§1. The calculus of modules.

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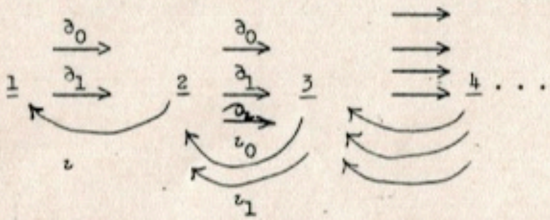
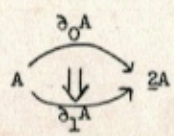
The full subcategory of $|\mathrm{Cat}|$ consisting of the non-empty finite ordinals is generated under composition by the cosimplicial complex



This cosimplicial complex is in fact a cocategory in Cat; that is, at each stage, the outside two ∂'s are the pushout of the previous outside two ∂'s.

In a <u>finitely cocomplete 2-category</u> $\mathcal{K}$, not only do we require the usual finite colimits (pushouts and an initial object 0 suffice) in the underlying category $|\mathcal{K}|$, but also that they be preserved by the category-valued hom-functors, and that, for each object A, there should be a 2-cell



composition with which yields an isomorphism of categories

$$\mathcal{K}(\underline{2}A, X) \cong [\underline{2}, \mathcal{K}(A,X)]$$

for all objects X. For any finitely presented category $\underline{C}$, we can then construct an object $\underline{C}A$ for which there is a 2-natural isomorphism of categories

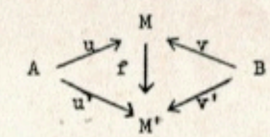
$$\mathcal{K}(\underline{C}A, X) \cong [\underline{C}, \mathcal{K}(A,X)] .$$

The assignments $\underline{C} \longrightarrow \underline{C}A$ and $A \longrightarrow \underline{C}A$ are 2-functorial and finite colimit preserving.

A <u>cospan</u> (u, M, v) <u>from</u> B <u>to</u> A in $\mathcal{K}$ is a diagram

$$A \xrightarrow{u} M \xleftarrow{v} B ;$$

when u, v are understood, (u, M, v) is abbreviated to M. An <u>arrow of cospans</u> $f : (u, M, v) \to (u', M', v')$ is a commutative diagram



A <u>2-cell</u> $\sigma : f \Rightarrow g$ between such arrows of cospans is a 2-cell $\sigma : f \Rightarrow g$ in $\mathcal{K}$ such that $\sigma u = 1_{u'}, \sigma v = 1_{v'}$. Write $\mathrm{Cospn}_{\mathcal{K}}(B,A)$, or $\mathrm{Cospn}\,(B,A)$, for the 2-category of cospans from B to A. There is a composition 2-functor

$$\mathrm{Cospn}\,(C,B) \times \mathrm{Cospn}\,(B,A) \to \mathrm{Cospn}\,(C,A)$$

$$(N,M) \longmapsto M \underset{B}{+} N$$

given on objects by pushout.

$$\begin{array}{ccccccc} A & & & B & & & C \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & M & & \text{p.o.} & & N & \\ & & \searrow & & \swarrow & & \\ & & & M +_B N & & & \end{array}$$

Note that the composition of cospans is associative and has identities $(1_A, A, 1_A)$ up to coherent natural isomorphisms. Thus we obtain a <u>bicategory</u> $\mathrm{Cospn}_{\mathcal{K}}$ (in the sense of Bénabou [ ]) which is enriched so that its homs are 2-categories.

For arrows $f : X \to A$, $g : X \to B$, the cospan $(\partial_0, \langle f,g\rangle, \partial_1)$ from B to A obtained as the composite of the three cospans $(1_A, A, f), (\partial_0 X, \underline{2}X, \partial_1 X), (g, B, 1_B)$ is called <u>the cocomma object of</u> f, g. The canonical arrow $\underline{2}X \to \langle f, g\rangle$ corresponds to a 2-cell

$$\begin{array}{ccc} X & \xrightarrow{g} & B \\ f\downarrow & \Rightarrow & \downarrow \partial_1 \\ A & \xrightarrow[\partial_0]{} & \langle f,g\rangle, \end{array}$$

and $\langle f, g\rangle$ can also be characterized as appearing in the universal such diagram with f, g fixed.

