Descent Theory

Ross Street

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Theme

Higher descent theory, non-abelian cohomology, and higher-order category theory are all one subject which might be called ***post-modern algebra*** (or even "post-modern mathematics" since geometry and algebra are handled equally well by higher categories).

Section Headings

§1. Outline of the program

What is *cohomology?* It involves a space and a coefficients object. My view [St2] is that a reasonable concept of *space* is a functor $R : \Delta^{op} \longrightarrow \mathcal{E}$; that is, a simplicial object in $\mathcal{E}$. For example, homotopy theorists are generally happy with simplicial sets as their spaces. We admit that sometimes a diagram of such functors may be needed, such as when $\mathcal{E}$ is a topos and R is a *hypercover*; then we need to take a colimit of the cohomology objects we are about to describe.

It is also my view that a reasonable concept of *coefficients object* is a *weak ω–category* A in the category $\mathcal{E}$. (Eventually, in our pursuit of stacks [Gk], we might consider a contravariant homomorphism A from $\mathcal{E}$ into the weak (n+1)-category of weak n-categories.) For the time being, we shall restrict to the case where A is an *ω-category* since these are very easy to define precisely. (In fact, the concepts that arise in dealing with this case provide some of the tools for the general case.) Cartesian closed categories n-Cat, $n \geq 0$, are defined recursively by:

$$0\text{-Cat} = \text{Set}, \qquad (n+1)\text{-Cat} = (n\text{–Cat})\text{-Cat}.$$

Objects of n-Cat are called *n-categories.* Let ω-Cat denote the union of the categories n-Cat; it is also cartesian closed. So, for our purposes here, an *ω-category* is just an n-category for some n ≥ 0. The *n-cells* in an ω-category can be defined recursively: the 0-cells of a set are its elements; the (n + 1)-cells of A are the n-cells of some hom n-category $A(a, b)$ for a, b objects of A. It is an important fact that n-categories are models for a finite limit theory, in fact, a 1-sorted finite limit theory where the one sort is "n-cell". In particular, this means that we can model n-categories in any finitely complete category $\mathcal{E}$.

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[1] This Section will be distributed as a separate paper *Fusion operators and cocycloids in monoidal categories.*

Take a space $R : \Delta^{op} \longrightarrow \mathcal{E}$ and a coefficients object $A \in \omega\text{-Cat}(\mathcal{E})$. Form the functor $\mathcal{E}(R, A) : \Delta \longrightarrow \omega\text{–Cat}$. We wish to construct *the cohomology ω-category* $\mathcal{H}(R, A)$ *of* R *with coefficients in* A. (Some people would have me call it the "cocycle ω-category" rather than cohomology, but the spirit of category theory has it that our interest in cells of any ω-category is only up to the appropriate equivalence, and this very equivalence is the appropriate notion of *cobounding.*)

Jack Duskin pointed out to me (probably in 1981) that the construction should be done for any cosimplicial ω-category $\mathcal{X} : \Delta \longrightarrow \omega\text{–Cat}$ and the result would be a *lax descent ω–category* $\operatorname{Desc}\mathcal{X}$. Then we would put

$$\mathcal{H}(R, A) = \operatorname{Desc} \mathcal{E}(R, A).$$

Duskin provided an informal description of $\operatorname{Desc}\mathcal{X}$ by drawing the low dimensional cells.

Let us look fairly explicitly at the *descent 2-category* $\operatorname{Desc}\mathcal{X}$ of a truncated cosimplicial 2-category $\mathcal{X}$*:*

$$\begin{tikzcd}
\mathcal{X}_0 \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & \mathcal{X}_1 \arrow[r, shift left=2, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=2, "\partial_2"'] & \mathcal{X}_2 \arrow[r, shift left=3, "\partial_0"] \arrow[r, shift left=1, "\partial_1"] \arrow[r, shift right=1, "\partial_2"] \arrow[r, shift right=3, "\partial_3"'] & \mathcal{X}_3
\end{tikzcd}$$

has objects (X, f, σ) where X is an object of $\mathcal{X}_0$, where

$$f : \partial_1 X \longrightarrow \partial_0 X$$

is an arrow of $\mathcal{X}_1$, and where

$$\begin{tikzcd}
{} \arrow[rr, "\partial_1 f"] \arrow[dr, "\partial_2 f"'] & {} \arrow[d, phantom, "\Downarrow \sigma" near start] & {} \\
& {} \arrow[ur, "\partial_0 f"'] &
\end{tikzcd}$$

is a 2-cell of $\mathcal{X}_2$, such that the following equation holds in $\mathcal{X}_3$ (commutative tetrahedron):

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] \arrow[dr, "\Downarrow\partial_1\sigma", "\Downarrow\partial_3\sigma"'] & {} & & {} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & {} \arrow[u] & = & {} \arrow[r] \arrow[ur, "\Downarrow\partial_2\sigma", "\Downarrow\partial_0\sigma"'] & {} \arrow[u]
\end{tikzcd}$$

(ignoring normalisation conditions; these involve the codegeneracy maps)

has arrows $(u, \theta) : (X, f, \sigma) \longrightarrow (Y, g, \tau)$ where $u : X \longrightarrow Y$ is an arrow of $\mathcal{X}_0$, and

$$\begin{tikzcd}
\partial_1 X \arrow[r, "f"] \arrow[d, "\partial_1 u"'] \arrow[dr, phantom, "\Downarrow \theta"] & \partial_0 X \arrow[d, "\partial_0 u"] \\
\partial_1 Y \arrow[r, "g"'] & \partial_0 Y
\end{tikzcd}$$

is a 2-cell of $\mathcal{X}_1$ such that the following equality holds in $\mathcal{X}_2$ (commutative triangular cylinder):

$$\begin{tikzcd}
{} \arrow[rr, "\partial_1 f"] \arrow[dr, "\partial_2 f"'] \arrow[dd] & & {} \arrow[dd] & & {} \arrow[rr, "\partial_1 f"] \arrow[dd] & & {} \arrow[dd] \\
& {} \arrow[ur, "\partial_0 f"'] \arrow[dd] \arrow[u, phantom, "\Downarrow \sigma"] & & = & & \Downarrow \partial_1 \theta & \\
{} \arrow[dr, "\partial_2 g"'] & \arrow[l, phantom, "\Downarrow \partial_2 \theta"] \arrow[r, phantom, "\Downarrow \partial_0 \theta"] & {} & & {} \arrow[rr, "\partial_1 g"] \arrow[dr, "\partial_2 g"'] & \Downarrow \tau & {} \\
& {} \arrow[ur, "\partial_0 g"'] & & & & {} \arrow[ur, "\partial_0 g"'] &
\end{tikzcd}$$

and has 2-cells $\alpha : (u, \theta) \Rightarrow (v, \phi) : (X, f, \sigma) \longrightarrow (Y, g, \tau)$ just 2-cells $\alpha : u \Rightarrow v : X \longrightarrow Y$ in $\mathcal{X}_0$ such that the following equality holds in $\mathcal{X}_1$ (commutative circular cylinder):

$$\begin{tikzcd}
\partial_1 X \arrow[r, "f"] \arrow[d, "\partial_1 u"] \arrow[d, bend right=70, "\partial_1 v"'] \arrow[d, phantom, shift right=5ex, "\overset{\partial_1 \alpha}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow \theta"] & \partial_0 X \arrow[d, "\partial_0 u"] & = & \partial_1 X \arrow[r, "f"] \arrow[d, "\partial_1 v"'] \arrow[dr, phantom, "\Downarrow \phi"] & \partial_0 X \arrow[d, "\partial_0 v"'] \arrow[d, bend left=70, "\partial_0 u"] \arrow[d, phantom, shift left=5ex, "\overset{\partial_0 \alpha}{\Leftarrow}"] \\
\partial_1 Y \arrow[r, "g"'] & \partial_0 Y & & \partial_1 Y \arrow[r, "g"'] & \partial_0 Y
\end{tikzcd}$$

Staring at these diagrams we see that the objects of Desc $\mathcal{X}$ are closely related to the *nerve* of an ω-category since the diagrams are all simplexes in ω-categories. In making precise this notion of nerve, which was suggested to me by John Roberts, I had introduced [St1] an n-category $\mathcal{O}_n$ for each $n \geq 0$, called the *n-th oriental.* I had defined what is meant by a *free n-category* and shown the sense in which $\mathcal{O}_n$ is the free n-category on the n-simplex. An n-functor $\mathcal{O}_n \longrightarrow A$ is a precise realisation of the concept of an n-simplex drawn in A. To see the relation to the descent construction, we note that the orientals themselves form a cosimplicial ω-category $\mathcal{O}_* : \Delta \longrightarrow \omega\text{–Cat}$ and the objects of Desc $\mathcal{X}$ are precisely morphisms of cosimplicial ω-categories $\mathcal{O}_* \longrightarrow \mathcal{X}$.

