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Mathematics Honours Essay

Species of Structure

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Table of Contents

Introduction

The theory of Species of Structure was introduced by the Canadian mathematician Andrè Joyal in 1980. Species are a structural approach to combinatorial generating series and allow us to lift these numerical identities to structural isomorphisms throwing light on how the identities arise. This is possible since we can associate to each species of structures a generating series which encodes information about the species and is consistent with operations on species. Species provide a model from combinatorics for the study of permutation representation of the permutation groups.

The algebraic operations on generating series (such as addition, multiplication, substitution, differentiation) are interpreted at the structural level. These operations allow us to combine species to form new species of structures by using set theoretical constructions.

The aim of this essay is to introduce and describe the concept of Species of Structure and then discuss Lagrange inversion as an example of how numerical identities can be looked at from this structural approach.

Chapter 1 first looks at the concept of structures. Species of Structure are then introduced and different examples of species are studied. The generating series associated to a species is introduced and the various concepts of equality of species are looked at.

Chapter 2 discusses some operations on species defining them by using set theoretical constructions and we look at how these operations allow us to form new species of structures.

Chapter 3 discusses Lagrange Inversion as an example of the application of this calculus of species, and shows how we can use species to transform recursive definitions of tree-like structures to functional or differential equations.

I would like to thank Professor Ross Street for his time, help and encouragement in supervising this essay.

Chapter 1

Species of structure

1.0. Structures

Structures are used in every area of mathematics and are fundamental in the development of mathematical concepts. To understand species of structures we will begin by looking at the notion of structures and the transport of structures.

A structure is a pair $s = (\gamma, U)$ where γ is an element of some construction performed on the set U. We call U the underlying set of the structure s.

For example, a rooted tree can be expressed as the structure $s = (\gamma, U)$ where

$U = \{a,b,c,d,e\}$

$\gamma = (\{b\},\{\{b,a\},\{b,c\},\{c.d\},\{c,e\}\})$

In this example U is the set of vertices of the tree and γ is the set of edges. The single element $\{b\}$ appearing as the first component of γ denotes the root of the tree. The other pairs of components tell us which two vertices are connected by an edge. We can picture this structure as in the following diagram.

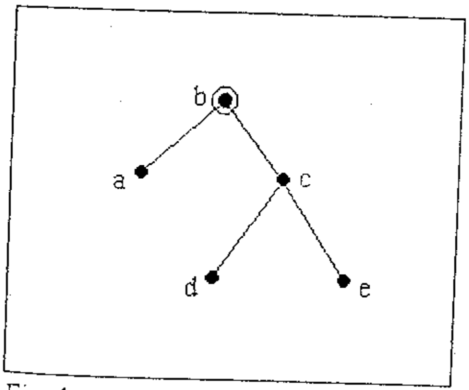


Fig. 1.

We should note that both γ and U are equally important in the definition of structure since the knowledge of γ alone does not necessarily tell us what the set U consists of. We can see this illustrated in the following example.

Example 1: Consider the structure $s = (\gamma, U)$ where $U = \{1,2,3,4,5\}$ and $\gamma = (\{1,3\},\{1,4\},\{3,4\})$

In this example the underlying set U consists of five elements but the construction γ on U is defined on only three of these elements. This is an example of a structure whose underlying set has some elements on which there is no construction. These elements are still part of the whole structure and therefore when describing s we need to look at both U and γ to give us the whole picture.

A fairly general approach to structure is to think of a structure γ on a set U as an element of some iterated power set $\wp \wp \ldots \wp U$ of U. Here are some examples that illustrate this.

Example 2: Topology

A topology $\Im$ on a set U deals with subsets of U which satisfy certain axioms. Hence $\Im \subset \wp U$ and we can therefore say $\Im \in \wp \wp U$ is a structure.

Example 3: Magma

A magma is a set U together with a binary operation $m : U \times U \to U$. An element $(x,y) \in U \times U$ can be identified with the element

$\{\{\{x\},\{x,y\}\} : x,y \in U\} \in \wp \wp U$.

Similarly $(x,y,z) \in U \times U \times U$ can be identified with

$\{\{\{x\},\{x,y\},\{x,y,z\}\} : x,y,z \in U\}$ which is an element of $\wp \wp U$. Hence the binary operation $m : U \times U \to U$ is determined by

$\{(x,y,m(x,y)) : x,y \in U\} \subseteq U \times U \times U \subseteq \wp \wp U$ and we can think of m as an element of $\wp \wp U$. Therefore a magma is a structure in this sense.

Example 4: Rooted trees

A rooted tree structure is the underlying set U together with a subset γ of the set of pairs of elements of U and hence $\gamma \in \wp \wp U$.

The way we will approach the concept of structure is by looking at transport of structure along bijections. This is a functorial approach to structures and will enable us to grasp the notion of species more rigorously. The transport of structures allows us to transport one structure to another along a bijection. In essence, we map a structure $s = (\gamma, U)$ to a structure $t = (\tau, V)$ via the given bijection $\sigma : U \to V$. Each element $u \in U$ appearing in γ is replaced by the corresponding element $\sigma(u) \in V$ in the new expression τ of γ. We say t is obtained by transporting s along the bijection σ and write $t = \sigma.s$. The following example will illustrate this concept.

Example 5. Consider the rooted tree structures s and t described by the following:

$s = (\gamma, U)$ where $U = \{a,b,c,d,e\}$ and $\gamma = (\{b\},\{\{b,a\},\{b,c\},\{c,d\},\{c,e\}\})$,

$t = (\tau, V)$ where $V = \{1,2,3,4,5\}$ and $\tau = (\{1\},\{\{1,2\},\{1,3\},\{3,4\},\{3,5\}\})$,

transported by the bijection $\sigma : U \to V$ described in the diagram below.

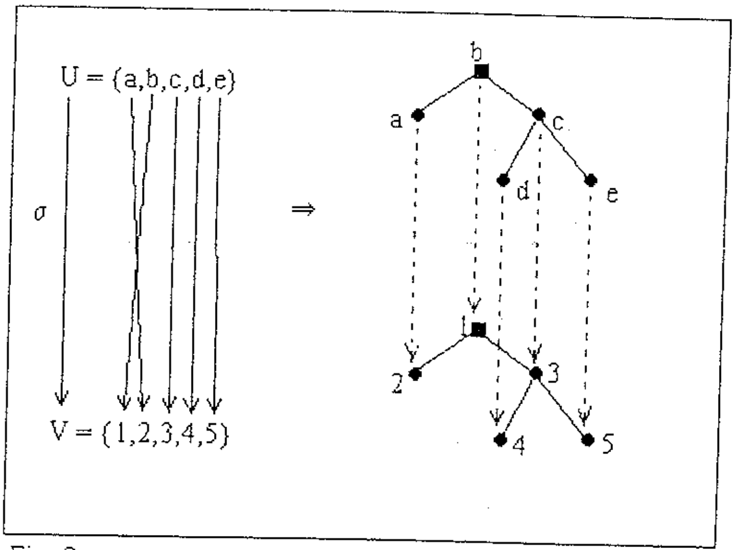


Fig. 2.

We can see that the rooted tree s is transported via the bijection σ to the rooted tree t by replacing each vertex $u \in U$ by the vertex $\sigma(u) \in V$ as defined by the bijection.
Since σ is a bijection, it is injective and surjective and we can also notice that it preserves the structure s. Hence it can be clearly seen that σ is an isomorphism of s to t. Note that if $U = V$ then σ is a permutation.

If we think of the construction γ as an element of an iterated power set , then we can think of the transport of structures in the following way. Suppose $\sigma : U \to V$ is a bijection between finite set U and V and γ is a structure on the set U where $\gamma \in \wp^n U$. Then the structure on V obtained by transporting γ along σ is the set V together with $(\wp^n \sigma)(\gamma)$, where $\wp^n U$ denotes the set U iterated n times and $\wp^n \sigma$ denotes the iterated direct image under σ. The transport of structures is of fundamental importance to species of structures.

1.1. Species of Structures

Definition 1: A species of structures F consists of the following functions.
i) For each finite set U, F produces a finite set $F[U]$,
ii) For each bijection $\sigma : U \to V$, F produces a function $F[\sigma] : F[U] \to F[V]$ which satisfies

1. $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$ where $\sigma : U \to V$ and $\tau : V \to W$
2. $F[Id_U] = Id_{F[U]}$ where $Id_U : U \to U$.

In other words, F is a functor from the category B whose objects are finite sets U,V,W,.... and whose arrows are bijections, to the category whose objects are sets and whose arrows are functions. We can depict properties 1 and 2 as:

1. For $\begin{tikzcd} U \arrow[r, "\sigma"] \arrow[rr, bend right, "\tau \circ \sigma"'] & V \arrow[r, "\tau"] & W \end{tikzcd}$ we have $\begin{tikzcd} F[U] \arrow[r, "F[\sigma]"] \arrow[rr, bend right, "F[\tau \circ \sigma]"'] & F[V] \arrow[r, "F[\tau]"] & F[W] \end{tikzcd}$

2. For $U \xrightarrow{Id_U} U$ we have $F[Id_U] = Id_{F[U]}$

F[U] is the set of F-structures on U so $s \in F[U]$ is called an F-structure on U or a structure of species F on U. The bijection $F[\sigma]$ is the transport of F-structures along σ. Note that there is also the empty species 0 which has no structure on any set. The diagram below can help us picture an F-structure on a set U where the dots represent the distinct elements of U and the structure itself is represented by the circular arc labeled F.

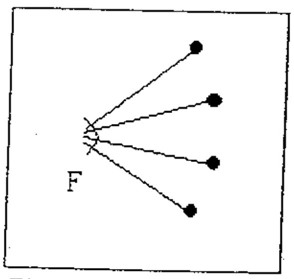


Fig. 1.

Example 1: Species of simple graphs

Consider a finite set U and let $G[U] = \{g : g = (\gamma, U), \gamma \subseteq \wp^{(2)}[U]\}$ where $\wp^{(2)}$ is the collection of unordered pairs of elements of U. G[U] is the set of structures of simple graphs on U and $g \in G[U]$ is a simple graph on U called a G-structure on U. Each bijection $\sigma : U \to V$ produces a map $G[\sigma] : G[U] \to G[V]$. This transport of structures along σ takes g to $\sigma.g$ which is just a relabelling of vertices and edges by σ. Hence we can deduce that
$G[\tau \circ \sigma] = G[\tau] \circ G[\sigma]$ where $\sigma : U \to V$ and $\tau : V \to W$ and
$G[Id_U] = Id_{G[U]}$ where $Id_U : U \to U$.

Example 2: Species S of permutations

The species S of permutations has S[U] equal to the set of permutations on the finite set U. The transport of structures along $\sigma : U \to V$ is the bijection $S[\sigma] : S[U] \to S[V]$ which takes $s \in S[U]$ to $\sigma s \sigma^{-1}$. So $S[\sigma](s) = \sigma s \sigma^{-1}$.

Looking at this for a particular example, take $U = [5] = \{1,2,3,4,5\}$, then $\sigma : [5] \to [5]$ and let $s = (124)(35)$ in cycle notation.
Then we have $S[\sigma](s) = S[\sigma]((124)(35)) = (\sigma(1)\ \sigma(2)\ \sigma(4))(\ \sigma(3)\ \sigma(5))$
Note that $S[0] = e$

$S[1] = e$

$S[2] = \{e,(12)\}$

$S[3] = \{e,(12),(13),(23),(123),(132)\}$

where e is the identity permutation.

Example 3 : Species L of linear orderings

For L, the species of linear orderings, we take $L[U]$ to be the set of all orderings of elements on the finite set U. For example $L[3] = \{123, 132, 213, 231, 321, 312\}$. For a bijection $\sigma : U \to V$, if $u_1u_2\ldots u_n$ is a linear ordering of elements of U then $L[\sigma](u_1u_2\ldots u_n) = \sigma(u_1)\ \sigma(u_2)\ldots\ \sigma(u_n)$. We can see that $L[\sigma]$ is a relabelling bijection.

Example 4: Species A of rooted trees

If A is the species of rooted trees on a finite set U then an A-structure is a rooted tree whose set of vertices is U. The transport of A-structures takes an A-structure on a set U to another isomorphic A-structure on the set V.

Example 5: Species E of sets

If E is the species of sets then for any finite set U, $E[U] = \{U\}$ is considered as a single element set. This means that each set U has one structure on it, a unique structure for each set U. The transport of E-structures is the bijection $E[\sigma] : E[U] \to E[V]$ where $E[\sigma](U) = V$ for any bijection $\sigma : U \to V$.

Example 6: Species End of endofunctions

For any finite set U, $End[U] = \{\ \varphi \mid \varphi : U \to U\}$ which is the set of all functions from U to U. We can describe the transport of End-structures by the map $End[\sigma] : End\ [U] \to End[V]$ for any bijection $\sigma: U \to V$ where $End[\sigma](\varphi) = (\sigma \circ \varphi \circ \sigma^{-1})$. This comes about because of the following:
Consider the bijection $End[\sigma] : End[U] \to End[V]$ and let $\theta = End[\sigma](\varphi) \in End[V]$.

So θ is a function from V to V where $V = \sigma(U)$.

Take any $v \in V$. This is of the form $v = \sigma(u)$ for some $u \in U$ and therefore $u = \sigma^{-1}(v)$. Also $End[\sigma]$ takes $\varphi \in End[U]$ to $\theta = End[\sigma](\varphi) \in End[V]$, and therefore is the function that maps $\sigma(u)$ to $\sigma(\varphi(u))$.

So for any $v \in V$, $v = \sigma(u)$ which is mapped to $\sigma(\varphi(u))$ which is the same as $\sigma(\varphi(\sigma^{-1}(v)))$. That is, $\theta : V \to V$ takes v to $(\sigma \circ \varphi \circ \sigma^{-1})(v)$.

