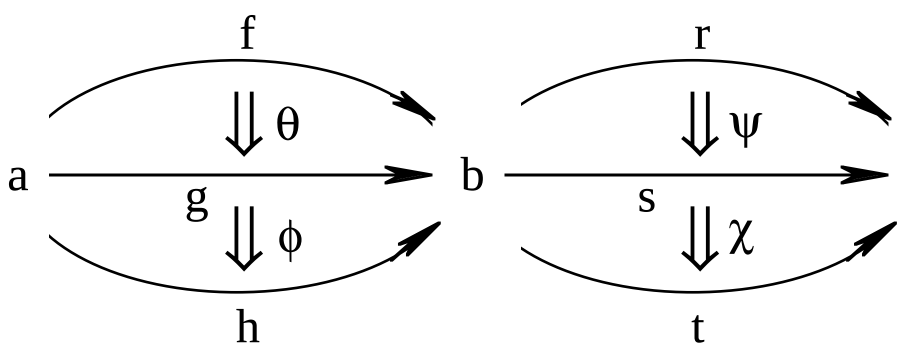
together with vertical and horizontal compositions





These compositions are required to be associative and unital; moreover, horizontal composition must preserve vertical units and the following *interchange law* is imposed.

$$(\chi\psi) \circ (\phi\theta) = (\chi \circ \phi)(\psi \circ \theta)$$



EXAMPLES

Cat is a 2-category:

objects are categories,

arrows are functors, and

2-arrows are natural transformations.

Surf is a 2-category:

objects are finite subsets of the real line,

arrows are progressive plane strings, and

2-arrows are deformation classes of progressive singular 3D surfaces.

A *weak 2-category* or *bicategory* consists of the data and conditions of a 2-category except that the associativity and unital equalities for horizontal composition are replaced by the extra data of invertible natural families of 2-arrows

$$\alpha_{f,r,m} : (m \circ r) \circ f \Rightarrow m \circ (r \circ f), \quad \lambda_f : 1_b \circ f \Rightarrow f, \quad \rho_f : f \circ 1_a \Rightarrow f,$$

called *associativity and unital constraints,* such that the *associativity pentagon* (or *3-cocycle condition*)

$$\alpha_{p,m,r\circ f}\ \alpha_{p\circ m,r,f} \quad = \quad (1_p \circ \alpha_{m,r,f})\ \alpha_{p,m\circ r,f}\ (\alpha_{p,m,r} \circ 1_f)$$

and *unit triangle* (or *normalisation condition*)

$$(1_r \circ \lambda_f)\ \alpha_{f,r,m} \quad = \quad \rho_r \circ 1_f$$

are imposed.

EXAMPLES

Each monoidal category $\mathcal{V}$ gives a one object bicategory $\Sigma\mathcal{V}$ whose arrows are objects of $\mathcal{V}$, whose 2-arrows are the arrows of $\mathcal{V}$, whose horizontal composition is the tensor product of $\mathcal{V}$, and whose vertical composition is the composition of $\mathcal{V}$.

Each topological space X has a *homotopy bicategory*

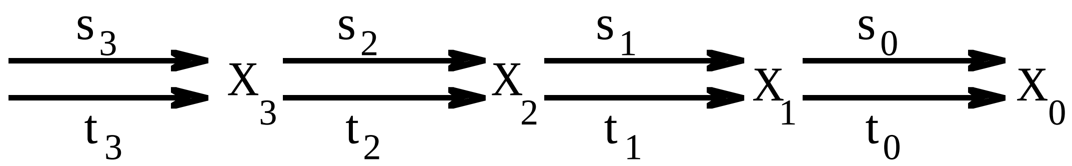
$$\Pi_2(X)$$

whose objects are points of X, whose arrows are paths in X, and whose 2-arrows are homotopy classes of homotopies.

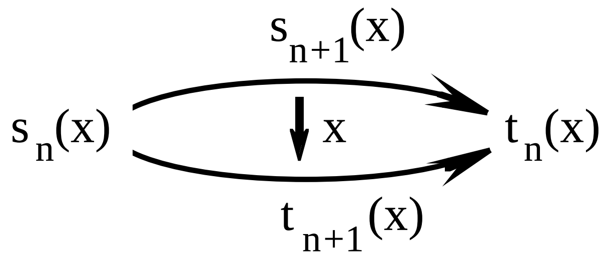
A *globular set* X is a sequence $(X_n)_{n \geq 0}$ of sets X_n together with functions

$$s_n, t_n : X_{n+1} \longrightarrow X_n$$

such that $s_n \circ s_{n+1} = s_n \circ t_{n+1}$, $t_n \circ s_{n+1} = t_n \circ t_{n+1}$.