For some reason it took me longer to realise that the pasting diagrams occurring in

Desc $\mathcal{X}$ were all ***products*** of “globs” with simplexes. This led me to “parity complexes” [St2] which were designed to allow me to redo what I had done for simplexes for a more general class of geometric structures closed under geometric product.

§2. Parity complexes

Free categories on circuit-free directed graphs have particularly simple descriptions. We generalise this to higher dimensions following [St2].

A parity complex C of dimension n cons[redacted] $= \Sigma_{0\le k\le n} C_k$ and functions $(–)^-$, $(–)^+ : C_k \longrightarrow \mathcal{P}(C_{k-1})$ for $0<k\le n$. There are some axi[redacted]ns such as

$$x^{--} \cup x^{++} = x^{-+} \cup x^{+-}.$$

The model for the free n-categ[redacted]scribed in a purel[redacted] combinatorial way. An n-cell of $\mathcal{O}C$ is a pair $(M,$[redacted] of non-empty fini[redacted]subsets M, P of C such that the following conditions hold (where S [redacted]eans the complemen[redacted]of S in C):

(i) each of M and P contains at most on[redacted]element of C_0 and[redacted]for all $x \ne y$ in C_k with $k>0$, if both $x, y \in M$ or if both $x, y \in P$, the[redacted]he set $(x^- \cap y^-) \cup$ [redacted]$^+ \cap y^+)$ is empty;

(ii) $P = (M\cup M^+)\cap\ M^-$, $M = (P\cup M^-)\cap\ M^+$, $P = (M\cup P^+)\cap\neg P^-$, $M = (P\cup P^-)\cap\ P^+$.

The *k-source* and *k-target* of (M,P) are defined as foll[redacted] $S^{(k)} = \Sigma_{h\le k} S_h$ for any subset S of C):

$$s_k(M,P) = (M^{(k)}, M_k \cup P^{(k-1)}), \quad t_k(M,P) = (M^{(k-1)} \cup P_k, P^{(k}\text{[redacted]}.$$

An ordered pair of cells $(M,P), (N,Q)$ is called *k-composable* when

$$t_k(M,P) = s_k(N,Q),$$

in which case their *k-composite* is defined by

$$(M,P) \circ_k (N,Q) = (M \cup (N\cap\ N_k), (P\cap\ P_k) \cup Q).$$

The *k-cells* of $\mathcal{O}C$ are the n-cells (M,P) with $s_k(M,P)$ [redacted]$M,P)$. The proof that [redacted]C is an n-category is non-trivial. There is a dimension preserving in[redacted]ctive function

$$x \longmapsto\ <x> : C \longrightarrow \text{[redacted]}C$$

given inductively as follows: for $x\in C_k$, put $<x> = (M,$[redacted]$)$ where

$$M_k = P_k = \{x\},$$

$$M_{r-1} = (M_r)^-\cap\ (M_r)^+, \quad P_{r-1} = (P_r)^-\cap\ (P_r)^+ \quad \text{for } 0<r\le k.$$

The notation I use for this particular M and P is $\mu(x)$ and $\pi(x)$ so that $<x> = (\mu(x), \pi(x))$. It is also non-trivial to prove that $\mathcal{O}C$ is the *free* n-category generated by the cells $<x>$, $x\in C$.

The *product* $C\times D$ of two parity complexes C, D is given by

$$(C\times D)_n = \sum_{p+q=n} C_p \times D_q, \quad (x,a)^\varepsilon = x^\varepsilon \times \{a\} \cup \{x\}\times a^{\varepsilon(p)}$$

for $x\in C_p$, $a\in D_q$, $\varepsilon\in\{–, +\}$, where $\varepsilon(p)\in\{–, +\}$ is ε for p even and is not ε for p odd.

Parity complexes can be regarded as combinatorial chain complexes. Each parity complex C gives rise to a chain complex FC by taking the free abelian groups on each C_n and using the differential $d(x) = x^+ - x^-$, where we have identified x^+ with the formal sum of its elements. It is easy to see that we have a canonical isomorphism of chain complexes:

$$F(C\times D) \cong FC \otimes FD\,,$$

where we remind readers that the tensor-product boundary formula is

$$d(x\otimes a) = dx\otimes a + (-1)^p\, x\otimes da \quad \text{for } x\in FC_p,\ a\in FD_q\,.$$

There are explicit formulas for $\mu(x, a), \pi(x, a)$ in terms of $\mu(x), \mu(a), \pi(x), \pi(a)$. To express these, write χ^r to denote $\chi\in\{\mu,\pi\}$ when r is even and to denote the other element of $\{\mu,\pi\}$ when r is odd. Then

$$\chi(x,a)_n = \bigcup_{r+s=n} \chi(x)_r \times \chi^r(a)_s\,.$$

The *join* $C\bullet D$ of two parity complexes C, D is given by

$$(C\bullet D)_n = C_n + \sum_{p+q+1=n} C_p\times D_q + D_n$$

in which the summands C and D are embedded as sub-parity complexes and the elements $(x, a)\in C_p\times D_q$ are written as xa with

$$(xa)^- = x^-a \cup xa^- \quad\text{and}\quad (xa)^+ = x^+a \cup xa^+ \quad \text{for } p \text{ odd,}$$

$$(xa)^- = x^-a \cup xa^+ \quad\text{and}\quad (xa)^+ = x^+a \cup xa^- \quad \text{for } p \text{ even,}$$

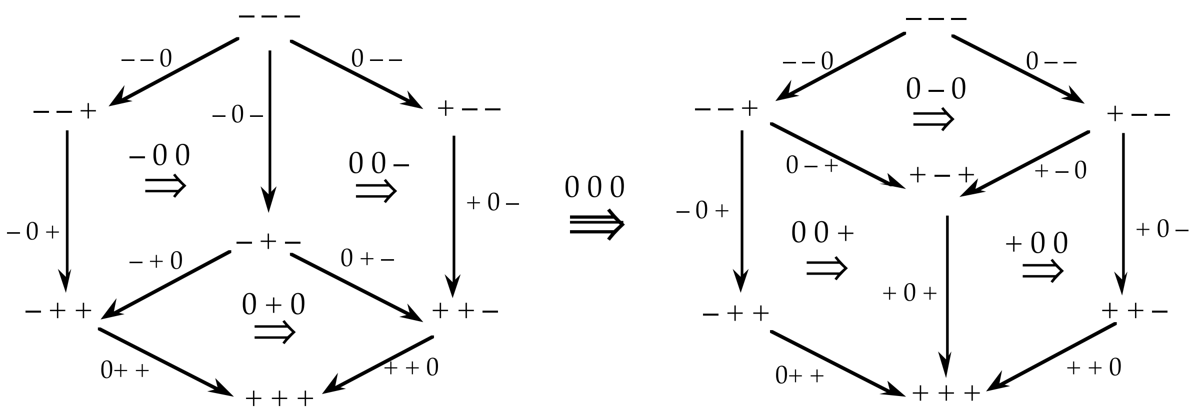
where, for example, $x^+a = \{\, ya : y\in x^+ \,\}$ is taken to mean $\{a\}$ when $p = 0$. In particular, when D consists of a single element ∞ in dimension 0, the join $C\bullet D$ is called the *right cone of* C and denoted by $C^>$. Also $D\bullet C$ is the *left cone of* C and denoted by $C^<$.

Let $\mathbb{1}$ denote the *parity point;* it is the parity complex C with $C_0 = \{0\}$ and $C_n = \varnothing$ for $n > 0$. The *parity interval* is the parity complex which is the join $\mathbb{I} = \mathbb{1}\bullet\mathbb{1}$.