Example 7: Species X of singletons.

For any finite set U, the species X is defined by

$$X[U] = \begin{cases} \{*\} & \text{if } \#U = 1 \\ \varnothing & \text{if } \#U \neq 1 \end{cases}$$

The transport of structure along $\sigma : U \to V$ induces the bijection $X[\sigma] : X[U] \to X[V]$. This is only valid when both $\#U = 1$ and $\#V = 1$. Therefore the bijection $X[\sigma]$ maps the singleton to the singleton.

Example 8: Species 1 of the empty set.

For any finite set U, the species 1 is the characteristic of the empty set and we define it by

$$1[U] = \begin{cases} \{U\} & \text{if } U = \varnothing \\ \varnothing & \text{otherwise} \end{cases}$$

Example 9: Species C of cyclic permutations

This is the species whose elements are those permutations which are cycles. For any finite set U, C[U] is the set of oriented cycles of elements of U. For any bijection $\xi : U \to V$, we can describe the transport of C-structures $C[\xi] : C[U] \to C[V]$ by the following:

$$\begin{array}{ccc} U & \xrightarrow{\xi} & V \\ \downarrow & & \downarrow \\ C[U] & \xrightarrow[C[\xi]]{} & C[V] \end{array}$$

The action on the elements is:

$$\begin{array}{ccc} (u_1, \dots, u_n) & \xrightarrow{\xi} & (\xi(u_1), \dots, \xi(u_n)) \\ \downarrow & & \downarrow \\ \sigma(u_1, \dots, u_n) & \xrightarrow[C[\xi]]{} & \sigma(\xi(u_1), \dots, \xi(u_n)) \end{array}$$

In the above diagrams, the set U contains elements $u_1, \dots, u_n$ exclusively. The cyclic permutation σ takes these elements to $\sigma(u_1, \dots, u_n)$ which is an oriented cycle of elements of U. Similarly, the map from V to C[V] takes $(\xi(u_1), \dots, \xi(u_n))$

to σ ($\xi(u_1), \ldots, \xi(u_n)$). We require the map from C[ξ] to apply ξ to an element of C[U]. Therefore we need $\xi\sigma\,(u_1, \ldots, u_n) = \sigma\xi\,(u_1, \ldots, u_n)$. That is, the cyclic permutation σ is mapped to $\xi\sigma\xi^{-1}$. Hence we can conclude that for any bijection $\xi : U \to V$, the transport of structures takes $\sigma \in C[U]$ to $\xi\sigma\xi^{-1}$. It is interesting to note that the species C of cyclic permutations is a subspecies of the species S of permutations. This is illustrated in the following "naturality" diagram.

$$
\begin{tikzcd}
C[U] \arrow[r, "i"] \arrow[d, "C[\xi]"'] & S[V] \arrow[d, "S[\xi]"] \\
C[V] \arrow[r, "i"'] & S[V]
\end{tikzcd}
$$

where i is the inclusion map.

Example 10: Species Der of derangements

Derangements are permutations without fixed points. For example, the derangements on a set of three elements is Der[3] = {(123),(321)}. Hence if a set U has order n, then a Der-structure is the permutations of size n on U. In a similar way to the previous example we can deduce that the transport of structures along $\xi : U \to V$ induces the bijection $Der[\xi] : Der[U] \to Der[V]$ where $\omega \in Der[U]$ is mapped to $\xi\omega\xi^{-1} \in Der[V]$. Notice that $\omega(i) \neq i$ for all i and this implies that $\xi\omega\xi^{-1}(i) \neq i$ for all i.

Example 11: Each finite set W gives a species defined by

$$
W[U] = \begin{cases} \varnothing & \text{for } U \neq \varnothing \\ W & \text{for } U = \varnothing \end{cases}
$$

for any finite set U. The transport of structures along $\sigma : U \to V$ induces the bijection $W[\sigma] : W[U] \to W[V]$ which is simply the identity map from W to itself.

Example 12: Species $\wp$ of subsets

The power set species is given by $\wp[U] = \wp U$ for any finite set U which is the set of all subsets of U. For any bijection $\sigma : U \to V$, the transport of structures $\wp[\sigma] : \wp U \to \wp V$ is given by the direct image.
That is, $\wp[\sigma](A) = \{\sigma(a) : a \in A\}$ where $A \subseteq U$.

We have seen that a species of structure is a rule F such that on any finite set it produces a set of structures on it. The number of structures is only dependent on the number of

elements in the set. That is, if there are two sets with the same number of elements then the species will have the same number of structures. So #F[U] only depends on #U. This is due to the fact that the F[σ] are always bijections. Since structure is only dependent on the number of elements in a set, we only need to consider sets of different sizes. Two sets U and V are considered the same if #U = #V. Hence we will turn our attention to sets of the form **n** = {1,2,...,n} which represents a set of n elements.
Now we can say that a species F gives a set F[**n**] for each **n** and S_n acts on F[**n**] where S_n is the group of bijections $\sigma : \mathbf{n} \to \mathbf{n}$. So $\sigma : \mathbf{n} \to \mathbf{n}$ produces $F[\sigma] : F[\mathbf{n}] \to F[\mathbf{n}]$ where $\sigma.s = F[\sigma](s)$ with the properties

1. $F[\mathbf{n}] \xrightarrow{F[\sigma]} F[\mathbf{n}] \xrightarrow{F[\tau]} F[\mathbf{n}]$ which takes $s \xrightarrow{F[\sigma]} F[\sigma](s) \xrightarrow{F[\tau]} F[\tau](F[\sigma]s)$
   so that $F[\tau](F[\sigma](s)) = F[\tau \circ \sigma](s) = F[\tau] \circ F[\sigma]$

2. $F[Id_\mathbf{n}] = Id_{F[\mathbf{n}]}$ which takes the F-structure s to itself.

Therefore a species of structures amounts to a sequence of sets F[**0**], F[**1**], F[**2**], ... with S_n acting on each F[**n**].

1.2. Generating Series

Generating Series can be used to encode all information concerning the enumeration of a species of structures. For each species of structures we are able to find generating series associated with it. We will look only at the formal power series although there are other types of series which can be used.

Definition 1: Let F be a species of structures. The generating series of F is the formal power series

$F(x) = \Sigma_{n \geq 0} f_n x^n/n!$

where f_n = #F[**n**] = the number of F-structures on a set of n elements.

Example1: Species of sets

Consider the species E of sets. We want to find its formal power series.
Now f_n = the number of E-structures on a set of n elements. Since an E-structure on any set U is the singleton consisting of the set itself, it is considered as a single element and therefore $f_n = 1$.

$$\begin{aligned} \text{Hence } E(x) &= \Sigma_{n\geq 0} 1. x^n/n! \\ &= \Sigma_{n\geq 0} x^n/n! \\ &= e^x \end{aligned}$$

Example 2: Species of subsets

Consider the species $\wp$ of subsets and look at f_n.
f_n = the number of $\wp$-structures on a set of n elements

f_n = the number of subsets of a set of n elements
$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$
$= (1+1)^n$
$= 2^n$

Hence $\wp(x) = \Sigma_{n\geq 0}\, 2^n\, x^n/n!$
$= \Sigma_{n\geq 0}\, (2x)^n/n!$
$= e^{2x}$

Example 3: Species of permutations

Consider the species of permutations S. To find its formal power series look at
f_n = the number of S-structures on a set of n elements
= the number of permutations on n elements
$= n!$

Hence $S(x) = \Sigma_{n\geq 0}\, n!\, x^n/n!$
$= \Sigma_{n\geq 0}\, x^n$
$= 1/(1-x)$ (sum of a geometric series)

Example 4: Species of linear orderings

Consider the species of linear orderings L. To find its formal power series we need to find the number of linear orderings on a set of n elements. The number of ways we can order n elements or arrange n elements is precisely n!.

Hence $L(x) = \Sigma_{n\geq 0}\, n!\, x^n/n!$
$= \Sigma_{n\geq 0}\, x^n$
$= 1/(1 - x)$

1.3. Combinatorial Equality

Looking at the previous two examples above, we should note that even though the formal power series of the species of permutations is the same as the formal power series of the species of linear orderings, these species are not the same. This can be further illustrated by noting that if $x_1x_2\ldots x_n$ and $y_1y_2\ldots y_n$ are linear orderings on the set of n elements, then there is an isomorphism $f : x_i \to y_i$ between the two orderings. So $L[U] \cong S[U]$ but $L[\sigma]$ and $S[\sigma]$ on σ do not correspond under this bijection; the bijection $L[U] \cong S[U]$ requires a choice of linear ordering of U. In fact there are strong and weak notions of combinatorial equality which we could apply to species of structures.

We will discuss various concepts equivalence for species of structures.

A strong definition of equality is to require that two species are equal if they have the same structures and the same transports. That is, two species F and G are equal if for all finite sets U, $F[U] = G[U]$ and for all bijections $\sigma : U \to V$, $F[\sigma] = G[\sigma]$. However this form of equality is very strict and too restrictive.

A weaker definition of equality is one of equipotence where we define the equipotence of structures by bijections. We say two species F and G are equipotent if there are the same number of F-structures as G-structures. More formally we have the following definition.

Definition 1: Two species F and G are equipotent if for each finite set U there exists a bijection $\alpha_U : F[U] \to G[U]$ and we write $F \equiv G$. A family of bijections α_U is called an equipotence.

So instead of requiring that $F[U] = G[U]$ we weaken this by requiring that there is a bijection from F[U] to G[U]. An example of two equipotent species are the species of permutations S and the species of linear orderings L since $\#S = \#L = n!$ on a set of n elements. Note that even though S and L are equipotent, we saw previously that they are different species. Hence equipotence is a weaker definition of equality and is only useful when we are looking at the enumeration of structures. Otherwise, it is too weak.

A "good" definition of equality is the concept of isomorphism of species. The concept of isomorphism of species lies half way between the strong and weak definitions of equality. For two species F and G to be isomorphic we require that in addition to the family of bijections $\sigma_U : F[U] \to G[U]$ for each finite set U, we also need an equivalent definition relating the transport of the structures F and G. This is called the *naturality condition*.

Definition 2: Two species F and G are isomorphic when for each finite set U, there is a bijection $\alpha_U : F[U] \to G[U]$ which satisfies the naturality condition: for each bijection $\xi : U \to V$, the following diagram commutes

$$\begin{array}{ccc} F[U] & \xrightarrow{\alpha_U} & G[U] \\ {\scriptstyle F[\xi]}\downarrow & & \downarrow{\scriptstyle G[\xi]} \\ F[V] & \xrightarrow[\alpha_V]{} & G[V] \end{array}$$

We write $F \cong G$ when F and G are isomorphic.

The diagram above represents the naturality condition and this is needed for two species to be isomorphic. This means that for any $s \in F[U]$, we must have $\xi.\alpha_U(s) = \alpha_V(\xi.s)$. We can see that if two species are isomorphic then they are equipotent but the converse is not true.

Contact of order n is a topological notion of equality between species of structures for $n \geq 0$ which looks at constructing species by approximations. To understand this notion we will first look at contact of order n on two formal power series.

Let $a(x) = \sum_{n \geq 0} a_n x^n$, $b(x) = \sum_{n \geq 0} b_n x^n$ and let $a_{\leq n}(x) = \sum_{0 \leq k \leq n} a_n x^n$, $b_{\leq n}(x) = \sum_{0 \leq k \leq n} b_n x_n$. We say two formal power series a(x) and b(x) have contact of order n, written $a(x) =_n b(x)$, if for all $k \geq n$, $a(x) =_n b(x)$ iff $a_{\leq n}(x) = b_{\leq n}(x)$. The definition of contact of order n for a species of structures is analogous to that of series.

Definition 3: Let F and G be two species of structures and let $n \geq 0$ be an integer. If $F_{\leq n} = G_{\leq n}$ then we say F and G have contact of order n, written $F =_n G$ where $F_{\leq n}$ is the restriction of F to sets of cardinality $\leq n$. This means that for finite sets U and V, and bijections $\sigma : U \rightarrow V$, we have

$$F_{\leq n}[U] = \varnothing \qquad \text{if } \#U > n$$

$$F_{\leq n}[U] = F[U] \text{ and } F_{\leq n}[\sigma] = F[\sigma] \qquad \text{if } \#U \leq n$$

Chapter2

Operations on Species

Operations can be performed on a species of structures to produce new species of structures. These operations can be defined by using set theoretical constructions and most of the operations we will look at are analogous to the usual arithmetic operations of addition, multiplication and so on. We will investigate these different operations on species of structures to see what types of species they produce and how we are able to construct them.

2.0. Addition of Species

We will look at adding two species F and G to get an (F+G)-structure.

Definition 1: Let F and G be two species. The sum of F and G is the species F+G where an (F+G)-structure on a finite set U is either an F-structure on U or a G-structure on U.

More precisely, suppose s is an (F+G)-structure on U, then
1. For any finite set U, $(F+G)[U] = F[U] + G[U]$ (disjoint union)
2. For any bijection $\sigma : U \rightarrow V$ between finite sets U and V we have

$$(F+G)[\sigma](s) = \begin{cases} F[\sigma] & \text{if } s \in F[U] \\ G[\sigma] & \text{if } s \in G[U] \end{cases}$$

In this definition $F[U] + G[U]$ represents the disjoint union of F-structures and G-structures. However when $F[U] \cap G[U] \neq \varnothing$ we have to approach this slightly differently. We first have to rewrite F and G in such a way that they are disjoint. This is done by replacing $F[U]$ by $F[U] \times \{1\}$ and $G[U]$ by $G[U] \times \{2\}$ and then setting $(F+G)[U] = (F[U] \times \{1\}) \cup (G[U] \times \{2\})$. Now we have two disjoint sets whose sum is their disjoint union. We should note that addition is associative and commutative up to isomorphism and that the empty species 0 is a neutral element for addition. That is, $F + 0 = 0 + F = F$. We can illustrate addition of species by the following diagram.