The formula $\mathcal{Z}M = \underline{2}A +_A M +_B \underline{2}B$ defines a 2-comonad $\mathcal{Z}$ on $\mathrm{Cospn}(B,A)$ using the 2-functionality of cospan composition and taking the counit and comultiplication to have components

$$\underline{2}A +_A M +_B \underline{2}B \xrightarrow[\iota A +_A 1 +_B \iota B]{} A +_A M +_B B \cong M, \quad \underline{3}A +_A M +_B \underline{3}B \xrightarrow[\partial_1 A +_A 1 +_B \partial_1 B]{} \underline{2}A +_A M +_B \underline{2}B .$$

Write $\mathrm{Cospl}_{\mathcal{K}}(B,A)$ for the 2-category of Eilenberg-Moore $\mathcal{Z}$-coalgebras; the objects are called split cofibrations from B to A. A $\mathcal{Z}$-coalgebra structure $d : M \to \mathcal{Z}M$ on a cospan M is called a <u>split cocleavage</u> for M.

<u>Proposition 1.</u> If $d : M \to \mathcal{Z}M$ <u>is a split cocleavage for a cospan</u> M from B <u>to</u> A <u>then, in the 2-category</u> $\mathrm{Cospn}(B,A)$:

(<u>a</u>) $d_0 = (M \xrightarrow[d]{} \underline{2}A +_A M +_B \underline{2}B \xrightarrow[1 +_B \iota B]{} \underline{2}A +_A M)$ <u>is a left adjoint for</u> $\iota A +_A 1$ <u>with identity unit;</u>

(<u>b</u>) $d_1 = (M \xrightarrow[d]{} \underline{2}A + M + \underline{2}B \xrightarrow[\iota A +_A 1]{} M +_B \underline{2}B)$ <u>is a right adjoint for</u> $1 +_B \iota B$ <u>with identity counit;</u>

<u>(c) the following diagram commutes</u>

$$\begin{array}{ccc} M & \xrightarrow{d_0} & \underline{2}A +_A M \\ d_1 \downarrow & \searrow^{d} & \downarrow 1 +_A d_1 \\ M +_B \underline{2}B & \xrightarrow[d_0 +_B 1]{} & \underline{2}A +_A M +_B \underline{2}B. \end{array}$$

□

Corollary 2. Split cocleavages on a cospan are unique up to isomorphism of arrows of cospans if they exist. □

For any arrow $f : A \to A'$, the 2-cell $A \xrightarrow{f} A' \overset{\partial_0}{\underset{\partial_1}{\Downarrow}} \underline{2}A'$ induces an arrow $\langle f, A\rangle \to \underline{2}A'$. Dually, an arrow $g : B \to B'$ induces a canonical arrow $\langle B, g\rangle \to \underline{2}B'$. Thus we obtain a canonical arrow

$$A' \underset{A}{+} M \underset{B}{+} B' = \langle f,A\rangle \underset{A}{+} M \underset{B}{+} \langle B,g\rangle \to \underline{2}A' \underset{A'}{+} A' \underset{A}{+} M \underset{B}{+} B' \underset{B'}{+} \underline{2}B' = \mathcal{Z}(A' \underset{A}{+} M \underset{B}{+} B').$$

Proposition 3. If $d : M \to \mathcal{Z}M$ is a split cocleavage for a cospan M from B to A then, for any two arrows $A \to A'$, $B \to B'$, a split cocleavage for $A' \underset{A}{+} M \underset{B}{+} B'$ is obtained by composing $A' \underset{A}{+} d \underset{B}{+} B'$ with the canonical arrow above. □