The *parity n-simplex* is the (n+1)-fold join $\mathbb{1}^{\bullet(n+1)} = \mathbb{1}\bullet\mathbb{1}\bullet \ldots \bullet\mathbb{1}$ of parity points. For $n = 3$:

$$\begin{tikzcd}
0 \arrow[rr, "03"] \arrow[dd, "01"'] \arrow[ddrr, "02"] & & 3 & & 0 \arrow[rr, "03"] \arrow[dd, "01"'] & & 3 \\
& \Downarrow 012 \quad \Downarrow 023 & & \overset{0123}{\Rrightarrow} & & \Downarrow 013 \quad \Downarrow 123 & \\
1 \arrow[rr, "12"'] & & 2 \arrow[uu, "23"'] & & 1 \arrow[rr, "12"'] \arrow[uurr, "13"] & & 2 \arrow[uu, "23"']
\end{tikzcd}$$

The *parity n-cube* is the n-fold product $\mathbb{I}^{\times n} = \mathbb{I}\times\mathbb{I}\times \ldots \times\mathbb{I}$ of parity intervals. For $n = 3$:

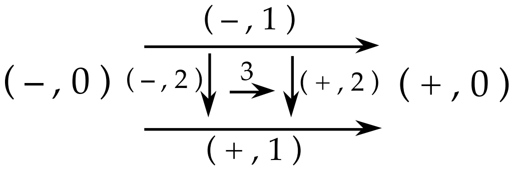


The *parity n-glob* is the parity complex $n\mathbb{G}$ defined by

$$n\mathbb{G}_{\mathbf{m}} = \{ (\varepsilon, m) : \varepsilon = - \text{ or } + \} \text{ for } m < n, \quad n\mathbb{G}_{\mathbf{n}} = \{ n \},$$

$$(\varepsilon, m)^- = \{ (-, m-1) \}, \ (\varepsilon, m)^+ = \{ (+, m-1) \}, \ n^- = (-, n-1), \ n^+ = (+, n-1) .$$

For $n = 3$:



A precise definition of the *n-th oriental,* that is, the free n-category on the n-simplex, is

$$\mathcal{O}_n = \mathcal{O}\,\mathbb{1}^{\bullet(n+1)} .$$

A precise definition of the nerve $N(A)$ of an ω-category A is then

$$N(A)_n = \omega\text{-Cat}\,(\mathcal{O}_n , A) .$$

This process is quite Kanonical: from the functor $\mathcal{O}_* : \mathbb{\Delta} \longrightarrow \omega\text{-Cat}$, we obtain the *nerve functor* $N : \omega\text{-Cat} \longrightarrow [\mathbb{\Delta}^{op}, \mathrm{Set}]$ and its left adjoint Φ. While the restriction of N to 1-categories is fully faithful, it is not true that N itself is full: simplicial maps $N(A) \longrightarrow N(B)$ amount to normal lax functors $A \longrightarrow B$.

§3. The Gray tensor product of ω-categories and the descent ω-category

We begin by reminding the reader of the technique of Brian Day [D1], [D2] for extending a monoidal structure on a small category $\mathcal{C}$ to a biclosed monoidal structure on a cocomplete category $\mathcal{X}$ using left Kan extension along a dense fully faithful functor $J : \mathcal{C} \longrightarrow \mathcal{X}$: the formula is

$$X \otimes Y = \int^{C,D} \big(\mathcal{X}(JC, X) \times \mathcal{X}(JD, Y)\big) \bullet J(C \otimes D)$$

where $S \bullet X$ means the coproduct in the category $\mathcal{X}$ of S copies of X, for S a set and $X \in \mathcal{X}$. The technique was already used by the author [St3] to construct the Gray tensor product of 2-categories.

The free ω-categories $\mathcal{O}\mathbb{I}^{\times n}$ on the parity cubes ($n \geq 0$) form a dense full subcategory $\mathcal{Q}$ of the category ω-Cat. The subcategory $\mathcal{Q}$ is monoidal via the obvious tensor product

$$(\mathcal{O}\mathbb{I}^{\times m}) \otimes (\mathcal{O}\mathbb{I}^{\times n}) \ = \ (\mathcal{O}\mathbb{I}^{\times (m+n)}) .$$

Hence, by Day, we induce a biclosed monoidal structure on ω-Cat. It is *not* the cartesian monoidal structure. We shall call it the *Gray monoidal structure* on ω-Cat, although it is not really what John Gray defined; his tensor product was on 2-Cat. The present structure was considered by Richard Steiner [Sn] and explored by Sjoerd Crans [C]. Dominic Verity [V] has another elegant approach using cubical sets. To obtain Gray's original tensor product [Gy1] we need to render all 3-cells identities, although his approach to coherence [Gy2] used the braid groups. To see the connection, consider the braid category $\mathbb{B}$ [JS2] which is the disjoint union of all the usual braid groups as 1-object categories. There is a 2-category $\Sigma\mathbb{B}$ with one object, with hom-category $\mathbb{B}$, and with addition of braids as composition. There is an ω-functor $P : \mathcal{O}\mathbb{I}^{\times\infty} \longrightarrow \Sigma\mathbb{B}$ which is universal with the property that it equates all objects, inverts all 2-cells, and takes all 3-cells to identities.

Dominic Verity has shown that, for a wide class of parity complexes C, D, we have

$$(\mathcal{O}C) \otimes (\mathcal{O}D) \ = \ \mathcal{O}(C \times D).$$

There is a connection between the Gray tensor product and ordinary chain complexes. Each chain complex R gives rise to an ω-category $\vartheta(R)$ whose 0-cells are 0-cycles $a \in R_0$, whose 1-cells $b : a \longrightarrow a'$ are elements $b \in R_1$ with $d(b) = a' - a$, whose 2-cells $c : b \longrightarrow b'$ are elements $c \in R_2$ with $d(c) = b' - b$, and so on. All compositions are addition. This gives a functor $\vartheta : DG \longrightarrow$ ω-Cat from the category DG of chain complexes and chain maps. In fact, $\vartheta : DG \longrightarrow$ ω-Cat is a *monoidal functor* where DG has the usual tensor product of chain complexes and ω-Cat has the Gray tensor product. By applying ϑ on homs, we obtain a (2-) functor $\vartheta_* :$ DG-Cat $\longrightarrow \mathcal{V}_2$-Cat, where $\mathcal{V}_2$ is ω-Cat with the Gray tensor product. In particular, since DG is closed, it is a DG-category and we can apply ϑ_* to it. The $\mathcal{V}_2$-category $\vartheta_*(DG)$ has chain complexes as 0-cells and chain maps as 1-cells; the 2-cells are chain homotopies and the higher cells are higher analogues of chain homotopies. In the next section we shall see the importance of $\mathcal{V}_2$-categories in the homotopy theory of topological spaces, not just the homotopy theory of chain complexes (which is homological algebra).

We return now to providing the definition of the descent ω-category. Notice that the functor $\mathrm{Cell}_n :$ ω-Cat $\longrightarrow$ Set, which assigns the set of n-cells to each ω-category, is represented by the free n-category $\mathcal{O}(n\mathbb{G})$ on the n-glob. Since the set of n-cells in an ω-category forms an n-category, it follows that $\mathcal{O}(n\mathbb{G})$ *is a co-n-category in the category* ω-Cat. As we pointed out earlier, n-categories are models of a finite-limit theory. So co-n-categories are taken to co-n-categories by right-exact functors. It follows that $\mathcal{O}(n\mathbb{G}) \otimes A$ is a co-n-category in ω-Cat for all ω-categories A.

In particular, $\mathcal{O}(\mathrm{n}\mathbb{G}) \otimes \mathcal{O}_{\mathrm{m}} = \mathcal{O}(\mathrm{n}\mathbb{G}) \otimes \mathcal{O}(\mathbb{1}^{\bullet(\mathrm{m}+1)}) = \mathcal{O}(\mathrm{n}\mathbb{G} \times \mathbb{1}^{\bullet(\mathrm{m}+1)})$ is a co-n-category in ω-Cat for all $\mathrm{m} \geq 0$. Allowing m to vary, we obtain a co-n-category $\mathcal{O}(\mathrm{n}\mathbb{G} \times \mathbb{1}^{\bullet *})$ in the category $[\mathbb{\Delta}, \omega\text{-Cat}]$ of cosimplicial ω-categories. Hence, for any cosimplicial n-category $\mathcal{X}$, we obtain an n-category

$$\mathrm{Desc}\,\mathcal{X} = [\mathbb{\Delta}, \omega\text{-Cat}](\mathcal{O}(\mathrm{n}\mathbb{G} \times \mathbb{1}^{\bullet *}), \mathcal{X}) .$$

We thus have our precise definition of Desc $\mathcal{X}$ in somewhat more detail than in [St2].

§4. Weak n-categories, files and homotopy

Significant progress has been made in 1995 by Trimble-Verity [TV] on obtaining a precise definition of *weak n-category.* A weak 2-category is a bicategory in the sense of Bénabou [Bu]. A weak 3-category is a tricategory in the sense of [GPS]. Trimble [T] has a complete definition of weak 4-category which we also call tetracategory.

We shall provide here the definition of tricategory and their homomorphisms much as in [GPS].