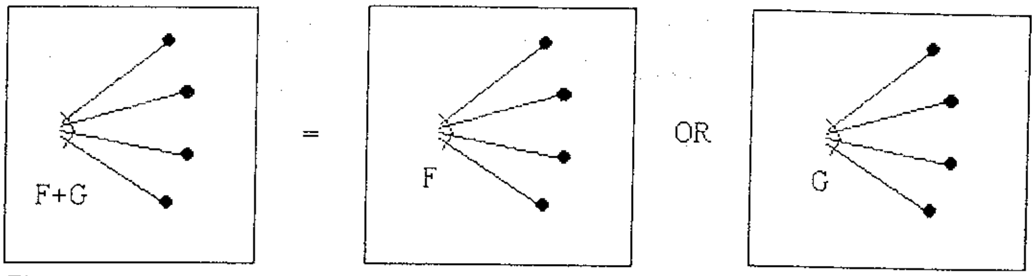


Fig. 1.

Example 1: Let G be the species of simple graphs, and denote G_c as the species of connected graphs and G_d as the species of disconnected graphs. On any finite set U, we can say the $G[U] = G_c[U] + G_d[U]$ since any simple graph is either connected or disconnected. Therefore a G-structure is either a G_c-structure or a G_d-structure and we write $G = G_c + G_d$.

Example 2: Consider the species E of sets. Let E_{even} be the species of sets which contain an even number of elements and let E_{odd} be the species of sets which contain an odd number of elements. So E_{even} is a one element set when E has an even number of elements and it is empty when it has an odd number of elements. Similarly, E_{odd} is a one element set when E has an odd number of elements and is empty when it has an even number of elements. Now any set has cardinality which is either even or odd. Therefore an E-structure on a finite set U is either an E_{even}-structure or an E_{odd}-structure. That is, $E = E_{even} + E_{odd}$.

Addition can be defined for a family of species $(F_i)_{i\in I}$.

Definition 2: A family of species $(F_i)_{i\in I}$ is summable if for any finite set U, $F_i[U] = \varnothing$ except for a finite number of $i\in I$. For any finite set U and bijection $\sigma : U \to V$ between finite sets U and V, the sum $\Sigma_{i\in I} F_i$ is defined as

$$(\Sigma_{i\in I} F_i)[U] = \Sigma_{i\in I} F_i[U] = \cup_{i\in I} F_i[U] \times \{i\}$$
$$(\Sigma_{i\in I} F_i)[\sigma](s,i) = (F_i[\sigma](s),i)$$

where $(s,i) \in (\Sigma_{i\in I} Fi)[U]$

We saw previously that a species of structures F gives rise to a sequence of sets F[**0**],F[**1**],F[**2**],....Hence each species gives rise to a countable family of species $(F_n)_{n\geq 0}$ defined by

$$F_n[U] = \begin{cases} F[U] & \text{if } \#U = n \\ \varnothing & \text{otherwise} \end{cases}$$

This family is clearly summable and we can write $F = F_0 + F_1 + F_2 + \ldots + F_k + \ldots$ called the canonical decomposition of the species F. The finite sum $F + F + F + \ldots + F$ is the sum of n copies of the same species F and we write this as nF.

Example 3: Consider the species E of sets and let E_n be the species of sets of cardinality n. Then we can write E as the decomposition $E = E_0 + E_1 + E_2 + \ldots + E_n + \ldots$. Note the $E_0 = 1$, $E_1 = X$ where X is the species of singletons, and so on.

Example 4: Consider the species S of permutations and let $S^{(k)}$ be the species of permutations having exactly k cycles. Then it is easy to see that we can write S as the decomposition $S = S^{(0)} + S^{(1)} + S^{(2)} + \ldots + S^{(k)} + \ldots$

We can also express the associated generating series of a species using the operation of addition.

Proposition 1: Given two species of structures F and G, the generating series of the species F + G satisfies $(F + G)(x) = F(x) + G(x)$.

Proof: Suppose the species F has generating series $F(x) = \Sigma_{n\geq 0}\, f_n\, x^n/n!$ and the species G has series $G(x) = \Sigma_{n\geq 0}\, g_n\, x^n/n!$.

Let the species F + G have series $(F + G)(x) = \Sigma_{n\geq 0}\, h_n\, x^n/n!$.

Now h_n = number of (F + G)-structures on n elements

= #F-structures + #G-structures (since $(F + G)[U]$ = disjoint union of $F[U]$ and $G[U]$ for any finite set U)

$= f_n + g_n$

Therefore
$$\begin{aligned}(F + G)(x) &= \Sigma_{n\geq 0}\, (f_n + g_n)\, x^n/n! \\ &= \Sigma_{n\geq 0}\, (f_n\, x^n/n! + g_n\, x^n/n!) \\ &= \Sigma_{n\geq 0}\, f_n\, x^n/n! + \Sigma_{n\geq 0}\, g_n\, x^n/n! \\ &= F(x) + G(x).\end{aligned}$$

■

2.1. Multiplication of Species

Definition 1: The product F.G of two species of structures F and G is defined as follows: an (F.G)-structure on a finite set U is an ordered pair s = (f,g) where

i) f is an F-structure on $U_1 \subseteq U$

ii) g is a G-structure on $U_2 \subseteq U$

and (U_1,U_2) is a decomposition of U meaning $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \varnothing$

More precisely, for any finite set U and bijection $\sigma : U \to V$ between finite sets U and V we have the following:

$(F.G)[U] = \Sigma_{(U1,U2)} F[U_1] \times G[U_2]$
which is a finite sum taken over all decompositions (U_1,U_2) of U.

$(F.G)[\sigma](s) = (F[\sigma_1](f), G[\sigma_2](g))$
where $s = (f,g)$ is an (F.G)-structure on U and σ_i is σ restricted to U_i for each $i = 1,2$.

When multiplying species we should note that 1 is the neutral element, ie: $F.1 = 1.F = F$, and 0 is the absorbing element, ie: $F.0 = 0.F = 0$. Although multiplication is associative and commutative up to isomorphism, in general F.G and G.F are not identical. Multiplication of species can be illustrated in the following diagram.

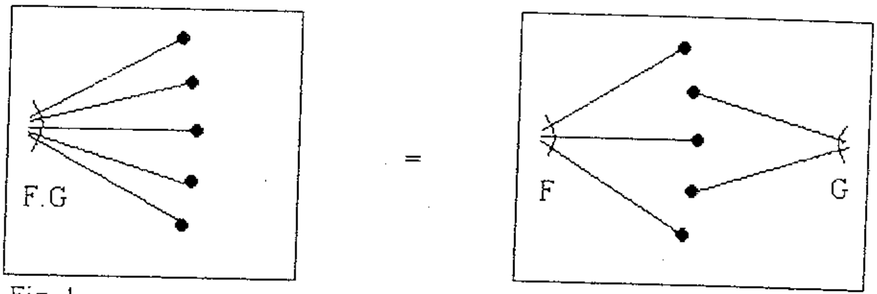


Fig. 1.

We saw earlier that $F + F + ... + F = nF$, but this is precisely the product of the species **n** with the species F. (Note that the species **n** is as in example 10 with $W = \mathbf{n}$). Hence addition of n copies of the species F is the same as multiplying the species **n** with the species F. This also justifies our use of identifying the integer n with the species **n**.

Example 1: Consider the following illustration of a permutation.

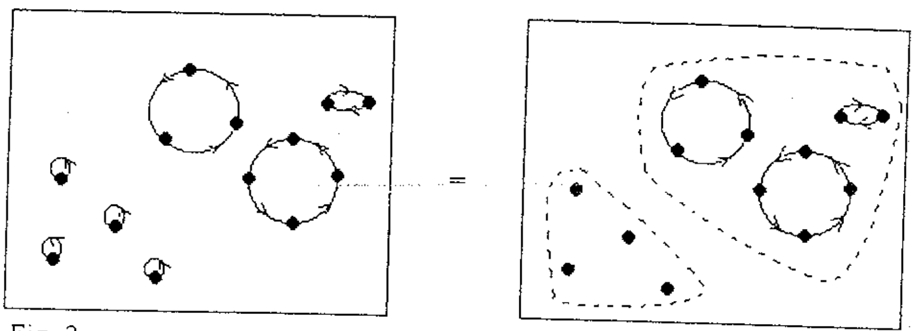


Fig. 2.

We can divide this structure into two disjoint structures one of which is the set of fixed points and the other is a structure of nontrivial cycles or derangements. This way of expressing a permutation can be done for all possible permutations. The second diagram above clearly illustrates this and we can write S = E.Der, where S is the species of permutations. E is the species of sets and Der is the species of derangements.

Example 2: Consider the species E of sets. An E-structure on a finite set U is the set U itself. We want to multiply two of these species E together. Let U be a finite set and let $U = U_1 \cup U_2$ where $U_1 \cap U_2 = \varnothing$. Then an E structure on U_1 is the set U_1 itself, and an E-structure on U_2 is the set U_2 itself. Therefore an (E.E)-structure on U divides U into two subsets U_1 and U_2. That is, we get a $\wp$-structure of cardinality two. Hence we can say that an (E.E)-structure is a $\wp$-structure where $\wp$ is the species of subsets (of sets). That is, $\wp = E.E$.

The generating series of a species can be expressed using the multiplication operation.

Proposition 1: Given two species of structure F and G, the associated series for the species F.G is given by $(F.G)(x) = F(x)G(x)$.

Proof: Suppose the species F has generating series $F(x) = \Sigma_{n\geq 0} f_n\, x^n/n!$ and the species G has series $G(x) = \Sigma_{n\geq 0} g_n\, x^n/n!$. Let the species F.G have series

$(F.G)(x) = \Sigma_{n\geq 0} h_n\, x^n/n!$

Now h_n = number of (F.G)-structures on n elements

= F-structure on k elements × G-structure on n-k elements for $0 \leq k \leq n$

$= {}^nC_0\, f_0 g_n + {}^nC_1\, f_1 g_{n-1} + {}^nC_2\, f_2 g_{n-2} + \ldots + {}^nC_n\, f_n g_0$

$= \Sigma_{0\leq k\leq n}\, {}^nC_k\, f_k g_{n-k}$

Then $F(x)G(x) = (\Sigma_{n\geq 0} f_n\, x^n/n!)(\Sigma_{n\geq 0} g_n\, x^n/n!)$

$= \Sigma_{n\geq 0} (\Sigma_{0\leq k\leq n}\, {}^nC_k\, f_k g_{n-k})\, x^n/n!$ (by usual series multiplication)

$= (F.G)(x)$

■

Example 3: Consider the species $\wp$ of power sets. We saw earlier that $\wp = E.E$ where E is the species of sets. Therefore using proposition 2, we can say that $\wp(x) = E(x)E(x)$. We saw earlier that $E(x) = e^x$ and hence $\wp(x) = e^x e^x = e^{2x}$ as we also saw earlier.

2.2. Substitution of Species

Definition 1: Suppose F and G are two species where $G[\varnothing] = \varnothing$ (ie: there is no G-structure on the empty set). The (partitional) composite of G in F, written $F \circ G$ (or F(G)) is defined as follows: an $(F \circ G)$-structure on a finite set U is a structure $s = (\pi, \varphi, \gamma)$ where

i) π is a partition of U

ii) φ is an F-structure on the set of classes of π

iii) $\gamma = (\gamma_p)_{p\in\pi}$ is a G-structure on p for each class p of π

More precisely, for any finite set U and bijection $\sigma : U \to V$ between finite sets U and V we have the following,

$$(F \circ G)[U] = \Sigma_\pi F[\pi] \times \Pi_{p \in \pi} G[p]$$
$$(F \circ G)[\sigma](s) = (\pi^*, \varphi^*, (\gamma^*_{p^*})_{p^* \in \pi^*})$$

where i) π^* is a partition of V obtained by the transport of π along σ

ii) for each $p^* = \sigma(p) \in \pi$, the structure $\gamma^*_{p^*}$ is obtained by the G-transport of γ_p along $\sigma|_p$

iii) the structure φ^* is obtained by the F-transport of φ along σ^* which is the map induced on π by σ.

Essentially, the partitional composite of a species G in F is a substitution of a G-structure into an F-structure. The following diagram represents this concept.

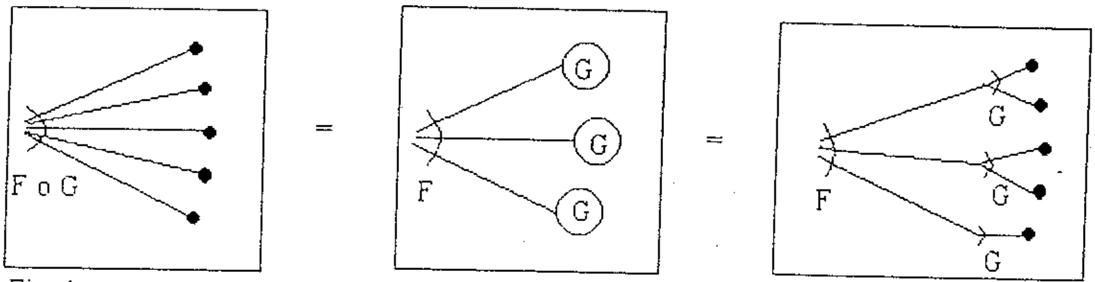


Fig. 1.