A cospan from 0 to A is just an object under A; that is, an arrow $u : A \to M$ with source A. A split cofibration from 0 to A (or from B to 0) is called a split left (or right) cofibration under A (or B). By Proposition 1(a), a split left cofibration M under A is such that $\iota A \underset{A}{+} 1 : \underline{2}A \underset{A}{+} M \to M$ has a left adjoint d; so the identity 2-cell $1_M \Rightarrow (\iota A \underset{A}{+} 1)(\partial_0 A \underset{A}{+} 1)$ corresponds to a 2-cell

$$M \underset{\partial_0 A \underset{A}{+} 1}{\overset{d}{\Downarrow}} \underline{2}A \underset{A}{+} M$$

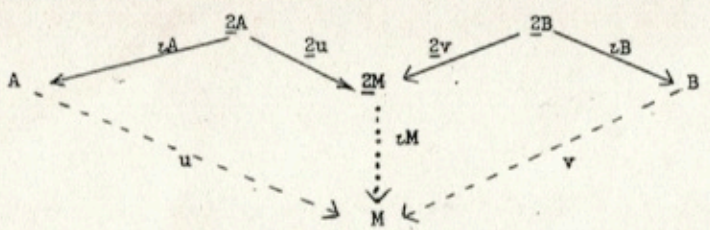
which, in turn, yields an arrow $\hat{d} : \underline{2}M \to \underline{2}A \underset{A}{+} M = \langle A,u\rangle$. (In fact this is a left adjoint with identity counit for the arrow $\langle A,u\rangle \to \underline{2}M$ corresponding to the 2-cell $A \xrightarrow{u} M \Downarrow \underline{2}M$ – dual Chevalley criterion.)

Proposition 4. A composite of split left (right) cofibrations is a split left (right) cofibration. More precisely, if $d : M \to \underline{2}A \underset{A}{+} M$, $d' : N \to \underline{2}M \underset{M}{+} N$ are split left cocleavages for objects $u : A \to M$, $r : M \to N$ under A, M, respectively, then the composite $N \xrightarrow{d'} \underline{2}M \underset{M}{+} N \xrightarrow{\hat{d} \underset{M}{+} 1} \underline{2}A \underset{A}{+} M \underset{M}{+} N \cong \underline{2}A \underset{A}{+} N$ is a split left cocleavage for the object $ru : A \to N$ under A. □

It should be pointed out that, if M is a split cofibration from B to A, then M is a split left (right) cofibration under $A(B)$ with the split left (right) cocleavage given by Proposition 1.

In any 2-category $\mathfrak{M}$ an object M is called codiscrete when every 2-cell $M \Downarrow M'$ is an identity. When $\underline{2}M$ exists this amounts to saying that $\iota M : \underline{2}M \to M$ is an isomorphism. Clearly, the full sub-2-category of $\mathfrak{M}$ consisting of the codiscrete objects has only identity 2-cells and so is essentially only a category.

An object M of Cospn (B,A) is codiscrete when the dotted arrows in the following diagram are the colimit of the diagram represented by the solid arrows.



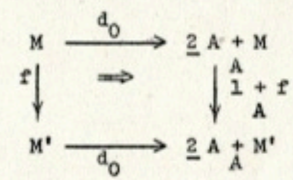
This is because the colimit of the solid arrows is $\underline{2}M$ for M as an object of Cospn (A,B); that is, $\underline{2}$ (u, M, v).

<u>A module from</u> B <u>to</u> A is a codiscrete cospan M from B to A which admits a split cocleavage. Since colimits in Cospl (B,A) are preserved by the underlying 2-functor into Cospn (B,A) we also have that modules are the underlying cospans of codiscrete split cofibrations.

<u>Proposition 5.</u> <u>A split cocleavage on a codiscrete cospan is unique (not just up to isomorphism) if it exists. Any arrows of cospans from a module from B to A to a split cofibration from B to A is an arrow in</u> Cospl (B,A).

