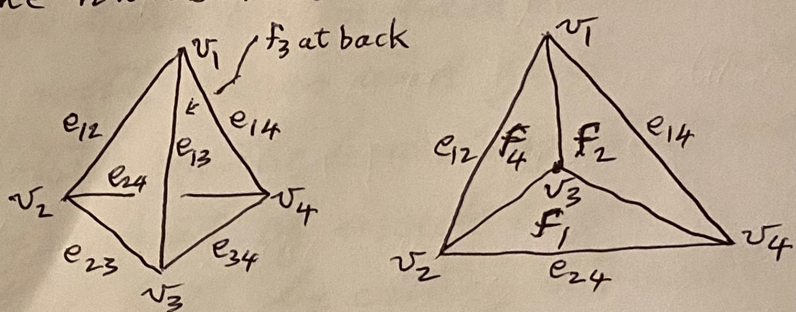
Euler's formula & Platonic solids

For any polyhedron (regular or not), we write

- V for the number of vertices
- E for the number of edges
- F for the number of faces.

Tetrahedron

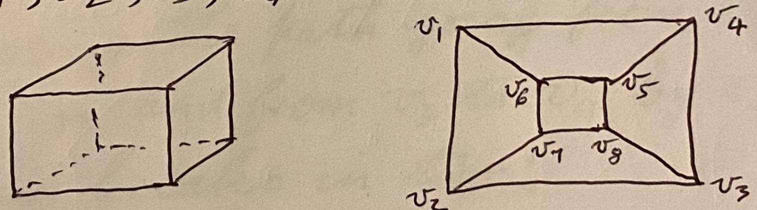


vertices are v_1, v_2, v_3, v_4 so $V=4$

edges are $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}$ so $E=6$

faces are f_1, f_2, f_3, f_4 so $F=4$

Cube



$V = 8 \qquad E = 12 \qquad F = 6$

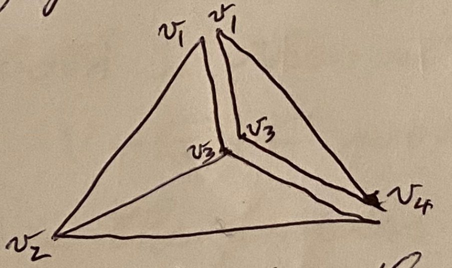
In each example, $V - E + F = 2$

Euler's formula For any polyhedron,

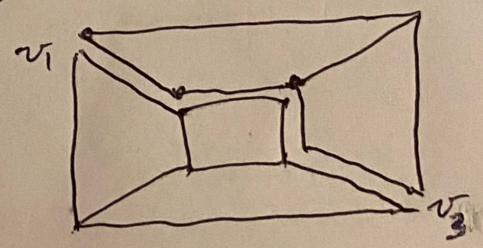
$$V - E + F = 2.$$

Proof of Euler's formula

Imagine the edges of the polyhedron to be made of flexible rods with universal joints at the vertices that can move to all angles. Pick a face and widen it as in the right-hand diagrams for the tetrahedron and cube. We have lost a face and obtained a polygon with edges and faces inside. Take two vertices on the boundary that are connected by a path of inside edges. Cut along that path by cutting the rods in half along their length. For example, if we choose v_1 and v_4 in the tetrahedron and cut along the path going from v_1 to v_3 by e_{13} and from v_3 to v_4 by e_{34}, the figure breaks in two:



If we do it for the cube along the path v_1 to v_6 to v_5 to v_8 to v_3:



Since we lost a face, we are trying to prove that $V-E+F=1$. Let V_1, E_1, F_1 be the new counts for one of the new configurations and let V_2, E_2, F_2 be the counts for the other. Let L be the number of edges we cut along so the cut involves duplication of $L+1$ vertices. We have

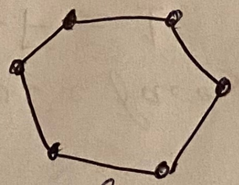
$$E_1+E_2 = E+L, \quad V_1+V_2 = V+L+1$$

and also $F_1+F_2 = F$. Then

$$\begin{aligned}
&(V_1-E_1+F_1)+(V_2-E_2+F_2)\\
&= (V_1+V_2)-(E_1+E_2)+(F_1+F_2)\\
&= V+L+1-(E+L)+F\\
&= (V-E+F)+1.
\end{aligned}$$

If we knew $V_1-E_1+F_1$ and $V_2-E_2+F_2$ were both 1 then it would follow that $V-E+F$ would also be 1.

Continue the process on the new pieces so long as there are internal edges. What remains is a collection of polygons like



which has one face and $V=E$, and so $V-E+F=1$.

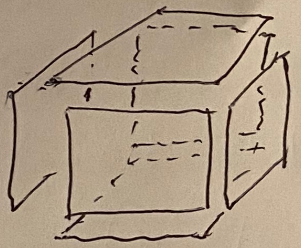
Q.E.D.

Proofs like that are called "descent" or "induction".

Now to Platonic (or regular) polyhedra.
The properties of regular solids that we want to use are:

- • each face has the same number s of boundary edges, and
- •• each vertex has the same number m of faces meeting at it.

Cut out the faces of the polyhedron.



Rough picture for the cube.

Each edge has become two.
So now we have $2E$ edges.

• and •• allow us to count the number of edges in two ways

• gives sF edges

•• gives mV edges

Now we do a little algebra:

$$2E = sF \qquad 2E = mV$$

$$V - E + F = 2$$

$$s \geq 3 \qquad m \geq 3$$

$V = \frac{2E}{m}$, $F = \frac{2E}{s}$ can be substituted into Euler's formula to give

$$\frac{2E}{m} - E + \frac{2E}{s} = 2,$$

Divide by $2E$:

$$\frac{1}{m} - \frac{1}{2} + \frac{1}{s} = \frac{1}{E} > 0.$$

So $\frac{1}{m} + \frac{1}{s} > \frac{1}{2}$. (*)

If $m > 5$ then $\frac{1}{m} + \frac{1}{s} \leq \frac{1}{6} + \frac{1}{s} \leq \frac{1}{2}$ for $s \geq 3$. Similarly for $s > 5$.

So we must have $3 \leq m, s \leq 5$.

Now you can check that, of the 9 possibilities, the only ones that give inequality (*) are

$\boxed{m=3,\ s=5}$ $\boxed{m=3,\ s=3}$

$\boxed{m=4,\ s=3}$ $\boxed{m=3,\ s=4}$

$\boxed{m=5,\ s=3}$

So there are at most 5 regular polyhedra. And the Greeks showed us the five.