Example 1: Consider the species of endofunctions End and take End[U] on a finite set U described by the following diagram below. We can separate the points in U by considering the recurrent points, ie: $x \in U$ such that there exists $k > 0$ with $\varphi^k(x) = x$, and the nonrecurrent points, ie. $x \in U$ such that $\varphi^k(x) \neq x$ for all $k > 0$. From the second diagram below we can see that the recurrent points form permutations and the nonrecurrent points form rooted trees where each point on a permutation forms the root of a rooted tree. Hence the function φ can be identified with a permutation of disjoint rooted trees. We can do this process no matter which endofunction is given. Hence every End-structure can be considered as a set of rooted tree structures placed on a structure of permutations. That is, an End-structure is an A-structure substituted into a S-structure and we see that End = S $\circ$ A where S is the species of permutations and A is the species of rooted trees.

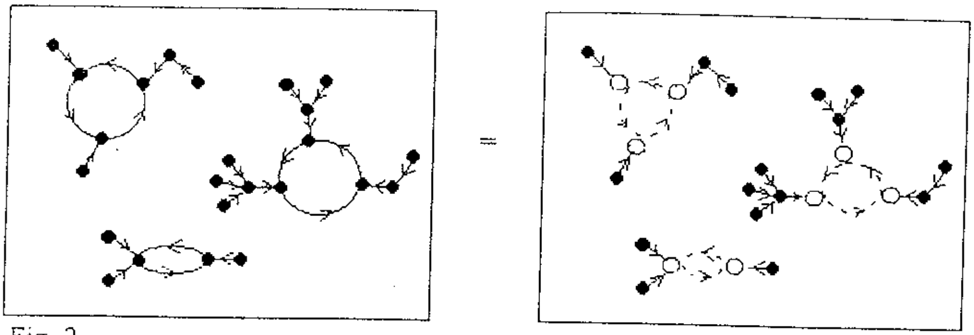


Fig. 2.

We now give a more formal proof to show that End $\cong$ S $\circ$ A.

Proposition 1: End $\cong$ S $\circ$ A

Proof: We can think of a rooted tree on a set U as a pair (a,ψ) where $a \in U$ is the root and $\psi : U\backslash\{a\} \to U$ is a function with no recurrent elements.
Then put $U_0 = \{a\}$, $U_1 = \psi^{-1} U_0$, $U_2 = \psi^{-1} U_1$, ... so that for some n, $U_n = \varnothing$. So we have surjective functions

$$U_n \xrightarrow{\psi} U_{n-1} \xrightarrow{\psi} \ldots \xrightarrow{\psi} U_1 \xrightarrow{\psi} U_0 = \{a\}.$$

Hence we get a rooted tree on U with root a where the edges are given by the function ψ. The following diagram illustrates this.

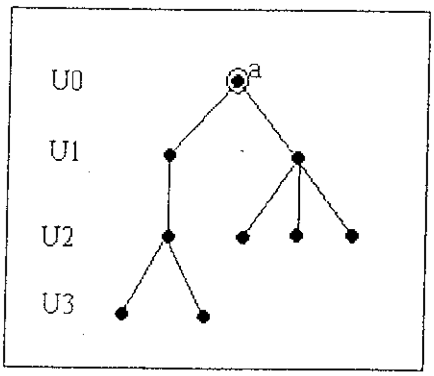


Fig. 3.

So we can write the species of rooted trees on a set U as
$A[U] = \{(a,\psi) : (a,\psi) \text{ is a rooted tree on } U\}$.
We can also write the species of permutations on a set U as
$S[U] = \{\sigma : U \to U : \sigma \text{ is invertible}\}$.
So $S(A)[U] = \{(\pi,\sigma,\gamma) : \pi \text{ is a partition of } U, \sigma : \pi \to \pi \text{ is invertible},$
$\gamma = (\gamma_p)_{p\in\pi}, \gamma_p = (a_p, \psi_p) \in A[p]\}$.

Here π is a partition, σ is a permutation on π and γ is a rooted tree structure on p for each class p of π.
To prove End = S $\circ$ A, define the bijection $\alpha_U : S \circ A[U] \to End[U]$ by $\alpha_U(\pi,\sigma,\gamma) = \varphi : U \to U$ by, for $u \in p \in \pi$,

$$\varphi(u) = \begin{cases} \psi_p(u) & \text{for } u \neq a_p \\ a_q & \text{for } u = a_p \text{ and } \sigma(p) = q \end{cases}$$

So if u is not the root, then $\varphi(u)$ is the element that ψ maps u to, and if u is the root then $\varphi(u)$ is the next element which the root gets mapped to under the permutation.
For the inverse of α, take a function $\varphi : U \to U$.
For each $u \in U$, there is a smallest $k>0$ such that $\varphi^k(u)$ is recurrent and let $r(u) = \varphi^k(u)$.
Therefore we get a partition π of U made up of the sets $p_x = \{x\} \cup r^{-1}(x)$ for x recurrent. So p_x contains the recurrent point together with all the other points which map to that recurrent point.
Define $\sigma : \pi \to \pi$ by $\sigma(p_x) = p_{\varphi(x)}$. So σ is a permutation of the partitions, it takes a rooted tree to the tree which the original σ would take the root to.
Define $\gamma \in \Pi_{x \text{ recurrent}} A[p_x]$ by $\gamma_x = (x, \psi_x)$ where $\psi_x : r^{-1}(x) \to p_x$ and $\psi_x(u) = \varphi(u)$.
So γ_x is a rooted tree structure on p_x and it is defined by the pair (x, ψ_x) where ψ_x defines the structure of the tree.
Therefore $\alpha_U^{-1}(\varphi) = (\pi,\sigma,\gamma)$ so α_U is an isomorphism.

■

Example 2: Consider the species G of graphs and let G_c be the species of connected graphs. Since every graph is an assembly of connected graphs we can say that $G = E \circ G_c$ where E is the species of sets. That is, for each element of an E-structure a G_c-structure is substituted into it giving a collection of connected graphs which is a G-structure.

We can also look at the effect of substitution of species on the generating series.

Proposition 2: Given two species of structure F and G, the generating series for the species F $\circ$ G is given by $(F \circ G)(x) = F(G(x))$.

Proof: Suppose the species F has generating series $F(x) = \Sigma_{n\geq 0} f_n x^n/n!$ and the species G has series $G(x) = \Sigma_{n\geq 0} g_n x^n/n!$.
Let the species F $\circ$ G have series $(F \circ G)(x) = \Sigma_{n\geq 0} h_n x^n/n!$
Now h_n = number of (F $\circ$ G)-structures on n elements. We can take a partition of n elements as $n_1 + n_2 + \ldots + n_k = n$ for $0 \leq k \leq n$ and let f_k be the number of

F-structures on k elements and g_{ni} be the number of G-structures on each class n_i of k where $0 \leq i \leq k$.

Then we have $h_n = \Sigma\, n!/(k!n_1! \ldots n_k!)\, f_k g_{n1} \ldots g_{nk}$ where this sum is taken over $0 \leq k \leq n$ such that $n_1 + \ldots + n_k = n$.

Now for any series $a(x) = \Sigma_{n\geq 0}\, a_n\, x^n/n!$ and $b(x) = \Sigma_{n\geq 0}\, b_n\, x^n/n!$ the coefficients c_n where $c(x) = (a \circ b)(x)$ is given by $c_n = \Sigma\, n!/(k!n_1! \ldots n_k!)\, a_k b_{n1} \ldots b_{nk}$ where this sum is taken over $0 \leq k \leq n$, $n_1 + \ldots n_k = n$.

Therefore we can conclude that $(F \circ G)(x) = F(G(x))$.

■

2.3. Derivative of a Species

Definition 1: Let F be a species if structures. The species F´ called the derivative of F is defined as follows: an F´-structure on a finite set U is an F-structure on the set $U^+ = U \cup \{*\}$ where $* = *_U$ is an element chosen outside the set U.

More precisely, for any finite set U and bijection $\sigma : U \to V$ between finite sets U and V we have the following:

$$F'[U] = F[U^+]$$

$$F'[\sigma](s) = F[\sigma^+](s) \text{ where } \sigma^+ : U \cup \{*\} \to V \cup \{*\} \text{ such that}$$

$$\sigma^+(u) = \sigma(u) \text{ if } u \in U \text{ and } \sigma^+(*) = *$$

This definition arises from the fact that we need $\#F[n] = \#F[n+1]$. We should note that $*$ is not an element of U but is chosen outside of this set and the set $U \cup \{*\}$ is not otherwise structured. It is always possible to choose $* = U$ itself since $U \notin U$ and therefore we will have $U^+ = U \cup \{U\}$. The derivative of a species of structures can be illustrated by the following diagram.

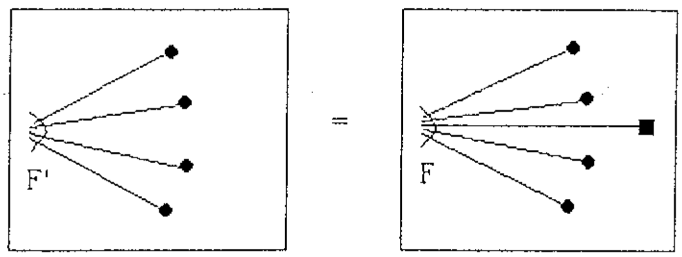


Fig. 1.

Example 1: Consider the species of cyclic permutations C and let U = {a,b,c,d,e}. A C´-structure on U is a C-structure on $U^+ = U \cup \{*\}$. Now a C-structure on U^+ is a cyclic permutation, so a C´-structure on U will be this cyclic permutation excluding the point $*$. Since removing a point on a cycle gives a linear ordering of the remaining elements, we can say the a C'-structure on U is a linear ordering on U and we write $C' = L$ where L is the species of linear orderings. This can be seen in the diagram below. So we have found an " antiderivative " for C.

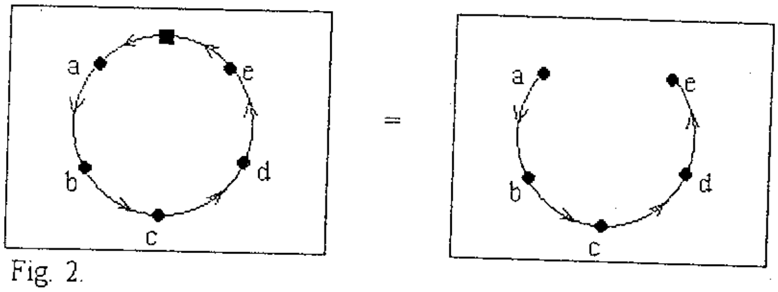


Fig. 2.

Example 2: Consider the species of rooted trees A. An A'-structure on a set U is an A-structure on the set $U+ = U \cup \{*\}$, which is a rooted tree with root $*$. Removing $*$ we get a number of rooted trees. So an A'-structure is an A-structure on the species of sets, E. In other words $A' = E \circ A$ showing that A satisfies a " differential equation ". This is illustrated in the diagram below.

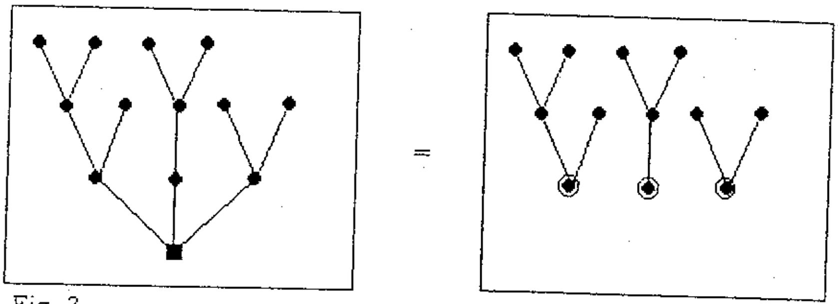


Fig. 3.

We can differentiate a species more than once. For example to find the second derivative of a species of structures we add two distinct elements $*_1$ and $*_2$ which do not belong to U. In general we have the following definition.

Definition 2: Let F be a species of structures. The species $F^{(k)}$ called the kth derivative of F is defined as follows: an $F^{(k)}$ -structure on a finite set U is an F-structure on the set $U \cup \{ *_1, *_2, \ldots, *_k\}$ where $*_i$, $i = 1,2,\ldots,k$ is an ordered sequence of k additional distinct elements.

We can also express the derivative of a species of structure as a generating series.

Proposition 1: Given a species of structure F, the generating series for the species F' is given by $F'(x) = d/dx\,(F(x))$.

Proof: Suppose the species F' has generating series $F'(x) = \Sigma_{n\geq 0} f_n x^n/n!$. We know that an F'-structure on n elements is an F-structure on n+1 elements. Then the generating series for the species F is $F(x) = \Sigma_{n\geq 0} f_n x^{n+1}/(n+1)!$.

$$\begin{aligned} \text{Now } d/dx\,(F(x)) &= d/dx\,(\Sigma_{n\geq 0} f_n x^{n+1}/(n+1)!) \\ &= \Sigma_{n\geq 0} f_n\, d/dx\,(x^{n+1}/(n+1)!) \\ &= \Sigma_{n\geq 0} f_n\,(n+1)x^n/(n+1)! \\ &= \Sigma_{n\geq 0} f_n\, x^n/n! \\ &= F'(x) \end{aligned}$$

■

2.4. Pointing Operation

The pointing operation (•) interprets combinatorially the operator x d/dx.

Definition 1: Let F be a species of structure. The species $F^\bullet$ called F dot is defined as follows:
an $F^\bullet$-structure on a finite set U is a pair s = (f,u) where
i) f is an F-structure on U,
ii) u is an element of U called a distinguished element.
We call the pair s = (f,u) a pointed F-structure (pointed at the distinguished element u).

So $F^\bullet$ is the rule which,
1. produces $F^\bullet[U] = F[U] \times U$ (set theory cartesian product)
2. produces $F^\bullet[\sigma] : F^\bullet[U] \to F^\bullet[V]$ for each bijection $\sigma : U \to V$ where
$F^\bullet[\sigma](s) = (F[\sigma](f), \sigma(u))$ for s = (f,u).