A *tricategory* $\mathcal{T}$ consists of the following data:

(TD1) a set ob$\mathcal{T}$ whose elements are called *objects* of $\mathcal{T}$;

(TD2) for objects S, T, a bicategory $\mathcal{T}(\mathrm{S},\mathrm{T})$ whose objects are called *arrows* of $\mathcal{T}$ with *source* S and *target* T, whose arrows and 2-cells are called *2-cells and 3-cells* of $\mathcal{T}$ (source and target preserving their meanings), whose vertical composition will be written as juxtaposition, whose horizontal composition will be denoted by $\circ$, and whose associativity and identity constraints will not be given explicit names nor, at times, explicit mention (allowable by the bicategory coherence theorem);

(TD3) for objects S, T, U of $\mathcal{T}$, a homomorphism of bicategories

$$\otimes : \mathcal{T}(\mathrm{T},\mathrm{U}) \times \mathcal{T}(\mathrm{S},\mathrm{T}) \longrightarrow \mathcal{T}(\mathrm{S},\mathrm{U})$$

whose constraints will not be named (allowable by the bicategory homomorphism coherence theorem);

(TD4) for each object S, an arrow $\mathrm{I}_{\mathrm{S}} : \mathrm{S} \longrightarrow \mathrm{S}$ of $\mathcal{T}$;

(TD5) for objects S, T, U, V, a strong transformation

$$\begin{tikzcd}
\mathcal{T}(\mathrm{U},\mathrm{V}) \times \mathcal{T}(\mathrm{T},\mathrm{U}) \times \mathcal{T}(\mathrm{S},\mathrm{T}) \arrow[rr, "\otimes \times 1"] \arrow[d, "1 \times \otimes"'] & & \mathcal{T}(\mathrm{T},\mathrm{U}) \times \mathcal{T}(\mathrm{S},\mathrm{T}) \arrow[d, "\otimes"] \\
\mathcal{T}(\mathrm{U},\mathrm{V}) \times \mathcal{T}(\mathrm{S},\mathrm{U}) \arrow[rr, "\otimes"'] & \Downarrow a & \mathcal{T}(\mathrm{S},\mathrm{V})
\end{tikzcd}$$

which is an equivalence in the bicategory $\mathrm{Hom}(\mathcal{T}(U,V)\times\mathcal{T}(T,U)\times\mathcal{T}(S,T),\ \mathcal{T}(S,V))$;

(TD6) for objects S, T, strong transformations

$$\begin{tikzcd}
\mathcal{T}(S,T)\times\mathcal{T}(S,S) & & \mathcal{T}(S,T) \arrow[ll, "1\times I_S"'] \arrow[rr, "I_T\times 1"] \arrow[dd, "1"] & & \mathcal{T}(T,T)\times\mathcal{T}(S,T) \arrow[ddll, "\otimes"] \\
 & \overset{r}{\Leftarrow} & & \overset{\ell}{\Leftarrow} & \\
 & & \mathcal{T}(S,T) & &
\arrow[from=1-1, to=3-3, "\otimes"']
\end{tikzcd}$$

which are equivalences in the bicategory $\mathrm{Hom}(\mathcal{T}(S,T),\ \mathcal{T}(S,T))$;

(TD7) for objects S, T, U, V, W, an invertible modification π whose component at $(A, B, C, D) \in \mathcal{T}(V,W)\times\mathcal{T}(U,V)\times\mathcal{T}(T,U)\times\mathcal{T}(S,T)$ has source and target as in the pentagon

$$\begin{tikzcd}
 & (A\otimes(B\otimes C))\otimes D \arrow[rr, "a"] & & A\otimes((B\otimes C)\otimes D) \arrow[dr, "1\otimes a"] & \\
((A\otimes B)\otimes C)\otimes D \arrow[ur, "a\otimes 1"] \arrow[drr, "a"'] & & \Downarrow \pi_{A,B,C,D} & & A\otimes(B\otimes(C\otimes D)) \\
 & & (A\otimes B)\otimes(C\otimes D) \arrow[urr, "a"'] & &
\end{tikzcd}$$

; and,

(TD8) for objects S, T, U, an invertible modification μ whose component at $(A, B) \in \mathcal{T}(T,U)\times\mathcal{T}(S,T)$ has source and target as in the square

$$\begin{tikzcd}
(A\otimes I_T)\otimes B \arrow[rr, "a"] & & A\otimes(I_T\otimes B) \arrow[d, "A\otimes\ell"] \\
A\otimes B \arrow[u, "r\otimes B"] \arrow[rr, "1"'] & \Downarrow \mu_{A,B} & A\otimes B
\end{tikzcd}$$

;

subject to the following three axioms:

(TA1) (5-simplex) for all $(A, B, C, D, E)\in\mathcal{T}(V,W)\times\mathcal{T}(U,V)\times\mathcal{T}(T,U)\times\mathcal{T}(S,T)\times\mathcal{T}(R,S)$, the equation

$$\begin{tikzcd}
 & & & A(((BC)D)E) \arrow[drr, "1\otimes(a\otimes 1)"] \arrow[dd, "1\otimes a"] & & & \\
 & (A((BC)D))E \arrow[urr, "a"] & & \Downarrow \pi_{A,BC,D,E} & & A((B(CD))E) \arrow[dr, "1\otimes a"] & \\
((A(BC))D)E \arrow[ur, "a\otimes 1"] \arrow[drr, "a"] & & & A((BC)(DE)) \arrow[ddrrr, "1\otimes a"] & & \Downarrow 1\otimes\pi_{B,C,D,E} & A(B((CD)E)) \arrow[dd, "1\otimes(1\otimes a)"] \\
 & & (A(BC))(DE) \arrow[ur, "a"] & & \Downarrow \pi_{A,B,C,DE} & & \\
(((AB)C)D)E \arrow[uu, "(a\otimes 1)\otimes 1"] \arrow[drr, "a"'] & \cong & & & & & A(B(C(DE))) \\
 & & ((AB)C)(DE) \arrow[uu, "a\otimes(1\otimes 1)"] \arrow[rr, "a"'] & & (AB)(C(DE)) \arrow[urr, "a"'] & &
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
 & & & A(((BC)D)E) \arrow[drr, "1\otimes(a\otimes 1)"] \arrow[d, phantom, "\cong"] & & & \\
 & (A((BC)D))E \arrow[urr, "a"] \arrow[rr, "(1\otimes a)\otimes 1"'] & & A(B(CD)))E \arrow[rr, "a"] & & A((B(CD))E) \arrow[dr, "1\otimes a"] & \\
((A(BC))D)E \arrow[ur, "a\otimes 1"] & \Downarrow \pi_{A,B,C,D}\otimes 1 & & & \Downarrow \pi_{A,B,CD,E} & & A(B((CD)E)) \arrow[dd, "1\otimes(1\otimes a)"] \\
 & & ((AB)(CD))E \arrow[uur, "a\otimes 1"] \arrow[rr, "a"] & & (AB)((CD)E) \arrow[urr, "a"] \arrow[dd, "(1\otimes 1)\otimes a"] & & \\
(((AB)C)D)E \arrow[uu, "(a\otimes 1)\otimes 1"] \arrow[urr, "a\otimes 1"] \arrow[drr, "a"'] & & & \Downarrow \pi_{AB,C,D,E} & & \cong & A(B(C(DE))) \\
 & & ((AB)C)(DE) \arrow[rr, "a"'] & & (AB)(C(DE)) \arrow[urr, "a"'] & &
\end{tikzcd}$$

holds in the bicategory $\mathcal{T}(R,W)$ (where we have omitted some of the $\otimes$ symbols for economy);

(TA2) (left-degenerate 5-simplex) for all $(A, B, C) \in \mathcal{T}(U,V) \times \mathcal{T}(T,U) \times \mathcal{T}(S,T)$, the equation

$$\begin{tikzcd}
 & & & A(I_U B))C \arrow[drr, "a"] & & & \\
 & ((AI_U)B)C \arrow[urr, "a\otimes 1"] \arrow[dr, "a"] & & \Downarrow \pi_{A,I,B,C} & & A((I_U B)C) \arrow[dl, "1\otimes a"'] \arrow[ddr, "1\otimes(\ell\otimes 1)"] & \\
 & & (AI_U)(BC) \arrow[rr, "a"] & & A(I_U(BC)) \arrow[drr, "1\otimes \ell"'] & \Leftarrow 1\otimes\lambda_{B,C} & \\
(AB)C \arrow[uur, "(r\otimes 1)\otimes 1"] \arrow[drr, "a"'] & \cong & & \Downarrow \mu_{A,BC} & & & A(BC) \\
 & & A(BC) \arrow[uu, "r\otimes(1\otimes 1)"] \arrow[urrrr, "1"'] & & & &
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
 & & & A(I_U B))C \arrow[drr, "a"] \arrow[d, "(1\otimes\ell)\otimes 1"] & & & \\
 & ((AI_U)B)C \arrow[urr, "a\otimes 1"] & \Downarrow \mu_{A,B}\otimes 1 & (AB)C \arrow[drrr, "a"] & \cong & A((I_U B)C) \arrow[ddr, "1\otimes(\ell\otimes 1)"] & \\
 & & & & & & \\
(AB)C \arrow[uur, "(r\otimes 1)\otimes 1"] \arrow[uurrr, "1"'] \arrow[drr, "a"'] & & & \cong & & & A(BC) \\
 & & A(BC) \arrow[urrrr, "1"'] & & & &
\end{tikzcd}$$

holds in the bicategory $\mathcal{T}(S, V)$, where the invertible modification λ, with components

$$\begin{tikzcd}
(I_U B)C \arrow[rr, "\ell\otimes 1"] \arrow[dr, "a"'] & \Downarrow \lambda_{B,C} & BC \\
& I_U(BC) \arrow[ur, "\ell"'] &
\end{tikzcd} ,$$

is defined by the particular case of the equality for which $U = V$ and $A = I_U$; and,