<u>Proof.</u> If d , d' are split cocleavages for a codiscrete cospan M from B to A then, by Corollary 2, there is an isomorphism 2-cell $M \underset{d'}{\overset{d}{\Downarrow}} \underline{2}M$ in Cospn (B,A). Since M is codiscrete, this isomorphism is an identity.

Suppose f : M → M' is an arrow of cospans between split cofibrations. By Proposition 1, to see whether f preserves the split cocleavages d, we must see that it preserves both the d_0's and the d_1's. The identity 2-cell $\iota A \underset{A}{+} 1 \,.\, 1 \underset{A}{+} f \Leftarrow f \,.\, \iota A \underset{A}{+} 1$ corresponds under the adjunction of Proposition 1(a) to a 2-cell



in Cospn (B,A), which must be an identity when M is codiscrete. A similar argument applies to the d_1's. □

<u>Corollary 6.</u> <u>For any arrow</u> f : A → B <u>and module</u> (u, M, v) <u>from</u> B <u>to</u> A, <u>there is a bijection between arrows of cospans</u> (u, M, v) → $(\partial_0, \langle A,f\rangle, \partial_1$ <u>and arrows of cospans</u> (u, M, v) → (f, B, 1_B) <u>obtained by composing with the arrow</u> ⟨A,f⟩ → B <u>induced by the identity 2-cell of</u> f.

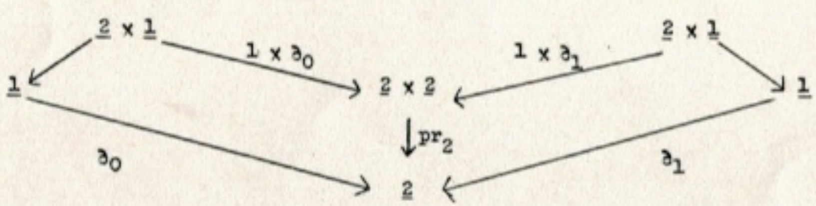
<u>Proof.</u> For the 2-comonad $\underline{2}A \underset{A}{+} -$ on Cospn (B,A), the free coalgebra on

$(f, B, 1_B)$ is $\langle A,f\rangle$. By Proposition 5, <u>any</u> arrow of cospans $M \to \langle A,f\rangle$ is an arrow of split cofibrations and so <u>a fortiori</u> is an arrow of $(\underline{2}A \underset{A}{+} -)$ - coalgebras. □

Let $\mathrm{Mod}_{\mathcal{K}}(B,A)$ denote the full sub-2-category of $\mathrm{Cospn}_{\mathcal{K}}(B,A)$ consisting of the modules from B to A; it is really just a category. By Proposition 5, Mod(B,A) is also the full sub-2-category of Cospl (B,A) consisting of the codiscrete objects.

<u>Proposition 7.</u> <u>The cospan</u> $A \xrightarrow{\partial_0 A} \underline{2}A \xleftarrow{\partial_1 A} A$ <u>is a module from</u> A <u>to</u> A. <u>If</u> M <u>is a codiscrete cospan from</u> B <u>to</u> A <u>and</u> $A \to A'$, $B \to B'$ <u>are arrows then</u> $A' \underset{A}{+} M \underset{B}{+} B'$ <u>is a codiscrete cospan from</u> B' <u>to</u> A'.

<u>Proof.</u> The split cocleavage for $\underline{2}A$ is just $(\partial_1\partial_1)A = (\partial_2\partial_1)A$: $\underline{2}A \to \underline{4}A$. That $\underline{2}A$ is a codiscrete cospan from A <u>to</u> A follows from the following colimit diagram of categories.



