Functorial calculus in monoidal bicategories

Ross Street

January 2002

Abstract

The definition and calculus of extraordinary natural transformations [EK] is extended to a context internal to any autonomous monoidal bicategory [DyS]. The original calculus is recaptured from the geometry [SV], [MT] of the monoidal bicategory $\mathcal{V}$-**Mod** whose objects are categories enriched in a cocomplete symmetric monoidal category $\mathcal{V}$ and whose morphisms are modules.

Mathematics Subject Classifications (2000): 18D05, 18D10, 18D20.

Key words: monoidal bicategory, enriched category, extraordinary natural transformation, bidual.

Introduction

Category theory officially began with the definition of naturality [EM1], [EM2]. A natural transformation mediates functors of the same variables of the same respective variances. A family of evaluation morphisms

$$X^A \otimes A \longrightarrow X$$

is natural, in this ordinary sense, in the variable X. However, Kelly [K; p. 21] identified the sense in which such a family is "natural" in the variable A. Then Eilenberg and Kelly [EK1] studied composition of these "extraordinary" natural transformations, finding it to be governed by loop freeness of an associated graph.

Unlike the more general dinatural transformations of Dubuc and the author [DcS], extraordinary natural transformations can be defined in enriched category theory. Indeed, the paper [EK1] was written with the needs of enriched category theory's seminal paper [EK2] in mind.

While the existence of "morphisms" such as evaluation in an autonomous[1] monoidal bicategory context was made clear in [DyS], that paper did not go the extra step of Kelly to abstract extraordinariness. The present paper takes that step. For a cocomplete symmetric closed monoidal category $\mathcal{V}$, the autonomous monoidal bicategory to obtain Kelly's naturality notion is $\mathcal{V}$-**Mod** (as defined in [DyS], for example). We see that the graph calculus of [EK1] is a relic, adequate for the purpose, of the surface-diagram geometry associated with an autonomous monoidal bicategory [SV], [MT]. The question of loops only arises in a pivotal monoidal bicategory where each bidual object A° is both a left and right bidual for A (such as in the symmetric case of $\mathcal{V}$-**Mod** implicitly considered by [EK1]). The failure to compose results from the fact that the composite of the unit $n' : I \longrightarrow A \otimes A^\circ$ for the left bidual with the counit $e : A \otimes A^\circ \longrightarrow I$ for the right bidual is not generally the identity of the tensor unit I.

[1] We use "right autonomous" for the existence of all right duals (in a monoidal category, or biduals in a monoidal bicategory) and "autonomous" for the existence of both left and right duals. The words "compact" and "rigid" also appear in the literature in this context.

1. Extraordinary 2-cells

Recall from [GPS] that every monoidal bicategory is appropriately equivalent to a Gray monoid (also see [DyS]). For the reader's comfort, we briefly repeat the definition here. While speaking as if we were working in a monoidal bicategory, we develop our general theory in a Gray monoid $\mathcal{M}$ (in the same way that we are allowed by coherence to assume a monoidal category is strict).

Consider the category 2-**Cat** whose objects are (strict) 2-categories and whose morphisms are (strict) 2-functors. There is a tensor product of 2-categories, named after Gray [G1], [G2], which defines a symmetric monoidal structure on 2-**Cat** ; it is actually a **closed** monoidal structure and the internal hom $\{\mathcal{A}, \mathcal{B}\}$ is easily described: it is the 2-category which has 2-functors $\mathcal{A} \longrightarrow \mathcal{B}$ as objects, pseudo-natural transformations as morphisms, and modifications as 2-cells. We write **Gray** for 2-**Cat** equipped with this monoidal closed structure. (Actually, Gray himself concentrated mainly on the larger internal hom where the morphisms are lax natural.) A *Gray monoid* is defined to be a monoid in the monoidal category **Gray**.