The number of F-structures on a set of n elements satisfies $\#F[\mathbf{n}] = n\#F[\mathbf{n}]$.

Example 1: Let us look at the species $\mathcal{a}$ of trees. When we point an $\mathcal{a}$-structure we simply add a distinguished element to this tree. Since a rooted tree is a tree with a distinguished element the root, then pointing the species $\mathcal{a}$ of trees gives us the species A of rooted trees, hence $\mathcal{a}^\bullet = A$. The diagram below describes this situation.

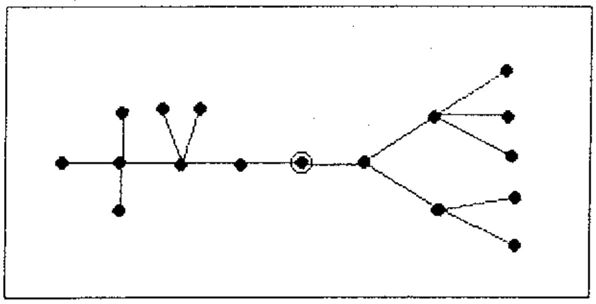


Fig. 1.

We should realise that there is a difference between u and the element $*$ which comes up when we look at the derivative of a species. The distinguished element u belongs to the underlying set U whereas $*$ does not belong to U. In fact the operations of pointing and derivation are related. The species $F^{\bullet}$ can also be expressed as the product of the species X of singletons together with the species F'. We can write the combinatorial equation $F^{\bullet} = X \,.\, F'$. This can be better understood by looking at the diagram below. The distinguished element in the $F^{\bullet}$-structure is can be moved aside and replaced by the element $*$ which is not in the underlying set, and hence we are left with an $X \,.\, F'$-structure.

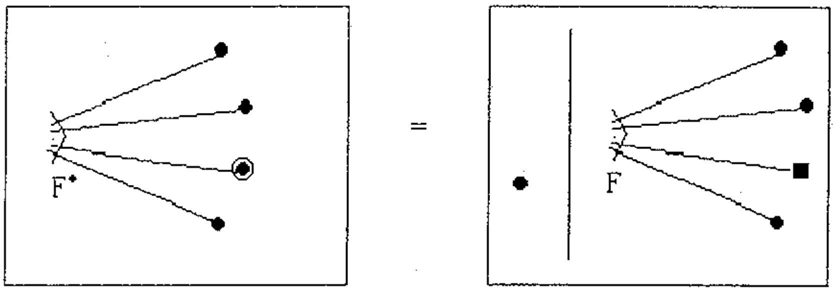


Fig. 2.

The effect of this pointing operation on the formal power series is expressed in the following proposition.

Proposition 1: Given a species of structure F, the generating series for the species $F^{\bullet}$ is given by $F^{\bullet}(x) = x\, d/dx\, F(x)$.

Proof: Suppose the species F has generating series $F(x) = \Sigma_{n\geq 0} f_n x^n/n!$ and let the species $F^{\bullet}$ have series $F^{\bullet}(x) = \Sigma_{n\geq 0} h_n x^n/n!$

where h_n = number of $F^{\bullet}$-structures on a set of n elements

$= n \times$ #F-structures on n elements

$= nf_n$

So $F^{\bullet}(x) = \Sigma_{n\geq 0} nf_n x^n/n!$.

Now $x\, d/dx(F(x)) = x\, d/dx(\Sigma_{n\geq 0} f_n x^n/n!)$

$= \Sigma_{n\geq 0}\, nf_n\, x^n/n!$
$= F^{\bullet}(x)$

■

Pointing is useful for counting techniques. Let us look at the following example to demonstrate this.

Example 2: Consider tree structures on a set of n elements. We want to determine the number of tree structures there are.

Let α_n = number of trees on a set of n elements.

Let $\mathcal{a}$ be the species of trees and A be the species of rooted trees.

Define $v = \mathcal{a}^{\bullet\bullet} = A^{\bullet}$ = species of vertebrates which is a pointed rooted tree.

Recall, in the previous example we saw that a pointed tree is a rooted tree.

Therefore a vertebrate is a pointed tree which has the pointing operation applied to it again.

An element of v is a "vertebral column" as in the diagram below.

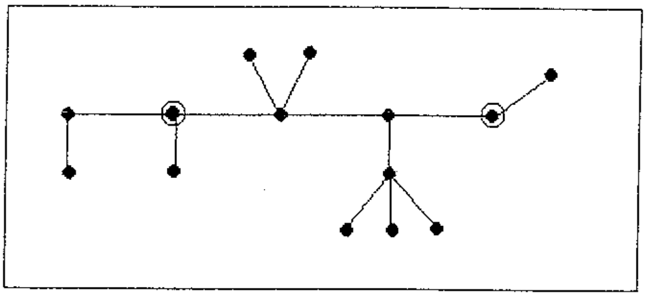


Fig. 3.

The first distinguished element is called the tail vertex and the second distinguished element is called the head vertex. Note that the tail and head can coincide.

Let v_n = number of vertebrates on a set of n elements.

Therefore $v_n = n^2\alpha_n$. The n^2 occurs since there are n possible choices for the head and n choices or the tail. Another way of calculating v_n is the following:

Notice that $v = L^+(A)$ where L^+ is the species of linear orders.

Therefore we can write $v = S^+(A)$ since L^+ and S^+ are equipotent (as seen in a previous example). Note that since we are only concerned with the number of structures in this example, we can use equipotency since two species are equipotent if they have the same number of structures.

But then we can write $v = End^+$ since we saw previously that End = S(A).

Therefore ν_n = number of endofunctions on n elements
$= n^n$ (since $\#End[n] = n^n$)

Hence $\nu_n = n^2\alpha_n = n^n$, and we can conclude that $\alpha_n = n^{n-2}$.

2.5. Cartesian Product

Definition 1: Suppose F and G are two species of structures. The cartesian product $F \times G$ of these two species is defined as follows: an $(F \times G)$-structure on any finite set U is a pair $s = (f,g)$ where

i) f is an F-structure on U, ie: $f \in F[U]$

ii) g is a G-structure on U, ie: $g \in G[U]$

In other words, the species $F \times G$ is a rule such that it

1. produces $(F \times G)[U] = F[U] \times G[U]$ for any finite set U,
2. produces $(F \times G)[\sigma] : (F \times G)[U] \to (F \times G)[V]$ where $\sigma : U \to V$ is a bijection such that $(F \times G)[\sigma](s) = (F[\sigma](f), G[\sigma](g))$ for any $s = (f,g) \in (F \times G)[U]$.

The cartesian product $F \times G$ is different to the normal multiplication of species F.G since an $(F \times G)$-structure consists of a pair $s = (f,g)$ where each f and g are structures on the entire set U, whereas in a (F.G)-structure we first have to decompose U into a disjoint union U_1 and U_2 and the pair $s = (f,g)$ has structures which are on U_1 and U_2 respectively. In other words an $(F \times G)$-structure is obtained by the superposition of both an F-structure on U and a G-structure on U. The diagram below illustrates this.

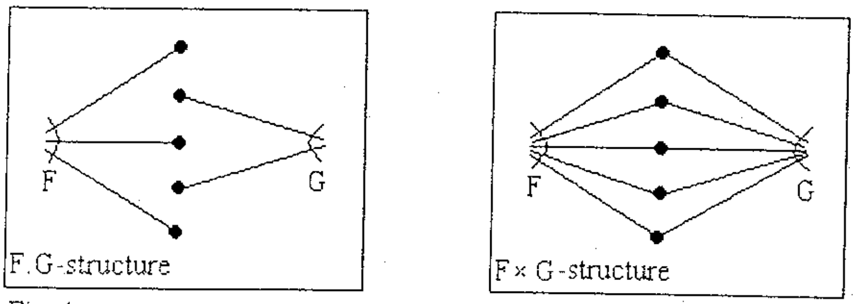
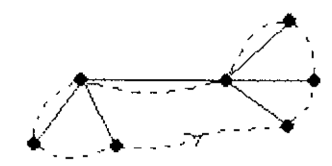
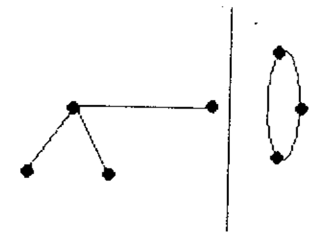


Fig. 1.

Example 1: Consider the species C of oriented cycles and the species $\mathcal{a}$ of trees. The diagrams below show the difference between an $(\mathcal{a} \times C)$-structure and an $(\mathcal{a}.C)$-structure. Notice that $(\mathcal{a} \times C)$-structure is obtained by the superposition of both an $\mathcal{a}$-structure on U and a C-structure on U. From the diagrams we can see that the set U remains the same but we get two very different structures from $\mathcal{a} \times C$ and $\mathcal{a}.C$.



$\mathcal{A} \times C$-structure



$\mathcal{A} . C$-structure

Fig. 2.

We can obtain the generating series for the cartesian product of two species.

Proposition 1: Given two species of structure F and G, the cartesian product $(F \times G)(x)$ of species F and G has series equal to the coefficient-wise product of their series

Proof: Suppose the species F has generating series $F(x) = \Sigma_{n \geq 0} f_n x^n/n!$ and species G has series $G(x) = \Sigma_{n \geq 0} g_n x^n/n!$.

Let the species $F \times G$ have generating series $(F \times G)(x) = \Sigma_{n \geq 0} h_n x^n/n!$ where h_n is the number of $(F \times G)$-structures on a set of n elements. This can be expressed as $\#(F \times G)[n] = \#F[n]\#G[n]$ and therefore $h_n = f_n g_n$.

So $(F \times G)(x) = \Sigma_{n \geq 0} f_n g_n x^n/n!$.

The cartesian product of species F and G corresponds to the coefficient-wise product of their series. That is,

$$\begin{aligned}(\Sigma_{n \geq 0} f_n x^n/n!) \times (\Sigma_{n \geq 0} g_n x^n/n!) &= \Sigma_{n \geq 0} f_n g_n x^n/n! \\ &= (F \times G)(x)\end{aligned}$$

•

The neutral element for the cartesian product is the species E of sets. That is, if F is any species of structure then $E \times F = F \times E = F$. This occurs because $F \times E$ is a pair $s = (f,g)$ where f is an F-structure on U and g is an E-structure on U, but an E-structure is just the set U itself. Therefore a $(F \times E)$-structure is the pair $s = (f,U)$ which is an F-structure on the set U together with the set U itself. This is just an F-structure on U. Hence we can conclude that $E \times F = F \times E = F$.

Example 2: Let C be the species of oriented cycles and $\wp$ be the species of subsets (of sets). For any finite set U, a $(C \times \wp)$-structure is the superposing of a C-structure on U and a $\wp$-structure on U. Since a $\wp$-structure on U is a subset of U, then a $(C \times \wp)$-structure is an oriented cycle on which is superimposed a subset of U. This is precisely an oriented cycle on U with certain elements which are distinguished. (The distinguished elements are those which belong to the $\wp$-structure, ie: a subset of U). The following diagram describes this.

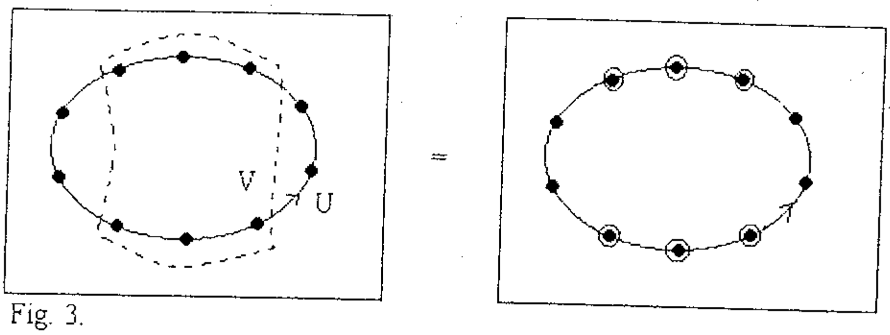


Fig. 3.

There is a relationship between the pointing operation and the cartesian product. Applying the cartesian product and then the pointing operation to two species F and G gives the following result:

$$(F \times G)^{\bullet} = F^{\bullet} \times G = F \times G^{\bullet}$$

In particular, $F^{\bullet} = (F \times E)^{\bullet} = F \times E^{\bullet}$

We can write F this way since E is the neutral element when applying the cartesian product. Since for any species F, $F^{\bullet} = X.F'$, then we can write $E^{\bullet} = X.E'$. Now an E'-structure on a set U is an E structure on a set $U \cup \{*\}$. Since E is the species of sets and an E-structure on a set is the set itself, then an E-structure on $U \cup \{*\}$ is $U \cup \{*\}$, and therefore an E'-structure on U is the set U. Therefore $E' = E$ and we can write $E^{\bullet} = X.E$. Now we can say $F^{\bullet} = F \times (X.E)$. In this way we have expressed the pointing operation in terms of the cartesian product and the normal product.

Note that the law of cancellation does not hold for the cartesian product. That is, if F and G were two species then $F \times G = F \times F$ does not imply that $G = F$. This can be seen by the following example.

Example 3: Consider the species L of linear orderings and the species S of permutations. The species $L \times L$ is isomorphic to the species $L \times S$ and therefore we can write $L \times L = L \times S$. However, it is not the case that $L = S$. Note that we can see that $L \times L$ is isomorphic to $L \times S$ by looking at the following diagram which shows that the naturality condition is satisfied.