The second sentence of the proposition follows from the fact that $\underline{2}(-)$ preserves colimits. □

<u>Corollary 8.</u> <u>For any arrows</u> $f : X \to A$, $g : X \to B$, <u>the cocomma cospan</u> $\langle f,g\rangle$ <u>is a module from</u> B <u>to</u> A. □

For a module M from B to A and a module N from C to B, <u>the tensor product</u> of M and N is defined to be the equalizer

$$M \underset{B}{\otimes} N \longrightarrow M \underset{B}{+} N \underset{M \underset{B}{+} d_0}{\overset{d_1 \underset{B}{+} N}{\rightrightarrows}} M \underset{B}{+} \underline{2}B \underset{B}{+} N$$

in Cospn $(\mathcal{C},A)$ when this equalizer exists. Note that the two arrows being equalized do have a common left inverse $M \underset{B}{+} \varepsilon B \underset{B}{+} N$.

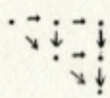
<u>Proposition 9.</u> <u>For any module</u> (u, M, v) <u>from</u> B <u>to</u> A, <u>the following diagrams are absolute equalizers:</u>

$$M \xrightarrow{d_0} \underline{2}A \underset{A}{+} M \underset{\underline{2}A \underset{A}{+} d_0}{\overset{d_1 \underset{A}{+} M}{\rightrightarrows}} \underline{2}A \underset{A}{+} \underline{2}A \underset{A}{+} M$$

$$M \xrightarrow{d_1} M \underset{B}{+} \underline{2}B \underset{M \underset{B}{+} d_0}{\overset{d_1 \underset{B}{+} \underline{2}B}{\rightrightarrows}} M \underset{B}{+} \underline{2}B \underset{B}{+} \underline{2}B$$

$$M \xrightarrow{d} \langle A,u\rangle \underset{M}{+} \langle v,B\rangle \underset{\langle A,u\rangle \underset{B}{+} d_0}{\overset{d_1 \underset{M}{+} \langle v,B\rangle}{\rightrightarrows}} \langle A,u\rangle \underset{M}{+} \underline{2}M \underset{M}{+} \langle v,B\rangle$$

<u>Proof</u>. The pairs (M,d_0), (M,d_1) are coalgebras for the 2-comonads $\underline{2}A \underset{A}{+} -$, $- \underset{B}{+} \underline{2}B$ on Cospn (B,A), so, by the general theory of comonads, the first two diagrams are split equalizers. The third diagram however is not as easily dismissed since, although $\mathcal{E}M \cong \langle A,u\rangle \underset{M}{+} \langle v,B\rangle$, we do not have $\mathcal{E}\mathcal{E}M$ isomorphic to $\langle A,u\rangle \underset{M}{+} \underline{2}M \underset{M}{+} \langle v,B\rangle$ (for example, in Cat with M taken to be $\underline{1} \xrightarrow{\partial_0} \underline{2} \xleftarrow{\partial_1} \underline{1}$, the former is $\underline{6}$ and the latter is the ordered set below).



By Paré [ ], to prove that the third diagram is an absolute equalizer it suffices to show that, for each X, the 2-functor $\mathcal{K}(-,X)$ takes it to a coequalizer of categories. The module $A \xrightarrow{u} M \xleftarrow{v} B$ is taken by such a 2-functor to a discrete fibration $\mathcal{K}(A,X) \longleftarrow \mathcal{K}(M,X) \longrightarrow \mathcal{K}(B,X)$. Re-naming, we suppose we have a discrete fibration $A \xleftarrow{p} E \xrightarrow{q} B$ of categories. Replacing this span by an isomorph, we may suppose that the span is obtained from a set-valued functor e on $A^{op} \times B$ as follows.

The objects of E are triples (a, x, b) where a, b are objects of A, B and $x \in e(a, b)$. The arrows $(\alpha, \beta) : (a, x, b) \to (c, y, d)$ consist of arrows $\alpha : a \to c$, $\beta : b \to d$ in A, B satisfying $e(1, \beta)x = e(\alpha, 1)x'$. The functors p, q are the obvious projections. The problem is now to prove that the diagram $Q \underset{v}{\overset{u}{\rightrightarrows}} P \xrightarrow{w} E$ is a coequalizer, where P is the pullback and Q is the comma category of the cospan $(A, p) \xrightarrow[pr_2]{} E \xleftarrow[pr_1]{}$ (c