Alternatively, a Gray monoid is a 2-category equipped with a pseudofunctor (in the terminology of [KS]; or "homomorphism of bicategories" in the terminology of [B])

$$\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$$

and an object I such that

- for all morphisms $f : A \longrightarrow A'$, $g : B \longrightarrow B'$ in $\mathcal{M}$, the composition-preservation constraints $(f \otimes B) \circ (A \otimes g) \cong f \otimes g$ are identity 2-cells (that is, $\otimes$ strictly preserves composites of pairs $(A, g) : (A, B) \longrightarrow (A, B')$, $(f, B') : (A, B') \longrightarrow (A', B')$ in $\mathcal{M} \times \mathcal{M}$);
- $\otimes$ is normal (that is, strictly preserves identity morphisms); and
- $\otimes$ is strictly associative with I as strict unit.

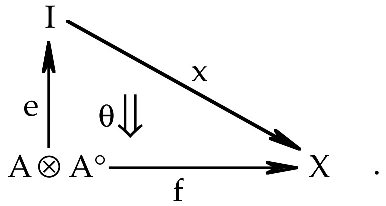
It is easy to see that this is the same as describing a 2-functor $\mathcal{M} \longrightarrow \{\mathcal{M}, \mathcal{M}\}$ making $\mathcal{M}$ a monoid in **Gray**.

Recall from [DyS] that a *right bidual* for an object A of $\mathcal{M}$ is an object A° together with a morphism $e : A \otimes A^\circ \longrightarrow I$ such that, for all objects B, C of $\mathcal{M}$, the functor

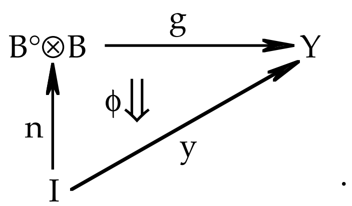
$$e^\# : \mathcal{M}(B, A^\circ \otimes C) \longrightarrow \mathcal{M}(A \otimes B, C),$$

given by $e^\#(f) = (e \otimes C) \circ (A \otimes f)$, is an equivalence of categories. Note that, up to isomorphism, e determines, and is determined by, a morphism $n : I \longrightarrow A^\circ \otimes A$ via the condition that $e^\#(n)$ is isomorphic to the identity $A \otimes I \longrightarrow A$. When every object has a right bidual, we call $\mathcal{M}$ *right autonomous.* In this case, the assignment $A \longmapsto A^\circ$ extends to a monoidal pseudofunctor $(\ )^\circ : \mathcal{M}^{\mathrm{oprev}} \longrightarrow \mathcal{M}$ called *bidualization.* When all objects have both left and right biduals, we call $\mathcal{M}$ *autonomous*. A right autonomous Gray monoid $\mathcal{M}$ is autonomous iff bidualization is a biequivalence.

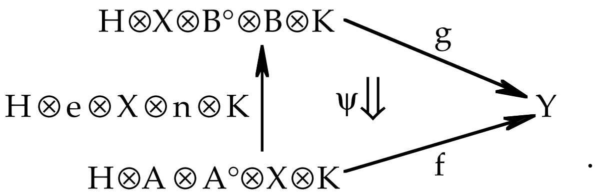
We work in a right autonomous Gray monoid $\mathcal{M}$. We define an *extraordinary 2-cell* θ from $x : I \longrightarrow X$ to $f : A \otimes A^{\circ} \longrightarrow X$ to be a 2-cell