Take the species L and S. Then for any finite set U, $(L \times L)[U] = L[U] \times L[U]$ and $(L \times S)[U] = L[U] \times S[U]$. There is a bijection $\alpha_U : L[U] \times L[U] \rightarrow L[U] \times S[U]$ such that for any bijection $\xi : U \rightarrow V$ we have the following :

$$\begin{array}{ccc} L[U] \times L[U] & \xrightarrow{\alpha_U} & L[U] \times S[U] \\ {\scriptstyle L[\xi]\times L[\xi]}\downarrow & & \downarrow{\scriptstyle L[\xi]\times S[\xi]} \\ L[V] \times L[V] & \xrightarrow[\alpha_V]{} & L[V] \times S[V] \end{array}$$

This is the naturality condition and to see this diagram commutes, we must look at what happens to the elements. This can be seen in the next diagram below.

$$\begin{array}{ccc} ((u_1, \dots ,u_n) , (\sigma(u_1), \dots , \sigma(u_n)) & \xrightarrow{\alpha_U} & ((u_1, \dots ,u_n) , \sigma) \\ {\scriptstyle L[\xi]\times L[\xi]}\downarrow & & \downarrow{\scriptstyle L[\xi]\times S[\xi]} \\ ((\xi(u_1), \dots , \xi(u_n)) , (\xi(\sigma(u_1)), \dots , \xi(\sigma(u_n)))) & \xrightarrow[\alpha_V]{} & ((\xi(u_1), \dots , \xi(u_n)) , \xi\sigma\xi^{-1}) \end{array}$$

In the diagram we have assumed that the set U has cardinality n. The elements of L are linear orderings which we will write as $(u_1, \dots ,u_n)$. Any other linear ordering on U is a permutation of this order which we will write as $(\sigma(u_1), \dots , \sigma(u_n))$ where σ denotes a permutation, and therefore belongs to the species S. The bijection ξ takes $(u_1, \dots ,u_n) \in L$ to $(\xi(u_1), \dots , \xi(u_n))$, and also takes $\sigma \in S$ to $\xi\sigma\xi^{-1}$. We get the element $((\xi(u_1), \dots , \xi(u_n)) , (\xi(\sigma(u_1)), \dots , \xi(\sigma(u_n))))$ in $L[V] \times L[V]$ and the second component is the same as applying $\xi\sigma\xi^{-1}$ to $(\xi(u_1), \dots , \xi(u_n))$ which is how we obtain the bijection between $L[V] \times L[V]$ and $L[V] \times S[V]$.

New species of structure can be created using these combinatorial operations we have looked at. We can apply these operations to known species to get new ones.

Example 4: Consider the species L of linear orderings and the species A of rooted trees. We will look at an $L \circ A$-structure. For any finite set U, an L-structure is a linear order on U and an A-structure is a rooted tree on U. Therefore an A-structure substituted into an L-structure gives a rooted tree on each element of the linear ordering. The diagram below describes an $L \circ A$-structure.

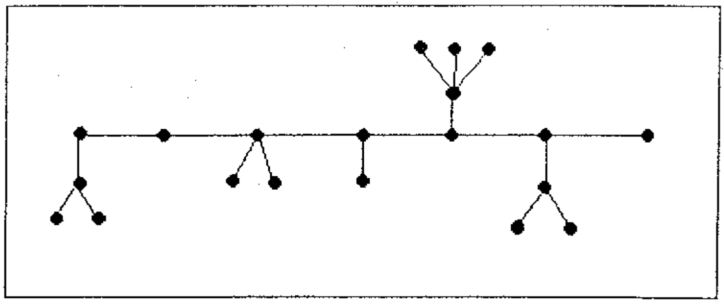


Fig. 4.

So substituting the species A into the species L produces a list of rooted tree structures. This species L ∘ A = H which is the species of hedges. We will see that H is indeed a species of structures by looking at H-structures and the transport of H-structures. For any finite set U, H[U] is the set of lists of rooted trees on U. For any bijection $\sigma : U \to V$, $H[\sigma] : H[U] \to H[V]$ where each H-structure on U is mapped to an H-structure on V where for all $v \in V$, $v = \sigma(u)$ for some $u \in U$.

Example 5: Consider the species E of sets and the species A of rooted trees. An E ∘ A-structure is an A-structure substituted into an E-structure which gives a collection of rooted trees all having the A-structure. We will write E ∘ A = $\mathcal{F}$ where $\mathcal{F}$ is the species of rooted forests; an $\mathcal{F}$-structure is a rooted forest. Note that there is a difference between rooted forests and a collection of rooted trees. A rooted forest is a collection of rooted trees where each rooted tree has the same structure, whereas a collection of rooted trees is an assembly of rooted trees where each tree can have a different structure. This is illustrated in the diagram below where the first diagram shows a rooted forest (ie: an $\mathcal{F}$-structure) and the second diagram shows a collection of rooted trees.

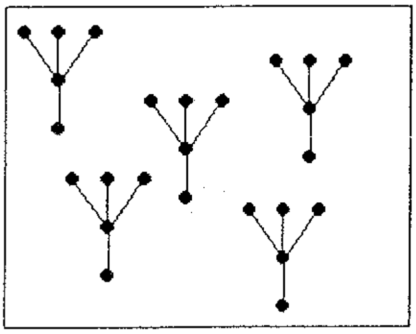


Fig.5. rooted forest

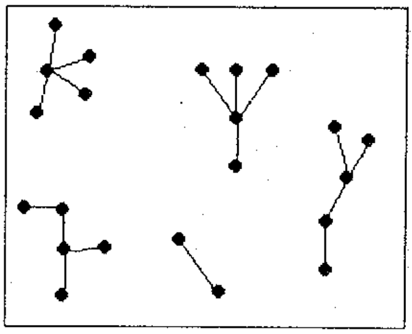


Fig. 6. collection of rooted trees

Example 6: Consider the two examples above where the species $\mathcal{F}$ and H are described. We want to look at the species $\mathcal{F}$.H. For any finite set U an $\mathcal{F}$.H-structure is an $\mathcal{F}$-structure on U_1 together with an H-structure on U_2 where (U_1,U_2) is a decomposition of U. This gives a rooted forest on U_1 together with a list of rooted trees on U_2. Therefore we have a collection of rooted trees on the set $U = U_1 \cup U_2$, and so can conclude that an $\mathcal{F}$.H-structure is a collection of rooted trees. Now consider the species of rooted trees A. An A'-structure on a set U is an A-structure on the set $U+ = U \cup \{*\}$, which is a rooted tree with root $*$. Removing $*$ we get a number of rooted trees. So an A'-structure is a collection of rooted trees. Hence we can conclude that

$$A' = \mathcal{F}.H = \mathcal{F}.L(A)$$

Chapter 3

Lagrange Inversion

3.0. Introduction

There are combinatorial functional equations which arise in the study of the algebra of species and we want to find a general way of solving an equation of this type. In particular, the combinatorial functional equation we will study is $Y = X.R(Y)$ where R is a given species of structures and X is the species of singletons. The method of Lagrange inversion is solving the equation $Y = X.R(Y)$.

Before looking at these equations we will first study some properties of formal power series to understand from where this type of equation is obtained.

Suppose $\mathbf{K}$ is a field of characteristic zero. Consider $\mathbf{K}[\![x]\!]$ whose elements are formal power series where the coefficients come from the field $\mathbf{K}$. The set of formal power series $T(x) \in \mathbf{K}[\![x]\!]$ where $T(0) = 0$ forms a group under the operation of substitution. The identity element of this group is x and each $T(x) \in \mathbf{K}[\![x]\!]$ has an inverse. We want to find this inverse.

Proposition 1: Suppose $G(x)$ is a formal power series over $\mathbf{K}$ with $G(0) = 0$. Then there exists a formal power series $T(x)$ with $T(0) = 0$ and $G(T(x)) = x$.

Proof: Given $G(x) = g_1x + g_2x^2 + g_3x^3 + \ldots$, we wish to find $T(x) = t_1x + t_2x^2 + t_3x^3 + \ldots$ such that $G(T(x)) = x$.
The coefficient of x_k in the power series $G(T(x))$ is $s_k = \Sigma\, g_n \times t_{m1} \times \ldots \times t_{mn}$ where the sum is over all $n, m_1, \ldots, m_n > 0$ with $m_1 + \ldots + m_n = k$.
We require $s_1 = 1$ and $s_k = 0$ for $k > 1$.
Now $s_1 = g_1t_1$ so $t_1 = g_1^{-1}$.
Also $s_2 = g_1t_2 + g_2t_1t_1$ so $t_2 = -g_1^{-1}g_2t_1^{2}$.
Generally, for $k > 1$, $0 = s_k = g_1t_k + \Sigma_{n>1}\, g_nt_{m1}\ldots t_{mn}$ inductively determines t_k since each m_i is $< k$.

■

So, as asserted, $\{G(x) \in \mathbf{K}[\![x]\!] : G(0) = 0\}$ is a group under substitution.

Proposition 2: For each power series $R(x)$ with constant term $r_0 \neq 0$, there exists a unique power series $T(x)$ with

$$T(x)R(x) = x. \qquad (*)$$

Moreover, $t_0 = 0$ where t_0 is the constant term of the series $T(x)$.

Proof: Let $T(x) = t_0 + t_1x + t_2x^2 + \ldots$ and $R(x) = r_0 + r_1x + r_2x^2 + \ldots$.
When $(t_0 + t_1x + t_2x^2 + \ldots)(r_0 + r_1x + r_2x^2 + \ldots) = x$ then we get
$t_0 = 0$, $t_1 = 1/r_0$, $t_2 = -r_1/r_0^2$, $t_3 = r_1^2/r_0^3 - r_2/r_0^2$, .. and so on.

■

Now the power series $T(x)$ mentioned in the above proposition has constant coefficient equal to zero and therefore belongs to those polynomials in $\mathbf{K}[\![x]\!]$ which form a group under substitution. So $T(x)$ has an inverse $A(x)$ under substitution, that is, $T(A(x)) = x$. Substituting $A(x)$ for x in (*) we get an expression for $A(x)$:

$$T(A(x))R(A(x)) = A(x)$$

ie: $$xR(A(x)) = A(x)$$

This is how to find $A(x)$ as a uniquely determined power series satisfying $A(x) = xR(A(x))$.

Looking at this more closely we have the following,
$A(x)$ is the inverse of $T(x)$ so

$$A(x) = \Sigma_{n\geq 1} [(d/dt)^{n-1} (t/T(t))^n]_{t=0}\, x^n/n!$$
$$= \Sigma_{n\geq 1} [(d/dt)^{n-1} R^n(t)]_{t=0}\, x^n/n! \qquad \textbf{(1)}$$

For any formal power series $F(x)$ we have

$$F(A(x)) = \Sigma_{n\geq 0} [(d/dt)^{n-1} F'(t)R^n(t)]_{t=0}\, x^n/n! \qquad \textbf{(2)}$$

And $$F(A(x)) / (1 - xR'(A(x))) = \Sigma_{n\geq 0} [(d/dt)^n F(t)R^n(t)]_{t=0}\, x^n/n! \qquad \textbf{(3)}$$

Writing **(1)**, **(2)** and **(3)** in terms of coefficient extraction we have

(1) $[x^n]\, A(x) = 1/n\, [t^{n-1}]R^n(t)$

(2) $[x^n]\, F(A(x)) = 1/n\, [t^{n-1}]F'(t)R^n(t)$

(3) $[x^n]\, F(A(x)) / (1 - xR'(A(x))) = [t^n]\, F(t)R^n(t)$

These are the statements of the Lagrange inversion for power series. We will interpret these formulas structurally, in terms of species of structure, by first looking at R-enriched rooted trees.

Let R be a given species of structures and X be the species of singletons. Interpreting the numerical equation $A(x) = xR(A(x))$ structurally, gives rise to the combinatorial equation $Y = X.R(Y)$. A solution to this equation is the species Y and in fact we will find that the species $Y = A_R$ satisfies this where A is the species of rooted trees. A_R is the species of R-enriched rooted trees and we will construct this species of structures.

3.1. R-enriched rooted trees

Before looking at R-enriched rooted trees, we will first develop a new way of thinking about rooted trees. This new definition expresses a rooted tree in terms of functions.

Definition 1: Suppose U is a finite set such that $U = \Sigma_{1 \le i \le n} T_i$, $T_i \subseteq U$ and $T_i \cap T_j \neq \varnothing \Rightarrow i = j$. A rooted tree T of height n is defined as a sequence of functions $\alpha_{i+1} : T_{i+1} \to T_i$ where T_i is the set of vertices at height i, for $i < n$, and $u \in T_{i+1}$ is connected to $v \in T_i$ by an edge if and only if $\alpha_{i+1}(u) = v$.

The following diagram illustrates this definition on a set U = {a,b,c,d,e,f,g,h,k,l,m,o,p}.

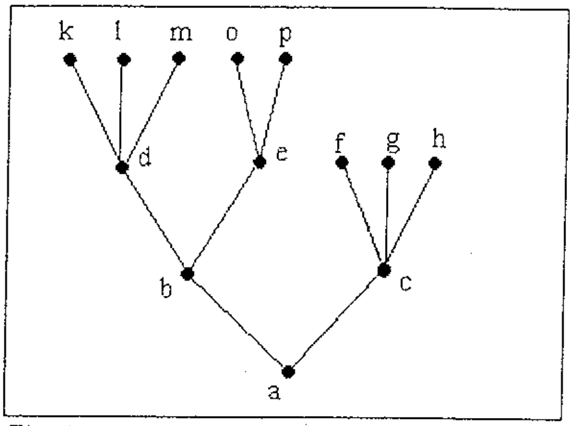


Fig. 1.

At height 0, we have the root {a}
At height 1, T_1 = {b,c}
At height 2, T_2 = {d,e,f,g,h}
At height 3, T_3 = {k,l,m,o,p}
Therefore we have the function $\alpha_1 : T_1 \to \{a\}$ which maps the elements b and c to a.
$\alpha_2 : T_2 \to T_1$ maps elements in T_2 to those elements of T_1 which are connected by an edge. So $\alpha_2(d) = b$, $\alpha_2(e) = b$, $\alpha_2(f) = c$ and so on.