and where $w(a' \xrightarrow{\theta} a, a, b \xrightarrow{\varphi} b') = (a', e(\theta, \varphi)x, b')$ and u, v respectively take $(a' \xrightarrow{\theta} a \xrightarrow{\gamma} \bar{a}, x, \bar{x}, b \xrightarrow{\delta} \bar{b} \xrightarrow{\bar{\varphi}} \bar{b}')$ to $(a' \xrightarrow{\gamma\theta} \bar{a}, \bar{x}, \bar{b} \xrightarrow{\bar{\varphi}} \bar{b}')$, $(a' \xrightarrow{\theta} a, x, b \xrightarrow{\bar{\varphi}\delta} \bar{b}')$. Define $s : E \to P$ by $s(a, x, b) = (a \xrightarrow{1} a, x, b \xrightarrow{1} b$ and $t_1, t_2, t_3 : P \to Q$ at $(a' \xrightarrow{\theta} a, x, b \xrightarrow{\varphi} b')$ respectively to be $(a' \xrightarrow{\theta} a \xrightarrow{1} a, x, e(1,\varphi)x, b \xrightarrow{\varphi} b' \xrightarrow{1} b')$, $(a' \xrightarrow{1} a' \xrightarrow{\theta} a, e(\theta, \varphi)x, e(1, \varphi)x$, $b' \xrightarrow{1} b' \xrightarrow{1} b')$, $(a' \xrightarrow{1} a' \xrightarrow{1} a', e(\theta, \varphi)x, e(\theta, \varphi)x, b' \xrightarrow{1} b' \xrightarrow{1} b')$. The equations $ws = 1$, $vt_1 = 1$, $ut_1 = ut_2$, $vt_2 = vt_3$, $ut_3 = sw$ are satisfied and imply that w is an (absolute) coequalizer of u, v. □

<u>Corollary 10</u>. <u>For any module</u> (u, M, v) <u>from</u> B <u>to</u> A <u>and arrows</u> $f : A \to A'$, $g : B \to B'$, <u>there are canonical isomorphisms</u>

$$\langle f,A\rangle \underset{A}{\otimes} M \cong f_* M \quad (= A' \underset{A}{+} M),$$

$$M \underset{B}{\otimes} \langle B,g\rangle \cong M g_* \quad (= M \underset{B}{+} B'),$$

$$\langle f,u\rangle \underset{M}{\otimes} \langle v,g\rangle \cong f_* M g_* \quad (= A' \underset{A}{+} M \underset{B}{+} B').$$

<u>Proof</u>. Pushing out along f,g preserves the absolute equalizers of Proposition 9. □

A 2-category $\mathcal{K}$ is said <u>to admit the calculus of modules</u> when it satisfies the following three axioms:

- $\mathcal{K}$ is a finitely cocomplete 2-category;
- for objects M,N of Mod (B,A), Mod (C,B), the equalizer defining $M \underset{B}{\otimes} N$ exists and is an object of Mod(C,A);
- for objects P, M, N, Q of Cospl(A,0), Mod(B,A), Mod(C,B), Cospl(0,C), the functors $P \underset{A}{+} -$, $- \underset{C}{+} Q$ preserve the defining equalizer for $M \underset{B}{\otimes} N$.

<u>Proposition 11</u>. <u>For a 2-category</u> $\mathcal{K}$ <u>which admits the calculus of modules, there is a bicategory</u> $\mathrm{Mod}_{\mathcal{K}}$ <u>(in the sense of Bénabou [ ]) whose objects are the objects of</u> $\mathcal{K}$, <u>whose hom-categories are the categories</u> Mod(B,A), <u>and whose composition is tensor product of modules.</u>

<u>Proof</u>. Proposition 9 shows that the modules $\underline{2A}$ act as identities under tensor product (up to isomorphism). For associativity, consider the diagram which follows.