Another name might be *ω-graph:* the higher arrow notation is used:



$x \in X_{n+2}$

Each 2-category A has an underlying globular set X :
$X_0 = \{\text{objects}\}$, $X_1 = \{\text{arrows}\}$, $X_2 = \{\text{2-arrows}\}$, $X_3 = \emptyset$.

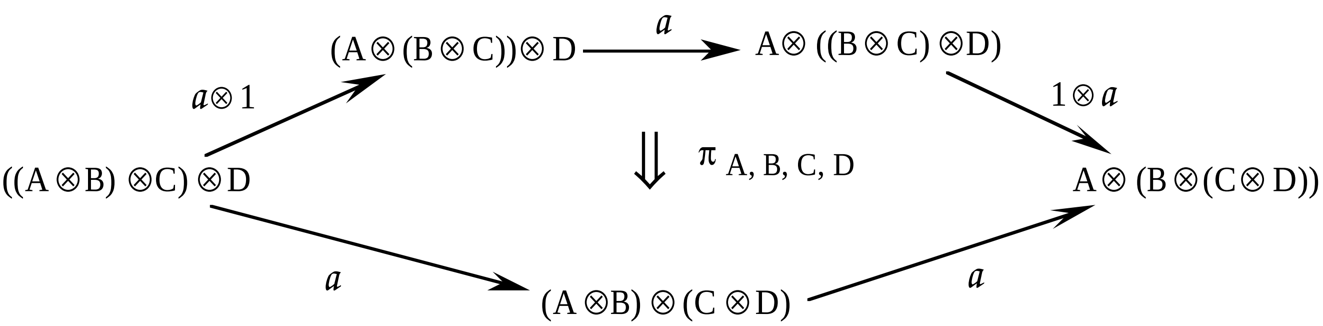
The definition of *ω-category* should now be fairly clear: we have a 2-category structure on each 2-graph of three consecutive sets X_n, X_{n+1}, X_{n+2} . There are no new kinds of conditions: just associative, unital and interchange laws.

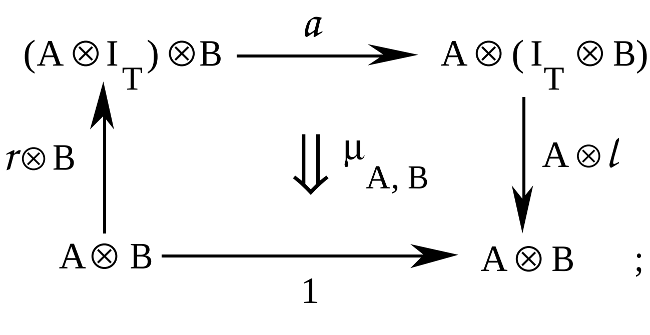
Tricategories

A *tricategory* $\mathcal{T}$ is a 3-graph

$$\mathcal{T}_0 \leftleftarrows \mathcal{T}_1 \leftleftarrows \mathcal{T}_2 \leftleftarrows \mathcal{T}_3$$

together with compositions like those for a 3-category, constraints making $\mathcal{T}_1 \leftleftarrows \mathcal{T}_2 \leftleftarrows \mathcal{T}_3$ a bicategory, constraints for $\mathcal{T}_0 \leftleftarrows \mathcal{T}_1 \leftleftarrows \mathcal{T}_2$ <u>like</u> those for a bicategory but merely equivalences (not necessarily isomorphisms) and, instead of the commutativity axioms on those constraints, further higher-dimensional constraints