(TA3) (right-degenerate 5-simplex) for all $(A, B, C) \in \mathcal{T}(U,V)\times\mathcal{T}(T,U)\times\mathcal{T}(S,T)$, the equation

$$\begin{tikzcd}
& & A((BI_T)C) \arrow[dr, "1\otimes a"] & & \\
& A(BC) \arrow[ur, "1\otimes(r\otimes 1)"] \arrow[drrr, "1"] & \Downarrow 1\otimes\mu_{B,C} & A(B(I_T C)) \arrow[dr, "1\otimes(1\otimes\ell)"] & \\
(AB)C \arrow[ur, "a"] \arrow[drrr, "1"'] & & \cong & & A(BC) \\
& & & (AB)C \arrow[ur, "a"'] &
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
& & & A((BI_T)C) \arrow[drr, "1\otimes a"] & & & \\
& A(BC) \arrow[urr, "1\otimes(r\otimes 1)"] & & \Downarrow \pi_{A,B,I,C} & & A(B(I_T C)) \arrow[ddr, "1\otimes(1\otimes\ell)"] & \\
& \cong & (A(BI_T))C \arrow[uur, "a"'] & & & & \\
(AB)C \arrow[uur, "a"] \arrow[urr, "(1\otimes r)\otimes 1"] \arrow[rrr, "r\otimes 1"'] \arrow[drrrr, "1"'] & & \rho_{A,B}\otimes 1 \Rightarrow & ((AB)I_T)C \arrow[ul, "a\otimes 1"'] \arrow[r, "a"'] & (AB)(I_T C) \arrow[uur, "a"'] \arrow[d, "(1\otimes 1)\otimes \ell"] & \cong & A(BC) \\
& & & \Downarrow \mu_{AB,C} & (AB)C \arrow[urr, "a"'] & &
\end{tikzcd}$$

holds in the bicategory $\mathcal{T}(S, V)$, where the invertible modification ρ, with components

$$\begin{tikzcd}
AB \arrow[rr, "1\otimes r"] \arrow[dr, "r"'] & \Downarrow \rho_{A,B} & A(BI_T) \\
& (AB)I_T \arrow[ur, "a"'] &
\end{tikzcd} ,$$

is defined by the particular case of the equality for which $S = T$ and $C = I_S$.

Suppose $\mathcal{T}$, $\mathcal{L}$ are tricategories. A *homomorphism* $M : \mathcal{T} \longrightarrow \mathcal{L}$ consists of the following data:

(HTD1) a function $M : \mathrm{ob}\mathcal{T} \longrightarrow \mathrm{ob}\mathcal{L}$;

(HTD2) for objects S, T of $\mathcal{T}$, a homomorphism of bicategories

$$M = M_{S,T} : \mathcal{T}(S,T) \longrightarrow \mathcal{L}(MS, MT)$$

(where again the constraints are given no special names);

(HTD3) for objects S, T, U of $\mathcal{T}$, a strong transformation

$$\begin{tikzcd}
\mathcal{T}(T,U)\times\mathcal{T}(S,T) \arrow[r, "M\times M"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow b"] & \mathcal{L}(MT,MU)\times\mathcal{L}(MS,MT) \arrow[d, "\otimes"] \\
\mathcal{T}(S,U) \arrow[r, "M"'] & \mathcal{L}(MS,MU)
\end{tikzcd}$$

which is an equivalence in Hom ($\mathcal{T}(T,U)\times\mathcal{T}(S,T)$, $\mathcal{L}(MS,MU)$);

(HTD4) for each object S of $\mathcal{T}$, an equivalence $s : I_{MS} \xrightarrow{\sim} MI_S$ in $\mathcal{L}(MS,MS)$;

(HTD5) for objects S, T, U, V of $\mathcal{T}$, an invertible modification ω whose component at (A, B, C)∈ $\mathcal{T}(U,V)\times\mathcal{T}(T,U)\times\mathcal{T}(S,T)$ is as in the hexagon

$$\begin{tikzcd}
 & M(A\otimes B)\otimes MC \arrow[r, "b"] & M((A\otimes B)\otimes C) \arrow[dr, "Ma"] & \\
(MA\otimes MB)\otimes MC \arrow[ur, "b\otimes 1"] \arrow[dr, "a"'] & \Downarrow \omega_{A,B,C} & & M(A\otimes(B\otimes C)) \\
 & MA\otimes(MB\otimes MC) \arrow[r, "1\otimes b"'] & MA\otimes M(B\otimes C) \arrow[ur, "b"'] &
\end{tikzcd}$$

; and,

(HTD6) for objects S, T of $\mathcal{T}$, invertible modifications γ, δ whose components at A∈ $\mathcal{T}(S,T)$ are

$$\begin{tikzcd}
MA\otimes I \arrow[r, "1\otimes s"] & MA\otimes MI \arrow[d, "b"] \\
MA \arrow[u, "r"] \arrow[r, "Mr"'] \arrow[ur, phantom, "\Uparrow \delta_A"] & M(A\otimes I)
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes MA \arrow[r, "s\otimes 1"] \arrow[d, "\ell"'] \arrow[dr, phantom, "\Leftarrow \gamma_A"] & MI\otimes MA \arrow[d, "b"] \\
MA & M(I\otimes A) \arrow[l, "M\ell"]
\end{tikzcd}$$

;

subject to the following two axioms:

(HTA1) (embellished associativity pentagon) (with obvious shorthand notation)

$$\begin{tikzcd}
 & & M((AB)C)D \arrow[r, "Ma\otimes 1"] \arrow[d, phantom, "\cong"] & M(A(BC))D \arrow[drr, "Ma"] & & \\
M(AB)CMD \arrow[urr, "b"] \arrow[rr] & & MA(BC)MD \arrow[ur, "b"] & \omega_{A,BC,D}\Downarrow & & MA((BC)D) \arrow[dd, "M1\otimes a"] \\
 & & & & MAM(BC)D \arrow[ur, "b"] \arrow[ddr, "1\otimes Ma"] & \cong \\
(MABMC)MD \arrow[uu, "b\otimes 1"] & \overset{\omega_{A,B,C}\otimes 1}{\Rightarrow} & (MAMBC)MD \arrow[uu, "b\otimes 1"'] \arrow[r, "a"] & MA(MBCMD) \arrow[ur, "1\otimes b"] & \Downarrow 1\otimes\omega_{B,C,D} & MA(B(CD)) \\
((MAMB)MC)MD \arrow[u, "(b\otimes 1)\otimes 1"] \arrow[rr, "a\otimes 1"] \arrow[dr, "a"'] & & (MA(MBMC))MD \arrow[u, "(1\otimes b)\otimes 1"'] \arrow[r, "a"] \arrow[ur, phantom, "\cong"] & MA((MBMC)MD) \arrow[u, "1\otimes(b\otimes 1)"] \arrow[dd, "1\otimes a"] & & MAMB(CD) \arrow[u, "b"'] \\
 & (MAMB)(MCMD) \arrow[drr, "a"'] & \Downarrow\pi & & MA(MB(MCD)) \arrow[ur, "1\otimes b"'] & \\
 & & & MA(MB(MCMD)) \arrow[ur, "1\otimes(1\otimes b)"'] & &
\end{tikzcd}$$

||

$$\begin{tikzcd}
 & & M((AB)C)D \arrow[rr, "Ma\otimes 1"] \arrow[dr, "Ma"'] & & M(A(BC))D \arrow[dr, "Ma"] & \\
M(AB)CMD \arrow[urr, "b"] & & & M(AB)(CD) \arrow[ddrr, "Ma"] & \Downarrow M\pi & MA((BC)D) \arrow[dd, "M1\otimes a"] \\
 & \Downarrow \omega_{AB,\,C,\,D} & & MABMCD \arrow[u] & & \\
(MABMC)MD \arrow[uu, "b\otimes 1"] \arrow[dr, "a"] & & & & & MA(B(CD)) \\
 & MAB(MCMD) \arrow[uurr, "1\otimes b"] & \cong & & \Downarrow \omega_{A,\,B,\,CD} & \\
((MAMB)MC)MD \arrow[uu, "(b\otimes 1)\otimes 1"] \arrow[dr, "a"'] & \cong & & (MAMB)MCD \arrow[uuu, "b\otimes 1"] \arrow[dr, "a"] & & MAMB(CD) \arrow[uu, "b"'] \\
 & (MAMB)(MCMD) \arrow[uu, "b\otimes 1"] \arrow[urr, "1\otimes b"] \arrow[drr, "a"'] & & \cong & MA(MB(MCD)) \arrow[ur, "1\otimes b"'] & \\
 & & & MA(MB(MCMD)) \arrow[ur, "1\otimes(1\otimes b)"'] & &
\end{tikzcd}$$