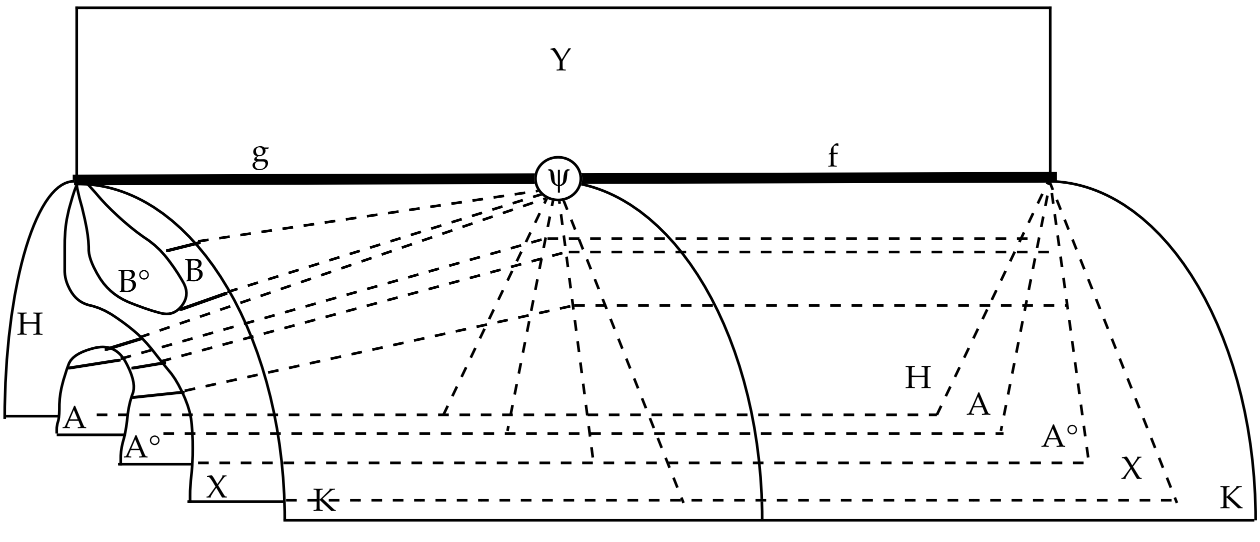
An *extraordinary 2-cell* ϕ from $g : B^{\circ}\otimes B \longrightarrow Y$ to $y : I \longrightarrow Y$ is a 2-cell



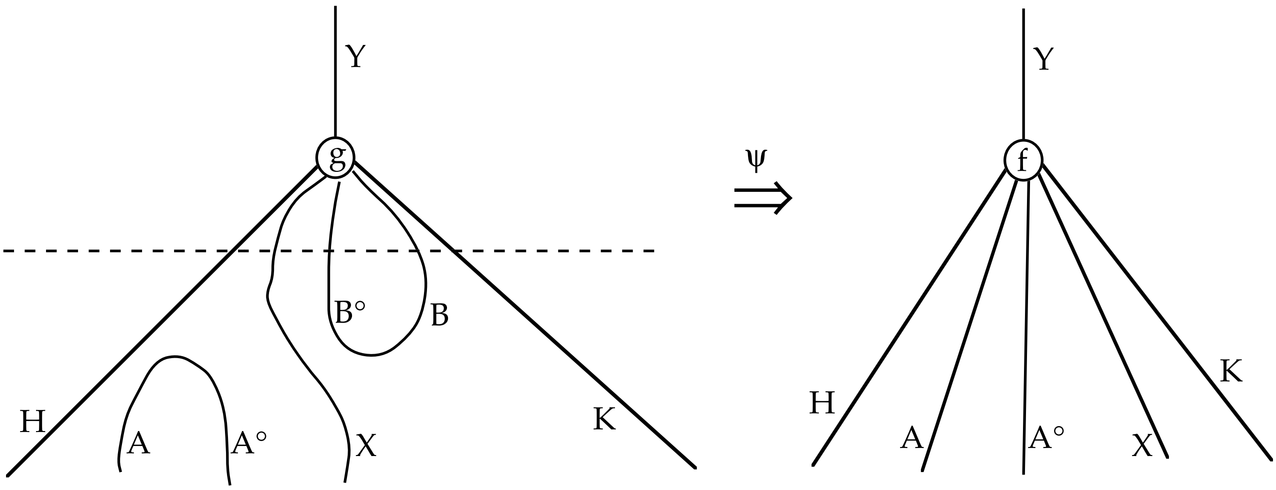
Generalizing both these cases, an *extraordinary 2-cell* ψ from $g : H\otimes X\otimes B^{\circ}\otimes B\otimes K \longrightarrow Y$ to $f : H\otimes A \otimes A^{\circ}\otimes X\otimes K \longrightarrow Y$ is a 2-cell



Such a 2-cell is represented in three-dimensional Euclidean space as a vertex, labelled by ψ, on a surface made up of sheets labelled by the objects H, K, X, Y, A, B, where the morphisms f and g label curves on the surface (see [SV] and [MT]).