Now we will look at R-enriched rooted trees. This is a rooted tree with a certain structure as defined in the definition below.

Definition 2: Let R be a species of structures. An R-enriched rooted tree on a finite set U is
i) an arbitrary rooted tree α on U
ii) an R-structure on the fiber of α over vertex $u \in U$ in α

Note that a fiber of a vertex u = the set $\alpha^{-1}(u)$
= immediate predecessors of u when all edges of the rooted tree are oriented towards the root.

It is possible for $\alpha^{-1}(u)$ to be an empty set.

So suppose we had a rooted tree with vertices {a,b,c,d,e,f} as shown in the diagram below.

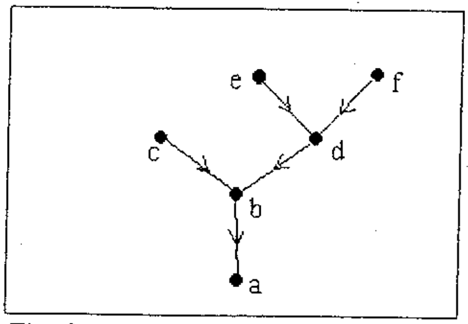


Fig. 2.

The fibre at b is $\alpha^{-1}(b) = \{c,d\}$ since these two vertices are the immediate predecessors of b.

In fact we have the following. $\alpha^{-1}(a) = \{b\}$

$$\alpha^{-1}(b) = \{c,d\}$$
$$\alpha^{-1}(c) = \varnothing$$
$$\alpha^{-1}(d) = \{e,f\}$$
$$\alpha^{-1}(e) = \varnothing$$
$$\alpha^{-1}(f) = \varnothing$$

We call those vertices with empty fibres, leaves. So in the example above, the vertices c,e and f are the leaves of α. Notice this definition of fibre can be better understood when taking into account Definition 1, which describes a rooted tree in terms of functions. From Definition 1 we can see that the inverse functions give us precisely the fibres of the tree.

Now consider the species A_R where A is the species of rooted trees and R is a given species. The species A_R satisfies the equation $Y = X.R(Y)$, that is, $A_R = X.R(A_R)$. This suggests a recursive definition for A_R and in fact this species A_R is the species of R-enriched rooted trees and we will prove this in the following theorem.

Theorem 1: Let R be a species of structures. Then the species A_R of R-enriched rooted trees is uniquely determined, up to isomorphism, by the combinatorial equation $A_R = X.R(A_R)$.

Proof: Let A_R be an R-enriched rooted tree structure on U. Then from the definition of R-enriched rooted trees, A_R is an arbitrary rooted tree on U with root $e \in U$ together with an R-structure on the fibre of each vertex $u \in U$, which is precisely an $R(A_R)$-structure attached to root e. Clearly this satisfies the equation $A_R = X.R(A_R)$.

Now we need to prove uniqueness. Suppose $Y = A$ and $Y = B$ are R-enriched rooted tree structures which satisfy the equation $Y = X.R(Y)$. Then there are isomorphisms $\eta: A \to X.R(A)$ and $\gamma : B \to X.R(B)$. We can construct an isomorphism $\psi : A \to B$ such that the diagram below commutes.

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d, "\psi"'] & X.R(A) \arrow[d, "X.R(\psi)"] \\ B \arrow[r, "\gamma"'] & X.R(B) \end{tikzcd}$$

Note that $X.R(\psi)$ is obtained from using transport of structures. To see this, we now want to show that we can construct ψ so that it is an isomorphism and this will be done by induction on the contact order between A and B.
Firstly recall that two species A and B have contact of order n if $A_{\leq n} = B_{\leq n}$, where $A_{\leq n}$ denotes A restricted to sets of cardinality $\leq$ n.
When $n = 0$, $A_0 = 0$ and $B_0 = 0$ so $A_0 = B_0$ = empty species, so A and B have contact of order 0. Therefore we can let $\psi_0 = Id_0$.
Suppose A and B have contact of order n via the isomorphism $\psi_n : A_{\leq n} \to B_{\leq n}$ for which the diagram above commutes when restricted to sets of order n.
Let $\psi_{n+1} = (\gamma^{-1} \circ X.R(\psi_n) \circ \eta)_{\leq n+1}$ be a map between $A_{\leq n+1}$ and $B_{\leq n+1}$.
[We want to show this is an isomorphism between $A_{\leq n+1}$ and $B_{\leq n+1}$].
Now the canonical decomposition of an $A_{\leq n+1}$-structure is an $X.R(A_{\leq n})$-structure. Since ψ_n is an isomorphism between $A_{\leq n}$ and $B_{\leq n}$, then there is an isomorphism between $X.R(A_{\leq n})$ and $X.R(B_{\leq n})$. That is, an isomorphism between $A_{\leq n+1}$ and $B_{\leq n+1}$, so ψ_{n+1} is an isomorphism. This is true for any n so there is an isomorphism ψ between A and B such that $\psi = \gamma^{-1} \circ X.R(\psi) \circ \eta$. Therefore A and B are isomorphic.

■

Therefore the R-enriched rooted tree species A_R satisfies the equation $A_R = X.R(A_R)$ and can be described by the following diagram.

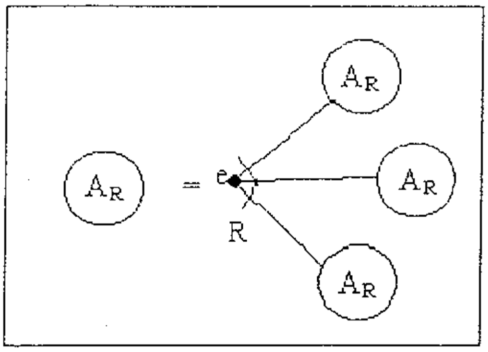


Fig. 3.

Looking at this for a particular example, take the set U = {a,b,c,d,e,f,g,h,j,k,l,m,o,p}. An A_R-structure on this set is described by the diagram below.

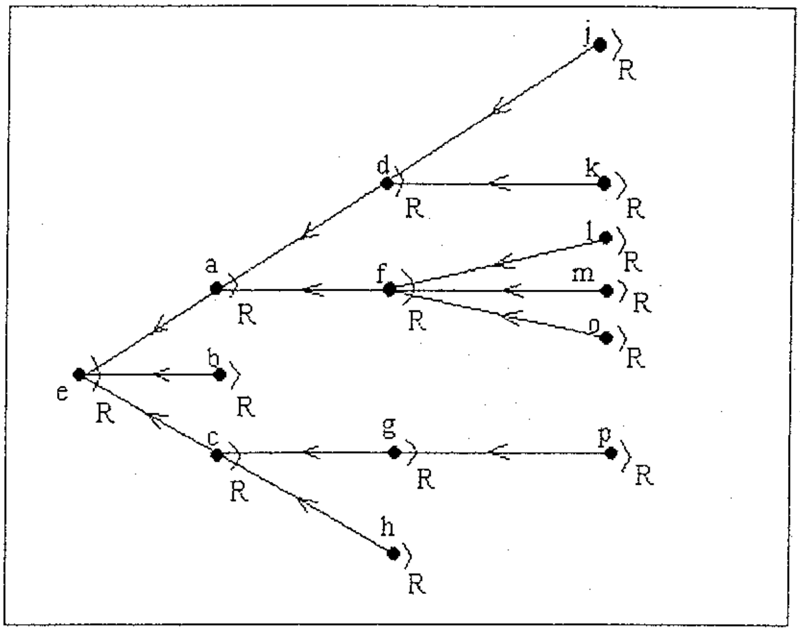


Fig. 4.

We will look at some examples of R-enriched rooted tree structures for certain species of structures R.

Example 1: Let R be the species E of sets. Then an A_E-structure on a finite set U is an arbitrary rooted tree on U together with an E-structure on the fibre of each vertex of the rooted tree. This is precisely an A-structure on U which satisfies the equation $A = X.E(A)$.

Example 2: Let R be the species L of linear orderings. Then an A_L-structure on a finite set U is an arbitrary rooted tree on U together with a linear ordering on each fibre of each vertex of the rooted tree. A way of placing a linear order on each fibre is to order the elements from top to bottom.

Consider the set U = {a,b,c,d,e,f,g,h,j,k,l,m} where an A_L-structure on U can be described by the diagram below. The elements of each fibre are ordered from top to bottom, and this ordering is represented in the diagram by the arrows at each fibre. This species of L-enriched rooted trees satisfies the equation $A_L = X.L(A_L)$.

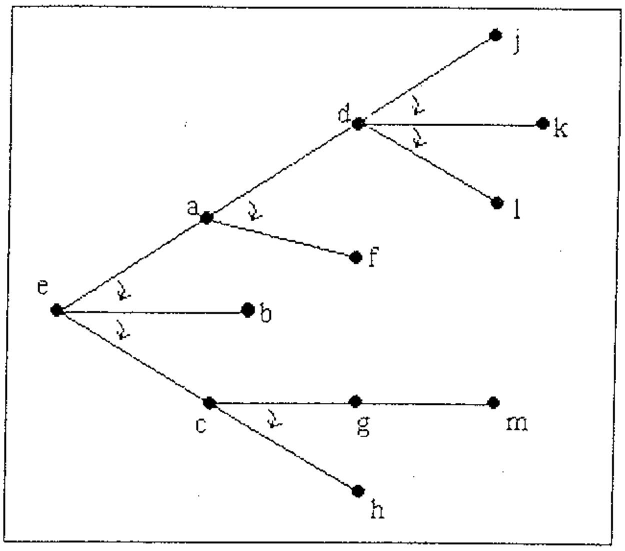


Fig. 5.

Example 3: Let R be the species C of cyclic permutations. Then an A_C-structure on a finite set U is an arbitrary rooted tree on U together with a C-structure on each of its fibres. A C-structure on a fibre is an oriented cycle of elements of the fibre, which can be done since the neighbours of a root can rotate freely around the root as long as they keep their respective positions.

If we take U to be the set U = {a,b,c,d,e,f,g,h,j,k,l,m} then we can picture an A_C-structure as in the diagram below, and this structure will satisfy the equation $A_C = X.C(A_C)$.

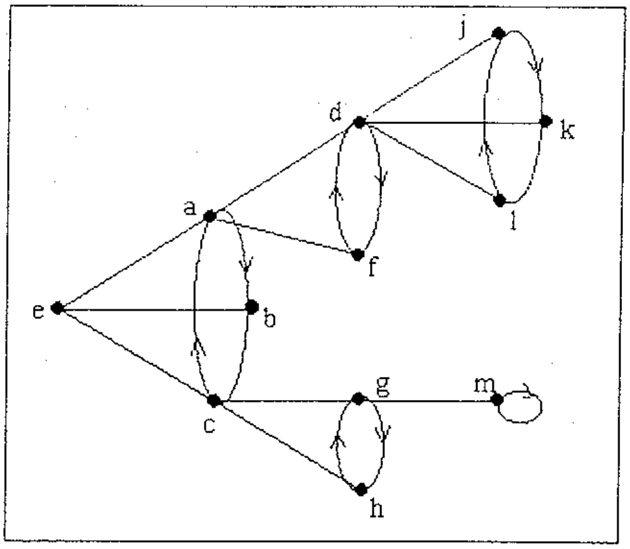


Fig. 6.

3.2. R-enriched Partial Endofunctions

We introduce R-enriched partial endofunctions to enable us to prove Lagrange Inversion combinatorially. Firstly, we will look at the powers R^{λ} of the species R for positive integer values of λ. Consider the species R which has generating series $R(x) = \Sigma_{n\geq 0}\, r_n\, x^n/n!$. Then the generating series of R^{λ} is of the form $R^{\lambda}(x) = (R(x))^{\lambda} = \Sigma_{n\geq 0}\, r_n(\lambda)\, x^n/n!$ where $r_n(\lambda) = \#\, R^{\lambda}[n]$ which is a function of λ.

Definition 1: The sequence of functions $(r_n(\lambda))_{n\geq 0}$ defined by $r_n(\lambda) = \#\, R^{\lambda}[n]$ is called the binomial type sequence associated to R.

The following example illustrates this.

Example 1: Let R be the species E of sets. Then $R(x) = e^x$ and

$$R^{\lambda}(x) = R(x).R(x)...R(x) \text{ (by multiplication of species)}$$
$$= e^{\lambda x}$$

and therefore $r_n(\lambda) = \lambda^n$

Example 2: Let R be the species L of linear orderings. Then $R(x) = (1-x)^{-1}$ and

$$R^{\lambda}(x) = (1-x)^{-\lambda}$$

Therefore using the binomial theorem $r_n(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-n+1)$

Definition 2: An R-enriched partial endofunction on a finite set U consists of
i) a subset $V \subseteq U$
ii) a function $f : V \to U$ of which each fiber $f^{-1}(u)$ for $u \in U$ is given an R-sturcture.
When $V = U$, $f : U \to U$ is called an R-enriched endofunction and we write End_R and $End_R^{\wp}$ for the set of R-enriched endofunctions and partial endofunctions respectively.

In other words the species of R-enriched partial endofunction consists of functions which map the subset V to U and give an R-structure on each fiber. Consider the set $U = [14]$ and $V = [11]$. The diagram below represents an R-enriched partial endofunction $f : V \to U$.
In this diagram we say for elements $v \in V$ and $u \in U$, $f(v) = u$ if and only if v and u are connected by an edge.