(HTA2) (embellished triangle for unit)

$$\begin{tikzcd}
 & & & M(AI)B \arrow[drrr, "Ma"] & & & \\
MAB \arrow[urrr, "Mb\otimes 1"] & & & & & & MA(IB) \arrow[ddd, "M1\otimes \ell"] \\
 & \cong & MAIMB \arrow[uur, "b"] & & \Downarrow \omega_{A,\,I,\,B} & MAMIB \arrow[ur, "b"] \arrow[dddddl, "1\otimes Mt"] & \\
 & & \overset{\delta_A\otimes 1}{\Rightarrow} & (MAMI)MB \arrow[ul, "b\otimes 1"'] \arrow[dr, "a"] & & \cong & \\
MAMB \arrow[uuu, "b"] \arrow[uurr, "Mr\otimes 1"] \arrow[dr, "r\otimes 1"'] \arrow[dddrrrr, bend right, "1"'] & & & \cong & MA(MIMB) \arrow[uur, "1\otimes b"] & & MAB \\
 & ((MA)I)MB \arrow[uurr, "(1\otimes s)\otimes 1"] \arrow[drr, "a"] & & & \Downarrow 1\otimes\gamma_B & & \\
 & & \Downarrow \mu & MA(IMB) \arrow[uur, "1\otimes(s\otimes 1)"] \arrow[dr, "1\otimes \ell"'] & & & \\
 & & & & MAMB \arrow[uuurr, "b"'] & &
\end{tikzcd}$$

$$||$$

$$\begin{tikzcd}
 & & M(AI)B \arrow[r, "Ma"] \arrow[dr, phantom, "\Downarrow M\mu"] & MA(IB) \arrow[dr, "M1\otimes \ell"] & \\
MAMB \arrow[r, "b"] & MAB \arrow[ur, "Mr\otimes 1"] \arrow[rrr] & & & MAB \quad .
\end{tikzcd}$$

It is well known what is meant for an arrow in a category to be an isomorphism (= 1-equivalence). It is well known what it means for an arrow in a bicategory to be an equivalence (= 2-equivalence). An arrow $f : a \longrightarrow b$ in a tricategory is called a *biequivalence* (= 3-equivalence) when there exists an arrow $g : b \longrightarrow a$ such that $f\,g$ and $g\,f$ are both equivalent to identity arrows. And so, recursively, we obtain the definition of *n-equivalence* in any weak n-category.

Now we can define homotopy sets for any weak n-category A. We define $\pi_0(A)$ to be the set of n-equivalence classes of 0-cells of A. Let a be any 0-cell of A and let AutEq(a) denote the full sub-(n–1)-category of $A(a, a)$ whose 0-cells are the n-equivalences $a \longrightarrow a$. We define the *fundamental group* $\pi_1(A, a)$ to be the set $\pi_0(\mathrm{AutEq}(a))$ equipped with the multiplication induced by the composition n-homomorphism $A(a,a) \times A(a,a) \longrightarrow A(a,a)$. We recursively define homotopy groups $\pi_n(A,a)$, $n > 1$, by

$$\pi_{n+1}(A,a) = \pi_n(\mathrm{AutEq}(a), 1_a).$$

Now we shall introduce the notion of *n-file* for all $n \geq 0$. Every n-category is an n-file, and every n-file is a weak n-category. In fact, an n-file is precisely an n-category for $n < 3$. A 3-file is a Gray-category in the sense of [GPS].

The definition of n-file is quite straightforward, using familiar concepts from category theory [EK], [D1], [D2].

We have already defined the Gray monoidal structure on ω-Cat. Let us denote this biclosed monoidal category by $\mathcal{V}_2$ to distinguish it from the cartesian closed category ω-Cat which might be denoted by $\mathcal{V}_1$. From the union of the equations (n+1)-Cat = (n–Cat)-Cat we obtain $\mathcal{V}_1$-Cat = $\mathcal{V}_1$. However, $\mathcal{V}_2$-Cat provides creatures more general than n-categories. A $\mathcal{V}_2$-category A consists of objects, and, for each pair of objects a, b, a hom-ω-category $A(a,b)$; however, we have composition ω-functors

$$A(a,b) \otimes A(b,c) \longrightarrow A(a,c)$$

defined on the Gray tensor product rather than $A(a,b) \times A(b,c) \longrightarrow A(a,c)$ defined on the cartesian product. Cells can be defined in A just as for n-categories, and let us suppose A is 3-dimensional (that is, all 4-cells in A are identities). There is a composition of 1-cells coming from the above displayed ω-functor, however, it does not extend to the "horizontal composition" of 2-cells $\sigma : f \Rightarrow f' : a \longrightarrow b$, $\tau : g \Rightarrow g' : b \longrightarrow c$ except when either σ or τ is an identity. Thus we obtain the boundary of a square of 2-cells

$$\begin{tikzcd} f\,g \arrow[r, "f\tau"] \arrow[d, "\sigma g"'] \arrow[dr, phantom, "\overset{c_{\sigma,\tau}}{\Rrightarrow}"] & f\,g' \arrow[d, "\sigma g'"] \\ f'\,g \arrow[r, "f'\tau"'] & f'\,g' \end{tikzcd}$$

in the 2-category $A(a,c)$. What the above displayed composition ω-functor does provide is the structural 3-cell $c_{\sigma,\tau}$ as shown in the square.

A *3-file* is a 3-dimensional $\mathcal{V}_2$-category in which all the structural 3-cells $c_{\sigma,\tau}$ are invertible. These are the Gray-categories of [GPS].

Theorem [GPS] *Every tricategory is 3-equivalent to a 3-file.*

Recall that the tensor product of $\mathcal{V}_2$ was induced from the dense full subcategory $\mathcal{Q}$ consisting of the ω-categories $\mathcal{O}(\mathbb{I}^{\times n})$. Every ω-category is certainly a $\mathcal{V}_2$-category and $\mathcal{Q}$ is a full subcategory of $\mathcal{V}_2$-Cat.

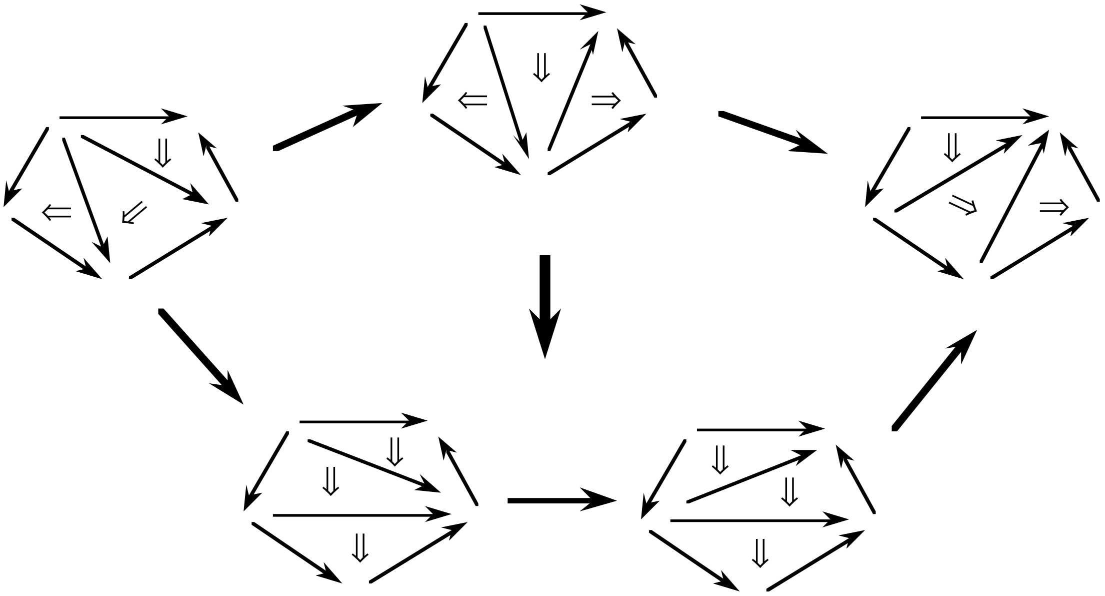
Conjecture 1. *$\mathcal{Q}$ is dense in $\mathcal{V}_2$-Cat, and, more generally, in all the categories $\mathcal{V}_n$ defined below.*

Day's construction now allows us to extend the monoidal structure on $\mathcal{Q}$ to a biclosed monoidal structure $\mathcal{V}_2$-Cat. Let $\mathcal{V}_3$ denote $\mathcal{V}_2$-Cat with this monoidal structure. Conjecture 1 recursively implies that the process continues providing biclosed monoidal $\mathcal{V}_n$ for all $n \geq 1$. By construction, each object of $\mathcal{V}_n$ has an underlying globular set (that is, cells make sense). An *n-file* is an n-dimensional object of $\mathcal{V}_n$ in which the structural cells are equivalences. A *file* is an n-file for some n. The following rather ambitious generalisation of the [GPS] theorem will require vastly new techniques.