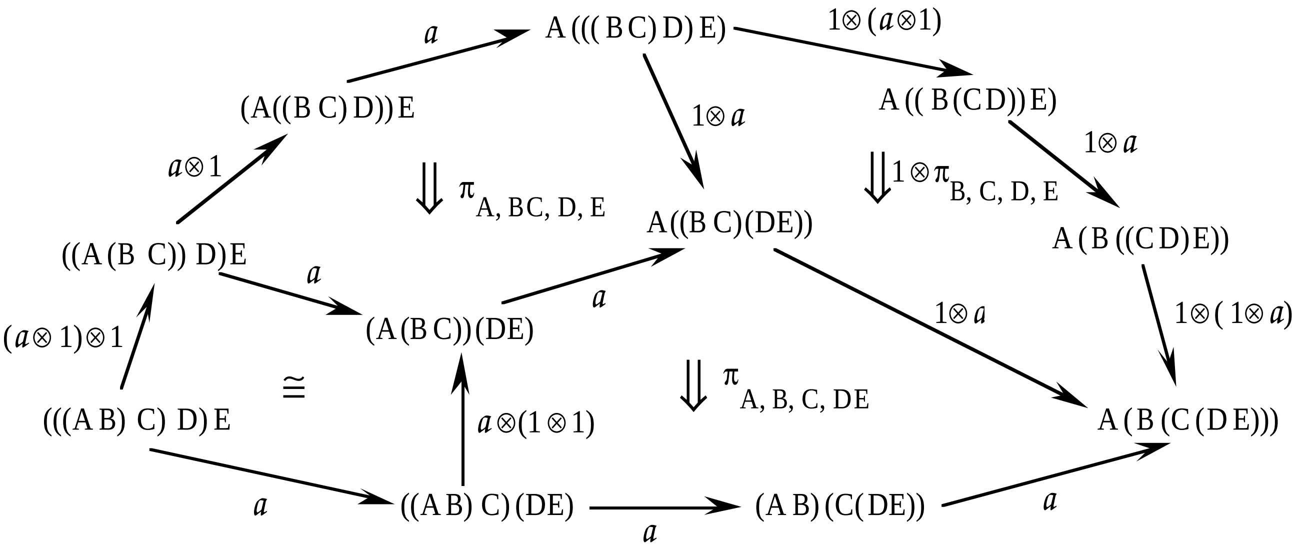




and even a further invertible 3-arrow constraint representing the failure of the precise interchange law:

$$\gamma : (f \otimes g) \circ (h \otimes k) \Rightarrow (f \circ h) \otimes (g \circ k) ;$$

subject to natural axioms including the equality:



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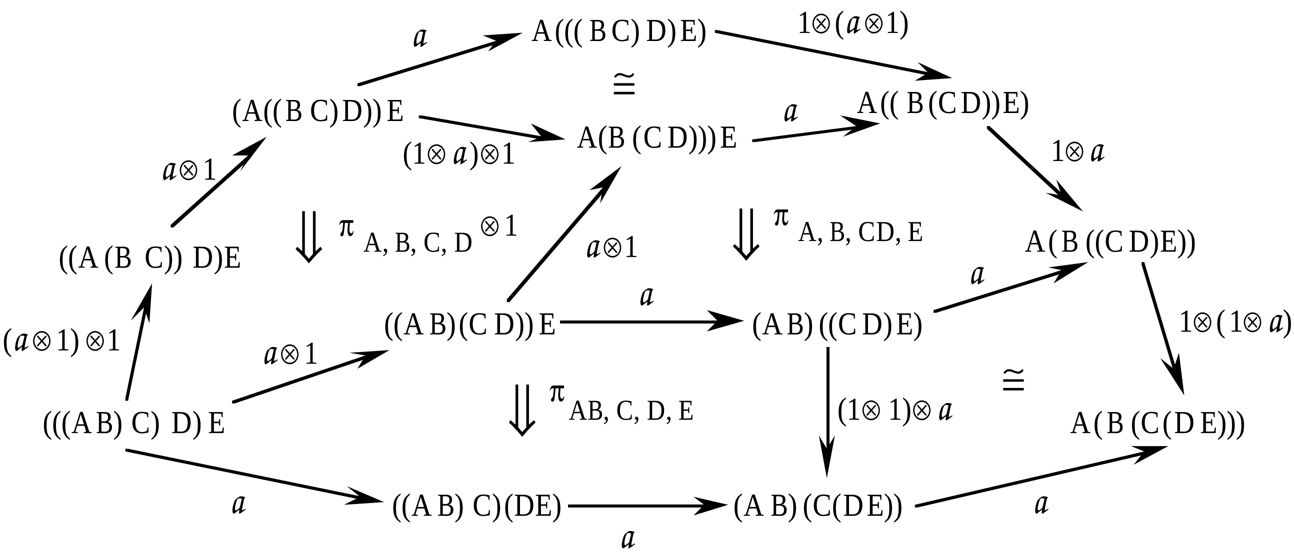
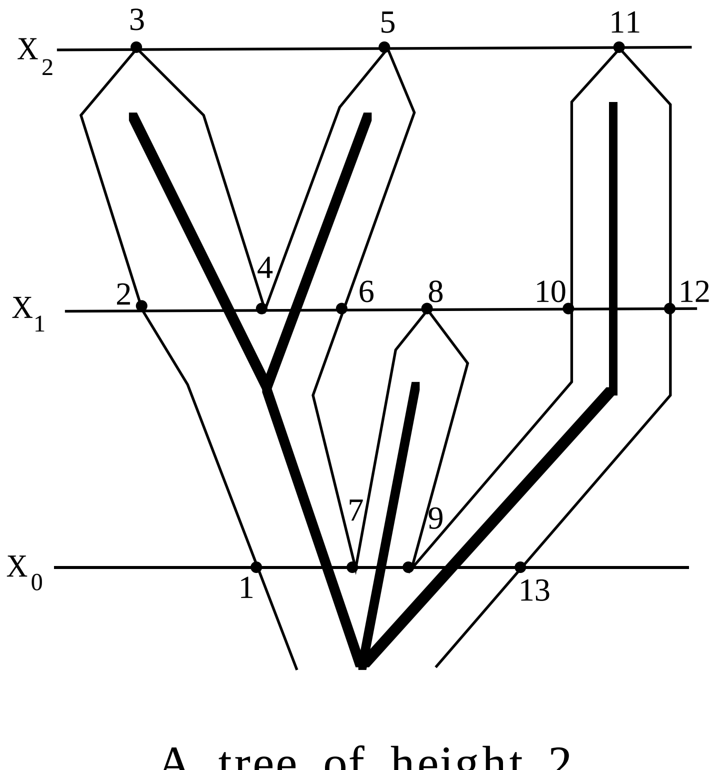