Each slice by a vertical plane perpendicular to the page gives a string diagram in the right autonomous monoidal category $c\mathcal{M}$ with the same objects as $\mathcal{M}$ and with isomorphism classes of morphisms as morphisms. The domain of ψ is obtained by taking the slice to the left of the slice containing ψ; the codomain, to the right.

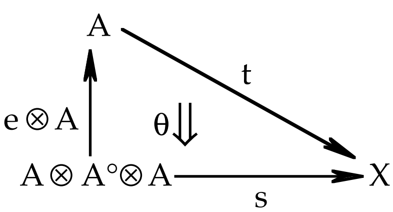


The information contained in the surface diagram can be completely reconstructed from the part of the last diagram to the left of ψ and below the dotted line. What we have then is a morphism in the free right autonomous monoidal category **RAut** on a single generating object.

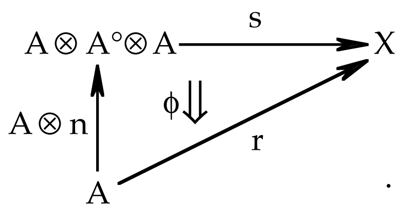
2. Composition of extraordinary 2-cells

The **calculus** of extraordinary 2-cells describes their composition. There are two basic types of composition: one involves composing two extraordinary 2-cells to get an ordinary 2-cell, while the other involves composing an extraordinary 2-cell with an ordinary 2-cell to get an extraordinary 2-cell.

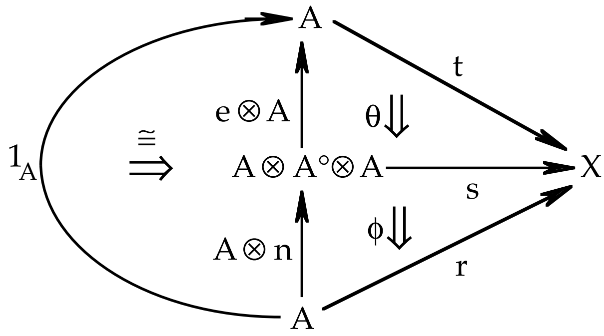
Suppose θ is an extraordinary 2-cell from $t : A \longrightarrow X$ to $s : A \otimes A^\circ \otimes A \longrightarrow X$, and suppose ϕ is an extraordinary 2-cell from $s : A \otimes A^\circ \otimes A \longrightarrow X$ to $r : A \longrightarrow X$. This means we have 2-cells



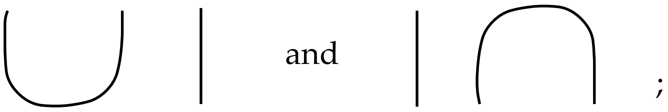
and



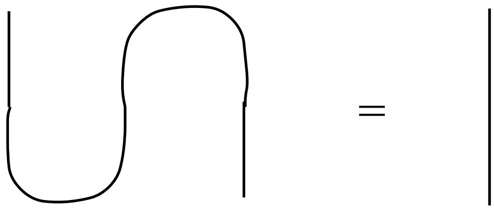
Pasting these two 2-cells together with the canonical isomorphism $(e \otimes A) \circ (A \otimes n) \cong 1_A$ as in the diagram below, we define the composite of θ and ϕ as the resultant ordinary 2-cell from t to r.