$$\begin{array}{ccccc}
L \underset{A}{\otimes} (M \underset{B}{\otimes} N) & \longrightarrow & L \underset{A}{+} (M \underset{B}{\otimes} N) & \rightrightarrows & L \underset{A}{+} \underline{2A} \underset{A}{+} (M \underset{B}{\otimes} N) \\
\vdots\downarrow & & \downarrow & & \downarrow \\
(L \underset{A}{\otimes} M) \underset{B}{+} N & \longrightarrow & L \underset{A}{+} M \underset{B}{+} N & \rightrightarrows & L \underset{A}{+} \underline{2A} \underset{A}{+} M \underset{B}{+} N \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
(L \underset{A}{\otimes} M) \underset{B}{+} \underline{2B} \underset{B}{+} N & \longrightarrow & L \underset{A}{+} M \underset{B}{+} \underline{2B} \underset{B}{+} N & \rightrightarrows & L \underset{A}{+} \underline{2A} \underset{A}{+} M \underset{B}{+} 2B \underset{B}{+} N
\end{array}$$

The third axiom in the definition of "to admit the calculus of modules" implies the fact that the second and third rows and columns are equalizers. The first row is an equalizer by definition. So the induced dotted arrow equalizes the pair in the first column (3 x 3 lemma). Thus $L \underset{A}{\otimes} (M \underset{B}{\otimes} N) \cong (L \underset{A}{\otimes} M) \underset{B}{\otimes} N$ and the coherence conditions are easily checked. □

For any arrow $f : A \to B$, the arrow $\langle f,f \rangle \to \underline{2B}$, corresponding to the identity 2-cell of f, composes with the isomorphism $\langle f,A \rangle \underset{A}{\otimes} \langle A,f \rangle \cong \langle f,f \rangle$ (Corollary 10) to yield a canonical arrow

$$m_f : \langle f,A \rangle \underset{A}{\otimes} \langle A,f \rangle \to \underline{2B}.$$

<u>Proposition 12</u>. <u>For any arrow</u> $f : A \to B$ <u>in a 2-category</u> $\mathcal{K}$ <u>admitting the calculus of modules; the arrow</u> m_f <u>in</u> Mod(B,B) <u>is a counit for an adjunction</u> $\langle f,A \rangle \dashv \langle A,f \rangle$ <u>in the bicategory</u> $\mathrm{Mod}_{\mathcal{K}}$.

Proof. Applying $\mathcal{F}$ to the cospan $A \xrightarrow{f} B \xleftarrow{f} A$, we obtain $\langle A,f\rangle \underset{B}{+} \langle f,A\rangle$ as the "cofree" $\mathcal{F}$-coalgebra on the cospan. So the arrow of cospans $f(\iota A) : \underline{2}A \longrightarrow (f, B, f)$ induces a $\mathcal{F}$-coalgebra arrow $\underline{2}A \longrightarrow \langle A,f\rangle \underset{B}{+} \langle f,A\rangle$ which can be seen to equalize the arrows $\langle A,f\rangle \underset{B}{+} \langle f,A\rangle \rightrightarrows \langle A,f\rangle + \underline{2}B + \langle f,A\rangle$ and hence factor through $\langle A,f\rangle \underset{B}{\otimes} \langle f,A\rangle$ via an arrow $n_f : \underline{2}A \longrightarrow \langle A,f\rangle \underset{B}{\otimes} \langle f,A\rangle$. It is left to the reader to show that n_f is the unit for an adjunction as stated. □

A module M from B to A is said to be cauchy when it has a right adjoint M^*. It is said to be convergent when there exists an arrow $g : B \to A$ such that $M \cong \langle g,B\rangle$. Proposition 12 implies every convergent module is cauchy. An object A is said to be cauchy complete (Lawvere [ ]) when, for all objects B, all cauchy modules from B to A are convergent. By the general theory of adjunctions, for a cauchy module M from B to A, there are natural bijections as indicated by the horizontal lines below.

