

Duality and well pointedness Sept 2017

\forall any monoidal category

$\rho: I \rightarrow A$ is a well-pointed object

$\sigma: B \rightarrow I$ is a well-copointed object

If $B \otimes \rho$ is a coretraction and $A \otimes \sigma$ is a retraction then $B \dashv A$ and ρ, σ are mates.

Proof

$$A \begin{array}{c} \textcircled{\rho} \\ | \\ A \end{array} \Big| \overset{A}{=} \Big| \textcircled{\rho} \overset{A}{A} ; \quad B \begin{array}{c} \textcircled{\sigma} \\ | \\ B \end{array} \Big| \overset{B}{=} \Big| \textcircled{\sigma} \overset{B}{B} ;$$

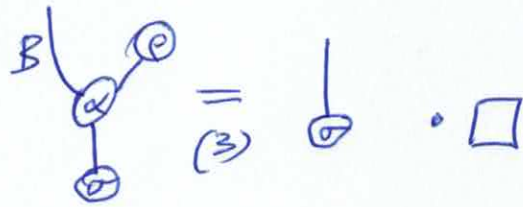
$$B \begin{array}{c} \textcircled{\rho} \\ | \\ \textcircled{\alpha} \\ | \\ B \end{array} \Big| \overset{A}{=} \Big| \textcircled{\alpha} \overset{A}{B} ; \quad A \begin{array}{c} \textcircled{\beta} \\ | \\ \textcircled{\gamma} \\ | \\ A \end{array} \Big| \overset{B}{=} \Big| \textcircled{\gamma} \overset{B}{A}$$

Put $\varepsilon = B \begin{array}{c} \textcircled{\rho} \\ | \\ \textcircled{\alpha} \\ | \\ B \end{array} \Big| \overset{A}{A} ; \quad \eta = A \begin{array}{c} \textcircled{\beta} \\ | \\ \textcircled{\gamma} \\ | \\ A \end{array} \Big| \overset{B}{B}$

$$A \begin{array}{c} \textcircled{\rho} \\ | \\ \textcircled{\alpha} \\ | \\ B \end{array} \Big| \overset{A}{A} \overset{(1)}{=} A \begin{array}{c} \textcircled{\rho} \\ | \\ \textcircled{\beta} \\ | \\ \textcircled{\alpha} \\ | \\ B \end{array} \Big| \overset{A}{A} \overset{(geom)}{=} A \begin{array}{c} \textcircled{\rho} \\ | \\ \textcircled{\beta} \\ | \\ \textcircled{\alpha} \\ | \\ \textcircled{\gamma} \\ | \\ B \end{array} \Big| \overset{A}{A} \overset{(3)}{=} A \begin{array}{c} \textcircled{\rho} \\ | \\ \textcircled{\beta} \\ | \\ \textcircled{\gamma} \\ | \\ B \end{array} \Big| \overset{A}{A} \overset{(4)}{=} A \Big| \overset{A}{A}$$

$$B \begin{array}{c} \textcircled{\rho} \\ | \\ \textcircled{\alpha} \\ | \\ A \end{array} \Big| \overset{B}{B} \overset{(2)}{=} B \begin{array}{c} \textcircled{\rho} \\ | \\ \textcircled{\beta} \\ | \\ \textcircled{\alpha} \\ | \\ A \end{array} \Big| \overset{B}{B} \overset{(geom)}{=} B \begin{array}{c} \textcircled{\rho} \\ | \\ \textcircled{\beta} \\ | \\ \textcircled{\alpha} \\ | \\ \textcircled{\gamma} \\ | \\ A \end{array} \Big| \overset{B}{B} \overset{(4)}{=} B \begin{array}{c} \textcircled{\rho} \\ | \\ \textcircled{\beta} \\ | \\ \textcircled{\gamma} \\ | \\ A \end{array} \Big| \overset{B}{B} \overset{(3)}{=} B \Big| \overset{B}{B}$$

The mate of $I \xrightarrow{p} A$ is $B \xrightarrow{B \otimes p} B \otimes A \xrightarrow{\varepsilon} I$; but this is σ since



Corollary For $X \in \mathcal{V}$, the following are equivalent:

- (i) $p \otimes X : X \rightarrow A \otimes X$ is invertible;
- (ii) $p \otimes X : X \rightarrow A \otimes X$ is a coretraction;
- (iii) $\sigma \otimes X : B \otimes X \rightarrow X$ is invertible;
- (iv) $\sigma \otimes X : B \otimes X \rightarrow X$ is a retraction.

Remark This was inspired by Theorems 6 and 7 of Barr's paper in Tbilisi Math J 10(3) 2017.