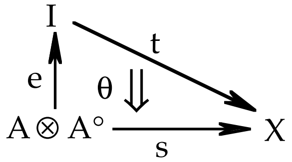
The morphisms of **RAut** representing θ and ϕ are

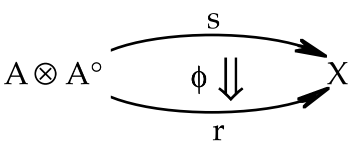


the morphism of **RAut** representing the composite of θ and ϕ is precisely the composite in **RAut** of the representing morphisms.

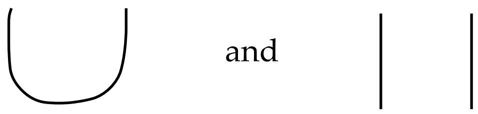


Suppose, as another example of the calculus, that θ is an extraordinary 2-cell from $t : I \longrightarrow X$ to $s : A \otimes A^\circ \longrightarrow X$, and suppose ϕ is an extraordinary 2-cell from $s : A \otimes A^\circ \longrightarrow X$ to $r : A \otimes A^\circ \longrightarrow X$. This means we have 2-cells



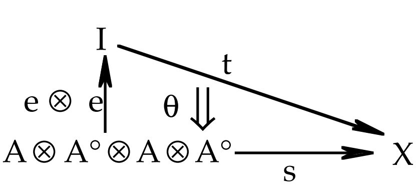


which paste along s to yield the composite extraordinary 2-cell from t to r. The morphisms in **RAut** representing these 2-cells are

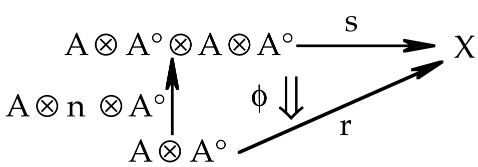


which compose to give the former (since the latter is an identity morphism of **RAut**).

As a final example of extraordinary composition, suppose θ is an extraordinary 2-cell from $t : I \longrightarrow X$ to $s : A \otimes A^\circ \otimes A \otimes A^\circ \longrightarrow X$, and suppose ϕ is an extraordinary 2-cell from s to $r : A \otimes A^\circ \longrightarrow X$. This means we have 2-cells

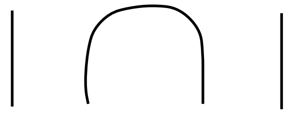


and

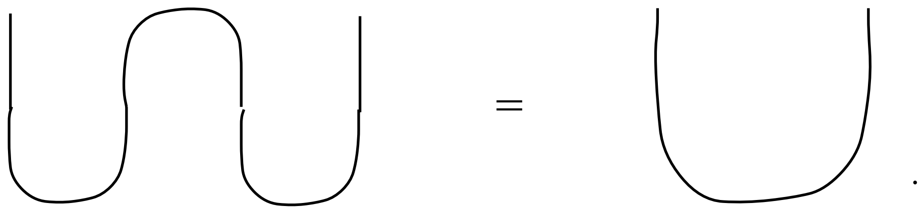


which paste along s and paste with the canonical isomorphism $(e \otimes e)\circ(A \otimes n \otimes A^\circ) \cong e$ to yield a composite extraordinary 2-cell from t to r. In **RAut** this is represented by the fact that the composite of

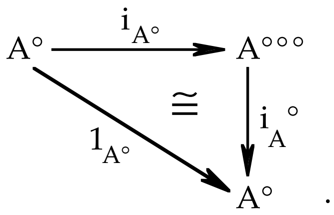
and



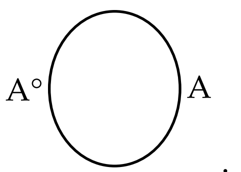
is



Of course, it is also possible to consider the phenomenon of extraordinariness in a left autonomous monoidal bicategory $\mathcal{M}$; for this, just reverse the tensor product in the right autonomous case. It becomes somewhat more interesting when $\mathcal{M}$ is autonomous; that is, is both left and right autonomous. The calculus here is governed by the free autonomous monoidal category **Aut** on a single generating object; the morphisms are represented by plane string diagrams which can make both left and right turns. However, the situation becomes much more interesting when $\mathcal{M}$ is a *pivotal* monoidal bicategory; that is, autonomous with the right and left duals coherently equivalent. Pivotal structure consists of an equivalence $i_A : A \longrightarrow A^{\circ\circ}$, pseudonatural in A, and an isomorphism