$$\frac{M \underset{B}{\otimes} P \longrightarrow Q}{P \longrightarrow M^* \underset{A}{\otimes} Q} \qquad\qquad \frac{R \underset{B}{\otimes} M^* \longrightarrow S}{R \longrightarrow S \underset{A}{\otimes} M}$$

Proposition 13. For any two objects A, B in a 2-category $\mathcal{K}$ admitting the calculus of modules, the inclusion functor

$$\mathrm{Mod}(B,A) \longrightarrow \mathrm{Cospn}\,(B,A)$$

has a right adjoint whose value at a cospan (f, X, g) is $\langle A,f\rangle \underset{X}{\otimes} \langle g,B\rangle$.

Proof. For a module M from B to A, we have bijections:

$$\begin{array}{ll} M \longrightarrow \langle A,f\rangle \underset{X}{\otimes} \langle g,B\rangle & \\ \hline & \text{(Proposition 12)} \\ M \underset{B}{\otimes} \langle B,g\rangle \longrightarrow \langle A,f\rangle & \\ \hline & \text{(Corollary 10)} \\ Mg_* \longrightarrow \langle A,f\rangle & \\ \hline & \text{(Corollary 6)} \\ Mg_* \longrightarrow (f, X, 1_X) & \\ \hline & \text{(pushout property)} \\ M \longrightarrow (f, X, g) & \end{array}$$

. □

For given arrows $g, h : B \to A$, 2-cells $\sigma : g \Rightarrow h$ are in bijection with arrows of spans $\langle g,B\rangle \to \langle h,B\rangle$ simply by the universal property of cocomma objects. These arrows of spans are moreover arrows of modules. This leads us to define a pseudo-functor

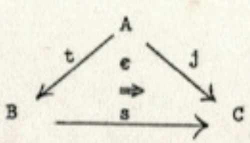
$$\Gamma : \mathcal{K} \to \mathrm{Mod}_{\mathcal{K}}$$

which is the identity on objects, which is given on arrows by $\Gamma g = \langle g,B\rangle$, and which is given on 2-cells by the bijection above. The coherent isomorphisms up to which Γ preserves identities and composition are provided by Corollary 10. As each Γg has a right adjoint $\Gamma^* g = \langle B,g\rangle$, we have an associated pseudo-functor

$$\Gamma^* : \mathcal{K}^{coop} \to \mathrm{Mod}_{\mathcal{K}}$$

which is also the identity on objects and bijective on 2-cells.

A 2-cell

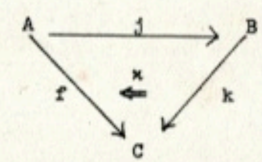


is said to be a relative counit for t as a right adjoint for s relative to j in the 2-category $\mathcal{K}$ when $\Gamma \epsilon$ exhibits Γt as a right lifting of Γj through Γs in the bicategory $\mathrm{Mod}_{\mathcal{K}}$. Since Γs has a right adjoint $\Gamma^* s$, this precisely amounts to saying that the arrow of modules

$\langle t,A\rangle \to \langle B,s\rangle \otimes_C \langle j,A\rangle$ induced by ϵ is an isomorphism. Since Γ is bijective on 2-cells and since liftings obtained by composition with an adjoint are respected by arbitrary arrows, we deduce the following result.

Proposition 14. Relative adjunctions are absolute liftings. That is, if the 2-cell ϵ (as above) is a relative counit for t as a right adjoint for s relative to j then, for all arrows $a : H \to A$, the 2-cell ϵa exhibits $t a$ as a right lifting of $j a$ through s in the 2-category $\mathcal{K}$. □

A 2-cell



is said to exhibit k as a pointwise-right extension of f along j when the 2-cell $\Gamma \kappa$ exhibits Γk as a right extension of Γf along Γj in the bicategory $\mathrm{Mod}_{\mathcal{K}}$. Since Γ is bijective on 2-cells, it is clear that any pointwise right extension is indeed a right extension.