EXAMPLE

Each braided monoidal category $\mathcal{V}$ gives a tricategory $\Sigma^2 \mathcal{V}$ with only one object, only one arrow, with 2-arrows the objects of $\mathcal{V}$, and with 3-arrows the arrows of $\mathcal{V}$.

What kind of algebraic theory is needed to describe these higher order structures?

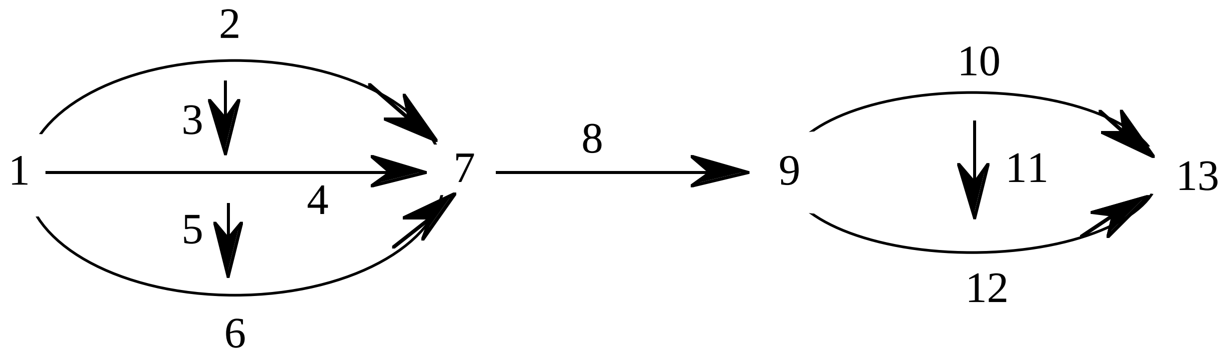
Instead of operations $A^n \longrightarrow A$ whose arities are natural numbers n, Batanin's idea was to use operations whose arities are *plane trees*.



A tree of height 2

Each tree T gives a globular set $X = |T|$ as illustrated in the diagram above:

$$X_0 = \{ 1, 7, 9, 13 \}, \quad X_1 = \{ 2, 4, 6, 8, 10, 12 \}, \quad X_2 = \{ 3, 5, 11 \}.$$



A *globular pasting diagram* in a globular set A is a pair (T, f) where T is a tree and $f : |T| \longrightarrow A$ is a map of globular sets.