In this case, we obtain not only a counit $e : A \otimes A^\circ \longrightarrow I$ and unit $n : I \longrightarrow A^\circ \otimes A$ expressing A° as a right bidual for each A, but a counit $e' : A^\circ \otimes A \longrightarrow I$ and unit $n' : I \longrightarrow A \otimes A^\circ$ expressing A° as a left bidual for A. The extraordinary composition calculus here is governed by the free pivotal monoidal category **Pvt** on a single generating object. The morphisms here are again string diagrams in the plane, but now may contain (directed) loops: for example, we may now compose $n : I \longrightarrow A^\circ \otimes A$ with $e' : A^\circ \otimes A \longrightarrow I$ to obtain an endomorphism of I represented by a loop:



We can either define these endomorphisms of I, represented by loops, as part of the composite of extraordinary 2-cells, or, restrict composition to those cases where no loops arise.

3. The example of extraordinary natural transformations

Let $\mathcal{V}$ denote a cocomplete symmetric closed monoidal category. We remind the reader of the autonomous monoidal bicategory $\mathcal{V}$-**Mod** of $\mathcal{V}$-modules. The objects are (small) $\mathcal{V}$-categories (in the sense of [EK]). The morphisms $M : A \longrightarrow B$ are $\mathcal{V}$-functors $M : B^{op} \otimes A \longrightarrow \mathcal{V}$. The $\mathcal{V}$-natural transformations $\theta : M \Rightarrow N : B^{op} \otimes A \longrightarrow \mathcal{V}$ are the 2-cells $\theta : M \Rightarrow N : A \longrightarrow B$ of $\mathcal{V}$-**Mod**; they are called module morphisms. Vertical composition of 2-cells is vertical composition of natural transformations. The horizontal composite $N\,M : A \longrightarrow C$ of $M : A \longrightarrow B$ and $N : B \longrightarrow C$ is given by the coend formula

$$(N\,M)(c,a) = \int^{b} M(b,a) \otimes N(c,b) \ ,$$

which is $\mathcal{V}$-functorial in M and N. Each $\mathcal{V}$-functor $f : A \longrightarrow B$ can be identified with the module $f_* : A \longrightarrow B$ having $f_*(b,a) = B(b,fa)$, and this gives an inclusion

$$\mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Mod}.$$

(We are writing $\mathcal{V}$-**Cat** for the usual 2-category of small $\mathcal{V}$-categories and writing $\mathcal{V}$-**CAT** when we drop the smallness requirement.) Recall too that $f_* : A \longrightarrow B$ has a right adjoint $f^* : B \longrightarrow A$ having $f^*(a,b) = B(fa,b)$. If $\mathcal{V}$ is furthermore complete, for each small $\mathcal{V}$-category B there is a $\mathcal{V}$-category $\mathcal{P}B$ and an equivalence of categories

$$\mathcal{V}\text{-}\mathbf{CAT}(A,\mathcal{P}B) \simeq \mathcal{V}\text{-}\mathbf{Mod}(A,B).$$

With this we see that $\mathcal{V}$-**Mod** is biequivalent to the sub-2-category of $\mathcal{V}$-**CAT** consisting of the objects of the form $\mathcal{P}A$ and the morphisms $\mathcal{P}A \longrightarrow \mathcal{P}B$ with right adjoints. The object assignments $(A,B) \longmapsto A \otimes B$ and $A \longmapsto A^{op}$ extend to pseudofunctors

$$\otimes : \mathcal{V}\text{-}\mathbf{Mod} \times \mathcal{V}\text{-}\mathbf{Mod} \longrightarrow \mathcal{V}\text{-}\mathbf{Mod} \quad \text{and} \quad (\ )^{op} : \mathcal{V}\text{-}\mathbf{Mod}^{op} \longrightarrow \mathcal{V}\text{-}\mathbf{Mod}.$$