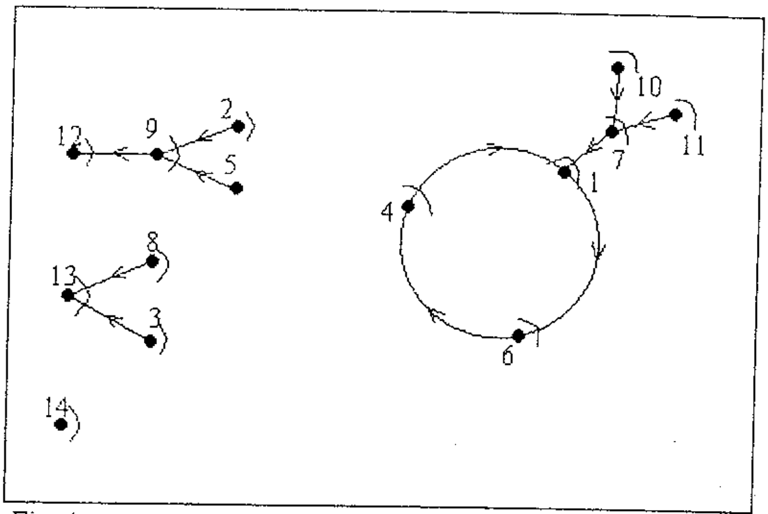


Fig. 1.

Example 3: Let R be the species E of sets. Then an E-enriched endofunction on U is the function $f : U \to U$ where each fiber of f has an E-structure. So f is an endofunction on U. Therefore when R = E, the species of R-enriched endofunction is the species End of (usual) endofunctions.

Example 4: Let R be the species 1 + X. Then $End_{1+X}[U]$ is a function $f : U \to U$ together with a (1+X)-structure on each fiber $f^{-1}(u)$, $u \in U$. A fiber with a (1+X)-structure is either empty or a single point. That is, $\#f^{-1}(u) = 1$ so f is injective. Therefore we can say $End_{1+X}[U] = \{f : U \to U : f \text{ is injective}\}$
$= \{f : U \to U : f \text{ is bijective}\}$ (by pigeon hole principle)
$= \{f : U \to U : f \text{ is invertible}\}$
$= \{\text{permutations on } U\}$
Therefore $End_{1+X} = S$.

Lemma 1: Let R be a species of structures with binomial type sequence $(r_n(\lambda))_{n\geq 0}$. Let U be a finite set with a fixed subset $V \subseteq U$ and $\#U = n$, $\#V = k$. Then the number of R-enriched partial endofunctions with domain V is $r_k(n) = \#R^n[k]$.

Proof. Without loss of generality let $V = \{1,2, \dots ,k\} \subseteq \{1,2, \dots ,n\} = U$. Let $f : V \to U$ be an R-enriched partial endofunction. Such a function is determined by a family $\{\xi_u\}_{u\in U}$ where ξ_u is an R-structure on the fiber $f^{-1}(u)$ of each $u \in U$. We can identify this family $\{\xi_u\}_{u\in U}$ with a list of n R-structures (since f has $n = \#U$ fibres). But this is precisely an R^n-structure on V (by definition of multiplying the species R n times). Therefore from Definition 1 we can conclude that the number of R-enriched partial endofunctions $r_k(n) = \#R^n[k]$.

∎

Lemma 2: (Lemma of R-enriched endofunctions)
Let R be a species of structures. Then we have the species isomorphisms

$$\mathrm{End}_R^{\wp} \cong E(A_R).\mathrm{End}_R$$

$$\mathrm{End}_R \cong S(X.R^{\prime}(A_R))$$

and the equipotence $\mathrm{End}_R \equiv L(X.R^{\prime}(A_R))$

Proof: An R-enriched endofunction naturally decomposes into two parts, the first is an assembly of R-enriched rooted trees whose roots are elements of U\V and the second is an R-enriched endofunction. (This is illustrated in Fig. 1.). Hence we can say $\mathrm{End}_R^{\wp} \cong E(A_R).\mathrm{End}_R$.

To prove $\mathrm{End}_R \cong S(X.R^{\prime}(A_R))$ recall that a rooted tree can be written as a pair (a,ψ) where a is the root and ψ is a function giving the structure of the tree. Therefore we have $A_R[U] = \{(a,\psi,r) : (a,\psi) \in A[U],\ r \in \Pi_{u\in U} R[\psi^{-1}(u)]\}$. So (a,ψ) defines a rooted tree on U and r is an R-structure on each fiber of the vertices of the tree.

Therefore $R^{\prime}(A_R)[U] = \{(\pi,s,b) : \pi$ is a partition of U, $s \in R[\pi+1]$, $b = (b_p)_{p\in\pi}$, $b_p = (a_p,\psi_p,r_p) \in A_R[p]\}$.

So an $R(A_R)$-structure is a triplet (π,s,b) where π is a partition of U, s is an R-structure on the set whose elements are π together with an extra element not belonging to π (by definition of the derivative of a species), and b is an R-enriched rooted tree structure on p for each class p of π.

$Q[U] = X.R^{\prime}(A_R)[U] = \{(u,\pi,s,b) : u \in U$, π is a partition of U, $s \in R[\pi+1]$, $b = (b_p)_{p\in\pi}$, $b_p = (a_p,\psi_p,r_p) \in A_R[p]\}$.

So an example of a Q-structure is the following diagram.

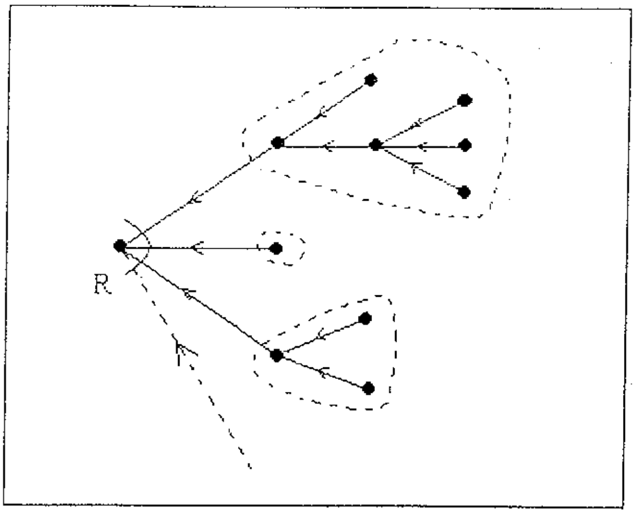


Fig. 2.

We can see that a Q-structure on U is a rooted tree on U with an R-structure on the fibre of each vertex except at the root where there is an R-structure on the fiber plus a point.

Now $S(Q)[U] = \{(\pi,\sigma,b) : \pi$ is a partition of U, $\sigma : \pi \to \pi$ is invertible,
$b \in \Pi_{p\in\pi} Q[p]\}$.

So a S(Q)-structure is a triplet (π,σ,b) where π is a partition of U, σ is a permutation of π and b is a Q-structure on each class p of π.

By the proof of Proposition 1 in section 2.2, each $(\pi,\sigma,b) \in S(Q)[U]$ gives an endomorphism $\varphi : U \to U$ by ignoring the R-structure. The roots u of the rooted trees underlying each b_p are recurrent elements for φ. In the above diagram, the dotted arrow into such a u gives the extra element needed to base the R-structure on the fibre of Q at u since that fiber includes the cycle that u is on. All the other structures are as they should be to make φ R-enriched. Therefore φ is an R-enriched endofunction and we have $End_R \cong S(X.R'(A_R))$.

Since the species L of linear orderings and the species S of permutations are equipotent it follows that End_R and $L(X.R'(A_R))$ are equipotent so we have $End_R \equiv L(X.R'(A_R))$.

■

3.3. Lagrange Inversion Theorem

Now we can interpret Lagrange Inversion for power series in terms of species. These numerical formulas (1), (2) and (3) can be described structurally by species and this is expressed in the Lagrange Inversion Theorem below. We will use the results discussed in section 3.2 to help prove the Lagrange Inversion Theorem.

Theorem 1: (Lagrange Inversion)

Let R and F be two species of structures. Then for any $n \geq 0$, there are bijections

1. $A_R^{\bullet}[n] \to (X.R^n)[n]$
2. $F(A_R)^{\bullet}[n] \to (F^{\bullet}.R)[n]$
3. $(F(A_R).End_R)[n] \to (F.R^n)[n]$

Proof. To prove this theorem we will start at 3, and end on the main result 1.

(3) Consider an $(F(A_R).End_R)$-structure on $U = [n]$.

Let W be the set of roots of the R-enriched rooted trees of the F assembly. The diagram below gives an example of this.

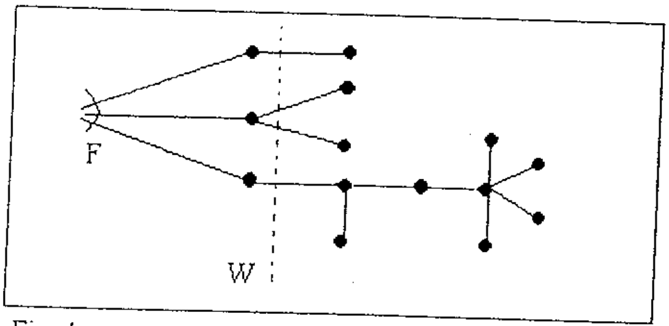


Fig. 1.

So we have an F-structure on W together with an R-enriched function on U\W which has codomain U. That is $f : U\backslash W \to U$ such that each fiber $f^{-1}(u)$, $(u \in U\backslash W)$ is an R-structure.

Since #U = n, we can identify this structure with an R^n-structure on U\W (by Lemma1). This is precisely an $(F.R^n)$-structure on U.

(2) Consider an $F(A_R)^{\bullet}$-structure on $U = [n]$. This is a pointed $F(A_R)$-structure so there is a distinguished element. The following diagram is an example of this.

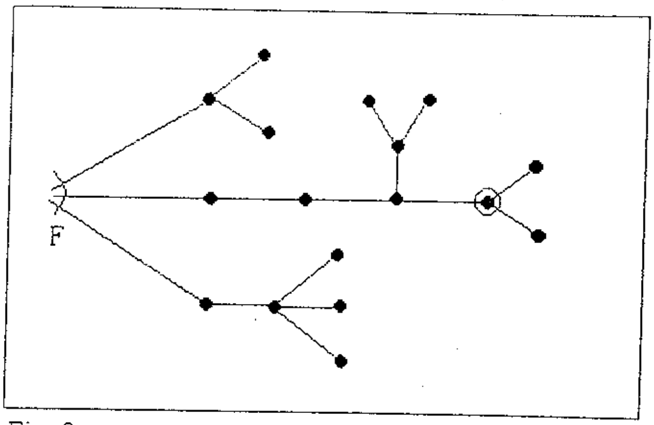


Fig. 2.

Now we can do the following bijective transformations:
1) separate from the F assembly the whole rooted tree α which contains the distinguished element.
2) separate α from the subtree α' whose root is the distinguished element.
3) reattach α' to the F-assembly replacing α.
Now $\alpha \setminus \alpha'$ is an $L(X.R'(A_R))$-structure and therefore using Lemma 2, we can transform this bijection into an R-enriched endofunction End_R.
The following diagrams apply the steps outlined above to Fig. 26.

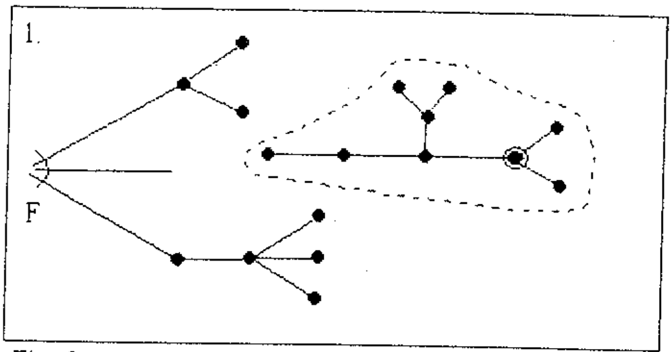


Fig. 3.

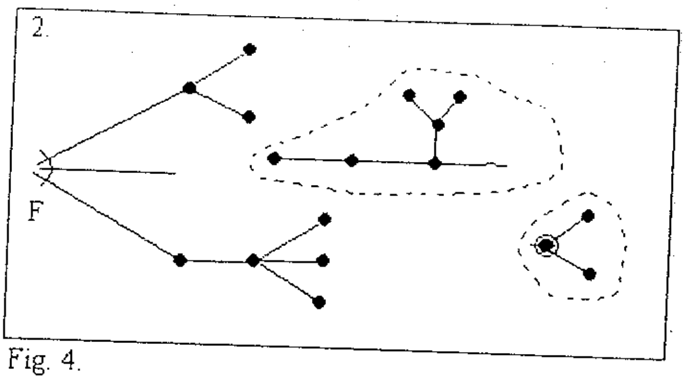


Fig. 4.

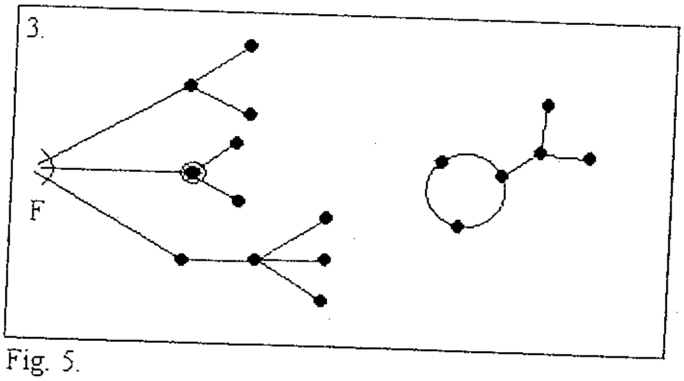


Fig. 5.

So now we have an $(F(A_R)^{\bullet}.End_R)$-structure. Therefore by the proof of (3) this is an $(F^{\bullet}.R)$-structure on U.

(1) This is a special case of (2) where $F = X$ since an $(X(A_R).End_R)$-structure is an $A_R{}^{\bullet}$-structure.

■

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