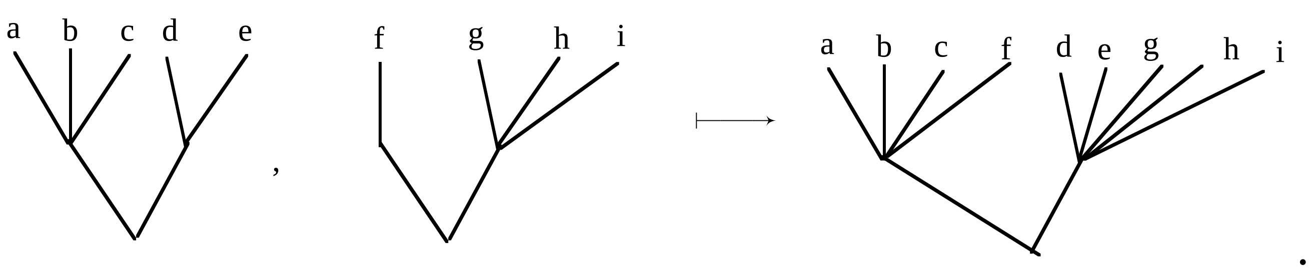
If A is an ω-category, each such pasting diagram has a unique n-arrow *paste*(T, f) $\in A_n$ obtainable using the compositions of A where n is the height of the tree T.

Trees form an ω-category **Tree:**

the n-arrows are the trees of height n;

the m-source and m-target of a tree are equal and are obtained by pruning off all the stuff above height m;

the compositions are illustrated by



<u>Theorem</u> **Tree** *is the free ω-category on the globular set with a single element in each dimension.*

The simplest kind of operad

Consider sets A equipped with a function $\alpha : A \longrightarrow \mathbf{N}$ into the natural numbers, called *arity*.

There is a substitution operation of graded sets:

given

$$\alpha : A \longrightarrow \mathbf{N}, \qquad \alpha : B \longrightarrow \mathbf{N},$$

put

$$B(A) = \{ (b, a_1, a_2, \ldots, a_k) : b \in B,\ a_i \in A \text{ and } k = \alpha(b) \}$$

where $\alpha(b, a_1, a_2, \ldots, a_k) = \alpha(a_1) + \ldots + \alpha(a_k)$.

A (non-permutative) *operad* is a graded set A together with a function

$$\text{sub} : A(A) \longrightarrow A,$$

written $\text{sub}(b, a_1, a_2, \ldots, a_k) = b(a_1, a_2, \ldots, a_k)$, and an element 1 of arity 1, such that

$$1(a) = a, \quad b(1, \ldots, 1) = b,$$

$$c(b_1(a_{11}, \ldots, a_{1j_1}), \ldots, b_m(a_{m1}, \ldots, a_{mj_m})) = c(b_1, \ldots, b_m)(a_{11}, \ldots, a_{1j_1}, \ldots, a_{m1}, \ldots, a_{mj_m}).$$

An A-*algebra* is a set X with an n-ary operation for each element of A of arity n subject to two obvious conditions.

Batanin operads

Consider globular sets A equipped with a globular function $\alpha : A \longrightarrow \mathbf{Tree}$, called *arity*.

There is a substitution operation of tree-graded sets:
given

$$\alpha : A \longrightarrow \mathbf{Tree}, \qquad \alpha : B \longrightarrow \mathbf{Tree},$$

put

$$B(A) = \{ (b,\ a : |T| \longrightarrow A) :\ b \in B,\quad a \textit{ is a globular pasting diagram in } A, \quad \textit{and} \quad T = \alpha(b) \}$$

where

$$\alpha(b, a) = \mathit{paste}\left(|T| \xrightarrow{\ a\ } A \xrightarrow{\ \alpha\ } \mathbf{Tree} \right).$$

A (Batanin) *operad* is a tree-graded set A together with a function

$$\mathrm{sub} :\ A(A) \longrightarrow A,$$

written $\mathrm{sub}(b, a) = b(a)$, and, for each n, an element $u_n \in A_n$ of arity the tree of height n having one node at each level up to n, satisfying the natural conditions.

An A-*algebra* is a globular set X with, for each $a \in A_n$, an assignment of an element $a(x) \in X_n$ to each globular pasting diagram $x : |\alpha(a)| \longrightarrow X$ in X subject to obvious conditions.

EXAMPLES

1) Take A to be the globular set with a single element in each dimension. There is a canonical operad structure on A. An A-algebra is an ω-category.

2) There is an operad K which is the free (initial) one generated by some basic operations and satisfying a contractibility condition. A K-algebra is a weak ω-category.