The object I of $\mathcal{V}$-**Mod** is the unit for the tensor of $\mathcal{V}$ regarded as a $\mathcal{V}$-category with one object 0. The modules $e : A \otimes A^{\circ} \longrightarrow I$ and $n : I \longrightarrow A^{\circ} \otimes A$ are given by

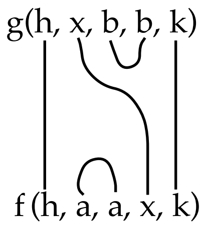
$$e(0,a,b) = A(b,a) \quad \text{and} \quad n(a,b,0) = A(a,b).$$

This gives what is required to make $\mathcal{V}$-**Mod** an autonomous monoidal bicategory. To actually get a Gray monoid, we need to choose a skeletal category of sets, a strict monoidal category equivalent to $\mathcal{V}$, and use the method above to replace $\mathcal{V}$-**Mod** by a 2-category (obtaining the $\mathcal{V}$-**Mog** of [DS; Section 7]). With care, this can all be done with $\mathcal{V}$ braided rather than merely symmetric.

Let us examine what extraordinary 2-cells are in $\mathcal{V}$-**Mod** when the morphisms they go between are in $\mathcal{V}$-**Cat**. So, suppose $x : I \longrightarrow X$ and $f : A \otimes A^{\circ} \longrightarrow X$ are $\mathcal{V}$-functors; we identify x with the object $x(0)$ of X. An extraordinary 2-cell θ from x_* to f_* is a module morphism $\theta : x_* e \Rightarrow f_*$. These are in natural bijection with module morphisms $e \Rightarrow x^* f_*$, and these amount to families of morphisms $A(a', a) \longrightarrow X(x, f(a, a'))$ $\mathcal{V}$-natural in objects a and a' of A. Using the Yoneda lemma, we see that such families are in bijection with families $x \longrightarrow f(a, a)$ extraordinarily $\mathcal{V}$-natural in the variable a.

Next suppose $g : B^\circ \otimes B \longrightarrow Y$ and $y : I \longrightarrow Y$ are $\mathcal{V}$-functors. An extraordinary 2-cell ϕ from g_* to y_* is a module morphism $\phi : g_* n \Rightarrow y_*$. These are in natural bijection with module morphisms $n \Rightarrow g^* y_*$, and these amount to families of morphisms $B(b, b') \longrightarrow Y(g(b', b), y)$ $\mathcal{V}$-natural in objects b and b' of B. Using the Yoneda lemma, we see that such families are in bijection with families $g(b, b) \longrightarrow y$ extraordinarily $\mathcal{V}$-natural in the variable b.

Similarly, in the more general case of $\mathcal{V}$-functors $g : H \otimes X \otimes B^\circ \otimes B \otimes K \longrightarrow Y$ and $f : H \otimes A \otimes A^\circ \otimes X \otimes K \longrightarrow Y$, an extraordinary 2-cell ψ from g to f amounts to a family of morphisms $\psi_{h, x, a, b, k} : g(h, x, b, b, k) \longrightarrow f(h, a, a, x, k)$ in Y which is natural in h, x and k, and extraordinary natural in a and b. The Eilenberg-Kelly graph for this situation is



which we see precisely is the string diagram which represents the same morphism as described above in the free right autonomous monoidal category **RAut** generated by a single object.