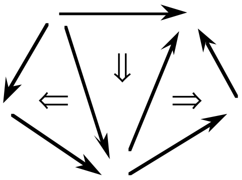
Conjecture 2. *Every weak n-category is n-equivalent to an n-file.*

Write Fil for the category of files. Each $\mathcal{V}_n$ is a full subcategory of Fil. In particular, n-categories are files. However, we need to look at morphisms between files which are weaker than those coming from enriched category theory, namely, those which do not necessarily preserve the structural cells like $c_{\sigma,\tau}$. Every file A has a *nerve:* the nerve N(A) of A is the simplicial set whose elements of dimension n are these weaker morphisms $\mathcal{O}_n \longrightarrow A$ where $\mathcal{O}_n$ is the n-th oriental.

By way of illustration we shall look at $N(A)_4$ in the case where A is a Gray category. Recall that $\mathcal{O}_4$ is the free 4-category on the parity 4-simplex.



A weak morphism $f : \mathcal{O}_4 \longrightarrow A$ will take the 4-cell in the middle of the big pentagon into an identity (since A is 3-dimensional). But f is not required to preserve horizontal composition of 2-cells; yet one such composite does occur in the top small pentagon



because the 2-cells pointing left and right are horizontally composable. What this means is that a structural isomorphism $c_{\sigma,\tau}$ is introduced into the pentagon when viewed in A ; the picture for $f : \mathcal{O}_4 \longrightarrow A$ is therefore a *hexagon* in A. This will be important to remember when we look at fusion operators in [St5].

Proposition *The nerve functor* $N : \mathrm{Fil} \longrightarrow [\mathbb{\Delta}^{op}, \mathrm{Set}]$ *commutes with* π_n *for all* $n \geq 0$.

We already remarked that $N : \omega\text{-Cat} \longrightarrow [\mathbb{\Delta}^{op}, \mathrm{Set}]$ is not full so neither can $N : \mathrm{Fil} \longrightarrow [\mathbb{\Delta}^{op}, \mathrm{Set}]$ be full. Simplicial maps $f : N(A) \longrightarrow N(B)$ are *normal lax functors* between the files A and B. Using the familiar process of replacing a map by an inclusion using a mapping cylinder, we see that each such normal lax functor gives rise to a *long exact homotopy sequence.*

$$\pi_n(A,a) \xrightarrow{f_*} \pi_n(B,f(a)) \longrightarrow \pi_n(f,a) \longrightarrow \pi_{n-1}(A,a) \xrightarrow{f_*} \pi_{n-1}(B,f(a))$$

Given Conjecture 2, we can define, up to homotopy, the *nerve of a weak n-category* to be the nerve of an n-equivalent n-file.

A similar approach can be taken to extending the definition of cohomology and descent as we have done for extending nerve to weak n-categories. If $\mathcal{X} : \mathbb{\Delta} \longrightarrow \mathrm{Fil}$ is a cosimplicial file, we define

$$\mathrm{Desc}\,\mathcal{X} = [\mathbb{\Delta}, \mathrm{Fil}](\mathcal{O}(n\mathbb{G} \times \mathbb{1}^{\bullet *}), \mathcal{X})\,.$$

However, I have not yet had time to investigate what the inclusion $\omega\text{-Cat} \longrightarrow \mathrm{Fil}$ does to pushouts. If it preserved pushouts then $\mathcal{O}(n\mathbb{G} \times \mathbb{1}^{\bullet *})$ would be a co-ω-category in Fil so that $\mathrm{Desc}\,\mathcal{X}$ would be an ω-category. My feeling is that $\mathrm{Desc}\,\mathcal{X}$ should be merely a file. Then we would, in the obvious way, define the cohomology file $\mathcal{H}(R, A)$ of R with coefficients in a file A. To generalise from files to weak n-categories again requires Conjecture 2.

A *homotopy type* is a file in which all n-cells, $n > 0$, are invertible. There is a category HoT of homotopy types and isomorphism classes of file morphisms. While we are speculating, we might as well submit a third conjecture.

Conjecture 3. *The restriction* $N : \mathrm{HoT} \longrightarrow [\mathbb{\Delta}^{op}, \mathrm{Set}]$ *of the nerve functor induces an equivalence between* HoT *and the usual homotopy category of simplicial sets.*

§5. Brauer groups

Let $\mathcal{M}$ denote a closed braided monoidal category which is finitely cocomplete. We have in mind that $\mathcal{M}$ is the category of modules over a commutative ring R, or the category of finite dimensional comodules for a quantum group. Consider the bicategory $\mathrm{Alm}\mathcal{M}$ whose objects are monoids (also called "algebras") in $\mathcal{M}$, whose arrows $M : A \longrightarrow B$ are left A- right B-modules, and whose 2-cells $f : M \Rightarrow M' : A \longrightarrow B$ are module homomorphisms $f : M \longrightarrow M'$; vertical composition is composition of functions and horizontal composition of modules $M : A \longrightarrow B$, $N : B \longrightarrow C$ is given by tensor product $M \otimes_B N : A \longrightarrow C$ over B ($M \otimes_B N$ is the coequalizer of the two arrows from $M \otimes B \otimes N$ to $M \otimes N$ given by the actions of B on M and on N).

Since $\mathcal{M}$ is braided, the tensor product $A \otimes B$ of algebras is canonically an algebra. This makes $\mathrm{Alm}\mathcal{M}$ into a monoidal bicategory. Let $\Sigma\mathrm{Alm}\mathcal{M}$ denote the 1-object tricategory whose hom bicategory is $\mathrm{Alm}\mathcal{M}$ and whose composition is tensor product of algebras.

In the particular case of the tricategory $\Sigma\mathrm{Alm}\mathcal{M}$, there it is an easy way to find a 3-equivalent Gray category (= 3-file). First replace $\mathcal{M}$ by an equivalent strict monoidal category (see [JS2]). Then identify modules $M : A \longrightarrow B$ with left adjoint functors $[A^{op}, \mathcal{M}] \longrightarrow [B^{op}, \mathcal{M}]$ where $[A^{op}, \mathcal{M}]$ is the category of right A-modules in $\mathcal{M}$. The point is that tensor product $M \otimes_B N$ of modules then becomes composition of functors.

Let $\mathcal{B}r(\mathcal{M})$ denote the sub-3-file of $\Sigma\ \mathrm{Alg}\mathcal{M}$ consisting of the arrows A which are bi-equivalences, the 2-cells M which are equivalences, and the 3-cells f which are isomorphisms. The arrows A of $\mathcal{B}r(\mathcal{M})$ are called *Azumaya algebras* in $\mathcal{M}$. The 2-cells M of $\mathcal{B}r(\mathcal{M})$ are called *Morita equivalences* in $\mathcal{M}$.

We can form the nerve $N\,\mathcal{B}r(\mathcal{M})$ of $\mathcal{B}r(\mathcal{M})$. It is a simplicial set whose homotopy objects are of special importance. In particular, $\pi_0 N\,\mathcal{B}r(\mathcal{M})$ is a singleton set, $\pi_1 N\,\mathcal{B}r(\mathcal{M})$ is called the *Brauer group* $\mathrm{Br}(\mathcal{M})$ *of* $\mathcal{M}$, and $\pi_2 N\,\mathcal{B}r(\mathcal{M})$ is the *Picard group* $\mathrm{Pic}(\mathcal{M})$ *of* $\mathcal{M}$. If $\mathcal{M}$ is equivalent to $\mathrm{Mod}(R)$ for a commutative ring R, these are the usual Brauer and Picard groups of R; also $\pi_3 N\,\mathcal{B}r(\mathcal{M})$ is then isomorphic to the group $\upsilon(R)$ of units of R. Compare the approach of Duskin [Dn1].

Now suppose $F : \mathcal{M} \longrightarrow \mathcal{N}$ is a right-exact braided strong-monoidal functor between finitely cocomplete closed braided monoidal categories. (We have in mind the functor $\mathrm{Mod}(f) : \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(S)$ induced by a commutative ring homomorphism $\phi : R \longrightarrow S$.) Such an F determines a homomorphism of tricategories $\mathrm{Alm}F : \mathrm{Alm}\mathcal{M} \longrightarrow \mathrm{Alm}\mathcal{N}$. Homomorphisms preserve n-equivalence for all n. So a homomorphism $\mathcal{B}r(F) : \mathcal{B}r(\mathcal{M}) \longrightarrow \mathcal{B}r(\mathcal{N})$ is induced, and thus we induce a simplicial map $N\mathcal{B}r(F) : N\mathcal{B}r(\mathcal{M}) \longrightarrow N\mathcal{B}r(\mathcal{N})$. This proves that we have the nine term exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Aut}(I_{\mathcal{M}}) \arrow[r, "F_*"] & \mathrm{Aut}(I_{\mathcal{N}}) \arrow[r] & \mathrm{Aut}(F) \arrow[r] & \mathrm{Pic}(\mathcal{M}) \arrow[r, "F_*"] & \mathrm{Pic}(\mathcal{N}) \\
{} \arrow[r] & \mathrm{Pic}(F) \arrow[r] & \mathrm{Br}(\mathcal{M}) \arrow[r, "F_*"] & \mathrm{Br}(\mathcal{N}) \arrow[r] & \mathrm{Br}(F) \arrow[r] & 1
\end{tikzcd}$$

in which $\mathrm{Aut}(I_{\mathcal{M}})$ denotes the abelian group of automorphisms of the unit $I_{\mathcal{M}}$ for the tensor product in $\mathcal{M}$. Compare with [DI] when $\mathcal{M} = \mathrm{Mod}(R)$.

§6. Giraud's H^2 and the pursuit of stacks

We use Duskin's [Dn2] amelioration of Giraud's theory [Gd] to show that Giraud's H^2 really fits into our general setting for cohomology. We work in a topos $\mathcal{E}$.

A groupoid B in $\mathcal{E}$ is *connected* when $\pi_0 B \cong 1$.

Lemma *Locally connected implies connected.*

Proof If $R \longrightarrow 1$ is an epimorphism ("a cover") then the functor $R \times - : \mathcal{E} \longrightarrow \mathcal{E}/R$ is reflects isomorphisms (that is, is conservative), and preserves terminal objects and coequalizers. Hence it also reflects coequalizers. So, to see whether

$$\begin{tikzcd}
B_1 \arrow[r, shift left] \arrow[r, shift right] & B_0 \arrow[r] & 1
\end{tikzcd}$$

is a coequalizer in $\mathcal{E}$, it suffices to see that

$$\begin{tikzcd}
R \times B_1 \arrow[r, shift left] \arrow[r, shift right] & R \times B_0 \arrow[r] & R
\end{tikzcd}$$

is a coequalizer in $\mathcal{E}/R$.**qed**

A functor $f : A \longrightarrow B$ in $\mathcal{E}$ is called *eso* (essentially surjective on objects) when the top composite of q and d_1 in the diagram below is an epimorphism $P \longrightarrow B_0$ (here $\mathbb{I}$ is the category with two objects and an isomorphism between them).

$$\begin{tikzcd}
P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\text{pull back}"] & B_0^{\mathbb{I}} \arrow[r, "d_1"] \arrow[d, "d_0"] & B_0 \\
A_0 \arrow[r, "f_0"'] & B_0 &
\end{tikzcd}$$

A groupoid B is called a *weak group* when there exists an eso $b : 1 \longrightarrow B$. In this case, if G denotes the full image of b, we have a weak equivalence (that is, eso fully faithful functor) $G \longrightarrow B$ where G is a group.

Lemma *A groupoid is connected iff it is a locally weak group.*

Proof By the last Lemma, "if" will follow from "weak group implies connected". Suppose $b : 1 \longrightarrow B$ is eso and form the pullback P as above with $A = 1$ and $f = b$. To prove

$$\begin{tikzcd} B_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & B_0 \arrow[r, "t"] & 1 \end{tikzcd}$$

is a coequalizer, suppose $h : B_0 \longrightarrow X$ has $h\, d_0 = h\, d_1$. Then

$$h\, d_1 q \;=\; h\, d_0 q \;=\; h\, b\, p \;=\; h\, b\, t\, d_1\, q$$

implies $h = h\, b\, t$ since $d_1 q$ is epimorphic. So h factors through t. But t is a retraction (split by b), so the factorization is unique.

Conversely, assume B is connected. Certainly $B_0 \longrightarrow X$ is epimorphic, so we pass to $\mathcal{E}/B_0$ where we pick up a global object $\Delta : B_0 \longrightarrow B_0 \times B$ over B_0 which we will see is eso.

$$\begin{tikzcd} B_1 \arrow[r] \arrow[d, "d_0"'] \arrow[dr, phantom, "\text{pull back}"] & B_0 \times B_1 \arrow[r, "1 \times d_1"] \arrow[d, "1 \times d_0"] & B_0 \\ B_0 \arrow[r, "\Delta"'] & B_0 \times B_0 & \end{tikzcd}$$

What we must see then is that $(d_0\, , d_1\,) : B_1 \longrightarrow B_0 \times B_0$ is epimorphic. Factor $(d_0\, , d_1\,) : B_1 \longrightarrow B_0 \times B_0$ as $B_1 \twoheadrightarrow K \rightarrowtail B_0 \times B_0$. Since B is a groupoid, K is an equivalence relation on B_0 . Since $\mathcal{E}$ is exact, K is a kernel pair of its coequalizer. The coequalizer is 1 since B is connected. So the kernel pair is $B_0 \times B_0$.**qed**

Recall that the category of groups in a category with finite products is actually a 2-category since group homomorphisms can be regarded as functors, so there are 2-cells amounting to natural transformations. (In fact, we can make it a 3-category by taking central elements of the target group as 3-cells, but this will not be needed here.) So we have a 2-functor

$$\mathrm{Gp} : \mathrm{Cat}_\times \longrightarrow \text{2-Cat}$$

from the 2-category $\mathrm{Cat}_\times$ of categories with finite products and product-preserving functors.

There is a homomorphism of bicategories $\mathcal{E}/- : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ taking an object X of $\mathcal{E}$ to the slice category $\mathcal{E}/X$ and given on arrows by pulling back along the arrow. It is easy to find an actual 2-functor $\mathbf{E} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ equivalent to $\mathcal{E}/-$. The composite 2-functor

$$\mathcal{E}^{\mathrm{op}} \xrightarrow{\mathbf{E}} \mathrm{Cat}_\times \xrightarrow{\mathrm{Gp}} 2-\mathrm{Cat}$$

defines a 2-category $\mathcal{G}$ in the category $[\mathcal{E}^{\mathrm{op}}, \mathrm{Set}]$.

It is natural then to look at the cohomology 2-category $\mathcal{H}(\mathcal{E}, \mathcal{G})$ of $\mathcal{E}$ with coefficients in $\mathcal{G}$. What I mean by this is the colimit of all the 2-categories $\mathcal{H}(R, \mathcal{G})$ over all hypercovers R in $\mathcal{E}$, which we regard, via the Yoneda embedding, as simplicial objects in the category $[\mathcal{E}^{\mathrm{op}}, \mathrm{Set}]$.

What Giraud actually looks at is obtained from $\mathcal{H}(\mathcal{E}, \mathcal{G})$ by lots of quotienting. First form the composite 2-functor

$$\begin{tikzcd}\mathcal{E}^{\mathrm{op}} \arrow[r, "\mathcal{G}"] & 2-\mathrm{Cat} \arrow[r, "\pi_{0*}"] & \mathrm{Cat}\end{tikzcd}$$

where π_{0*} is the 2-functor which applies π_0 to the hom categories of each 2-category. Let $\mathcal{L}: \mathcal{E}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ denote the associated stack of that composite 2-functor. The category $\mathcal{L}(X)$ is called *the category of* X*-liens of* $\mathcal{E}$; in particular, $\mathcal{L}(1)$ is the category of *liens* of $\mathcal{E}$.

The stack condition implies that each epimorphism $R \longrightarrow 1$ induces an equivalence between the category $\mathcal{L}(1)$ of liens and the descent category of the following truncated cosimplicial category.

$$\begin{tikzcd}\mathcal{L}(R) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathcal{L}(R \times R) \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \mathcal{L}(R \times R \times R) \arrow[l, shift left=2] \arrow[l, shift right=2]\end{tikzcd}$$

Each connected groupoid B determines a lien $\mathrm{lien}(B) \in \mathcal{L}(1)$ as follows. By the last Lemma, there exists an epimorphism $R \longrightarrow 1$ and $G \in \pi_{0*}\mathcal{G}(R)$. The quotient functor $\pi_{0*}\mathcal{G}(R) \longrightarrow \mathcal{L}(R)$ gives an R-lien $[G] \in \mathcal{L}(R)$ which can be enriched with descent data. These descent data are determined up to isomorphism by B. It follows that there is a lien $\mathrm{lien}(B) \in \mathcal{L}(1)$ taken to B by the functor $\mathcal{L}(1) \longrightarrow \mathcal{L}(R)$.

For any lien L, let $\mathcal{H}^2(\mathcal{E}, L)$ denote the category whose objects are connected groupoids B with $\mathrm{lien}(B) \cong L$, and whose arrows are weak equivalences of groupoids. We leave as an open problem to study the connection between the 2-category $\mathcal{H}(\mathcal{E}, \mathcal{G})$ and the categories $\mathcal{H}^2(\mathcal{E}, L)$.

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Centre of Australian Category Theory
Mathematics Department, Macquarie University