The bicategory $\mathcal{V}$-**Mod** is symmetric monoidal in the sense of [DyS] and hence pivotal. So loops can occur. The course of action taken in [EK1] was to forbid composition when loops occur. The full functorial calculus of Eilenberg-Kelly can now be seen to be identical with calculating in the free autonomous symmetric monoidal category **Autsym** on a single generating object (see [Dy2] for a construction, where autonomous symmetric monoidal categories are called "compact closed"); a model of **Autsym** was produced in [KL] where the morphisms are closely related to Feynman diagrams [BD]. The amazing coincidence is that this study of **Autsym** in [KL] was presumably in ignorance of its precise relationship to the Eilenberg-Kelly calculus.

References

[B] J. Bénabou, Introduction to bicategories, *Lecture Notes in Math.* **47** (Springer, Berlin, 1967) 1- 77.

[BD] J. Baez and J. Dolan, From finite sets to Feynman diagrams, *Mathematics Unlimited - 2001 and Beyond*, eds. Bjorn Engquist and Wilfried Schmid (Springer-Verlag, Berlin, 2001) 29-50.

[Dy1] B.J. Day, On closed categories of functors, *Lecture Notes in Math.* **137** (Springer, Berlin, 1970) 1- 38.

[Dy2] B.J. Day, Note on compact closed categories. *J. Austral. Math. Soc. Ser. A* **24** (1977) 309--311.

[DyS] B.J. Day and R. Street, Monoidal bicategories and Hopf algebroids, *Advances in Math.* **129** (1997) 99-157.

[DcS] E. Dubuc and R. Street, Dinatural transformations, *Lecture Notes in Math.* 137 (Springer, Berlin, 1970) 126-137.

[EM1] S. Eilenberg and S. Mac Lane, Natural isomorphisms in group theory, *Proc. Nat. Acad. Sci. U.S.A.* **28** (1942) 537-543.

[EM2] S. Eilenberg and S. Mac Lane, General theory of natural equivalences. *Trans. Amer. Math. Soc.* **58** (1945) 231-294.

[EK1] S. Eilenberg and G.M. Kelly, A generalization of the functorial calculus, *J. Algebra* **3** (1966) 366-375.

[EK2] S. Eilenberg and G.M. Kelly, Closed categories, *Proceedings of the Conference on Categorical Algebra at La Jolla* (Springer, 1966) 421-562.

[GPS] R. Gordon, A.J. Power and R. Street, Coherence for tricategories, *Memoirs of the American Math. Society* **117** #558 (1995) (ISBN 0-8218-0344-1).

[G1] J.W. Gray, *Formal Category Theory: Adjointness for 2-Categories* Lecture Notes in Math. **391** (Springer, Berlin, 1974).

[G2] J.W. Gray, *Coherence for the tensor product of 2-categories, and braid groups*, Algebra, Topology, and Category Theory (a collection of papers in honour of Samuel Eilenberg), (Academic Press, New York 1976) 63-76.

[JS1] A. Joyal and R. Street, Braided tensor categories, *Advances in Math.* **102** (1993) 20-78.

[JS2] A. Joyal and R. Street, The geometry of tensor calculus I, *Advances in Math.* **88** (1991) 55-112.

[K] G.M. Kelly, Tensor products in categories, *J. Algebra* **2** (1965) 15-37.

[KL] G.M. Kelly and M.L. Laplaza, Coherence for compact closed categories, *J. Pure Appl. Algebra* **19** (1980) 193-213.

[KS] G.M. Kelly and R. Street, Review of the elements of 2-categories, *Lecture Notes in Math.* **420** (Springer, Berlin, 1974) 75- 103.

[ML] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Math. **5** (Springer-Verlag, 1971).

[MT] M. McIntyre and T. Trimble, The geometry of Gray monoids (preprint).

[SV] R. Street and D. Verity, Low–dimensional topology and higher-order categories, *Proceedings of CT95, Halifax, July 9-15, 1995* <**http://www.mta.ca/~cat-dist/ct95.html**>.

Centre of Australian Category Theory
Macquarie University
N. S. W. 2109
AUSTRALIA
email: street@math.mq.edu.au