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HOMOTOPY CLASSIFICATION BY DIAGRAMS OF INTERLOCKING SEQUENCES

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Abstract

Functors which assign certain diagrams of interlocking sequences to certain diagrams of complexes of free abelian groups are shown to be full on homotopy classes of maps of diagrams. The kernels of these functors are provided by Ext in the category of diagrams of interlocking sequences. Some of the theory of differential graded categories is developed which provides the language and machinery for our results.

AMS Class 55U20, 18D20, 18G40, 55N20, 18E30.

Introduction

The homotopy classification theorem [3] provides a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}(HA,HB) \longrightarrow \mathrm{HCAb}(A,B) \longrightarrow \mathrm{GAb}(HA,HB) \longrightarrow 0$$

of graded abelian groups where $H : \mathrm{HCAb} \longrightarrow \mathrm{GAb}$ is the homology graded functor from the graded category HCAb of complexes of abelian groups and homotopy classes of chain maps (of all degrees) to the graded category GAb of graded abelian groups. The hypothesis that A is a complex of free abelian groups is required. (In fact the short exact sequence has an unnatural splitting.)

This leads one to ask whether there is such a theorem when A,B are replaced by a specified type of diagrams of complexes of abelian groups. Apart from abelianness, the essential attribute of the category Ab of abelian groups used to prove the theorem is that it has projective dimension 1 (a subgroup of a free abelian group is free). It is known [12] that only for trivial categories C does the functor category $[C,Ab]$ have projective dimension 1. In fact, $H : HC[C,Ab] \longrightarrow G[C,Ab]$ is not full on the complexes of projective objects even when C is the non-discrete ordered set 2 with two objects.

There is a famous type of diagram of complexes which has been studied successfully using homological data : filtered complexes using spectral sequences. In forming the spectral sequence of a filtered complex

$$A: \quad A^1 \leq A^2 \leq A^3 \leq \ldots,$$

one not only uses the graded groups HA^p but also the graded groups $H(A^p/A^{p-1})$. However, even if we restrict to filtered complexes for which each A^p/A^q is a complex of free abelian groups and the filtrations of length 3, the spectral sequence functor is not full. What is needed is to take the diagram involving all the objects $H(A^p/A^q)$. It is shown in [14] and [16] that one does then obtain a homotopy classification theorem for finite filtrations.

This last result seems to require a restriction on the kind of diagram considered, namely, that the maps therein should be monic. The diagram type for filtrations of length N is the linearly ordered set C with N objects. In this case, projectives in $[C,Ab]$ do take each map in C to a monic; so to say A is a complex of projective objects of $[C,Ab]$ is to say each $A^q \longrightarrow A^p$ is monic and each A^p/A^q is a complex of free abelian groups. In the homotopy classification theorem, B can be an arbitrary object of $C[C,Ab]$, but then we must replace $H(B^p/B^q)$ by the homology of the mapping cone of $B^q \longrightarrow B^p$.

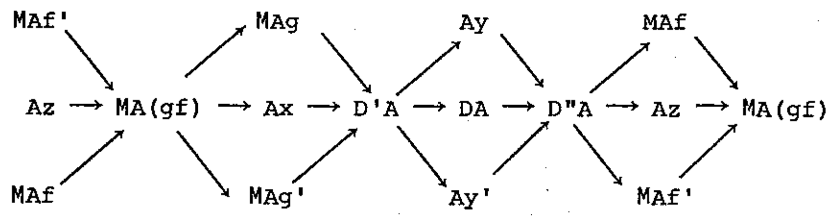
Early in 1968 the author [13] proved a homotopy classification theorem for the case where C is 2×2 :

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ y' & \xrightarrow[g']{} & z \end{array} .$$

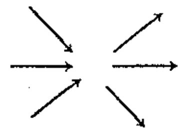
The result remained unpublished and did not appear in the author's thesis [14]. The diagram one assigns to a complex A in $[C, \mathrm{Ab}]$ is the homology of the diagram obtained in forming the mapping cones MAf, MAf', MAg, MAg', MA(gf) of all the values of A at arrows in C , then the mapping cones DA, D'A of the induced maps MAf $\longrightarrow$ MAg', Ay $\longrightarrow$ MAg' , and then the mapping cone D"A of the induced map Az $\longrightarrow$ DA . This can be depicted in a diagram:

$$\begin{array}{ccccccc} Ax & \longrightarrow & Ay & \longrightarrow & MAf & \longrightarrow & Ax \\ \downarrow & MA(gf) & \downarrow & D'A & \downarrow & D''A & \downarrow \\ Ay' & \longrightarrow & Az & \longrightarrow & Ag' & \longrightarrow & Ay' \\ \downarrow & D'A & \downarrow & D''A & \downarrow & MA(gf) & \downarrow \\ MAf' & \longrightarrow & MAg & \longrightarrow & DA & \longrightarrow & MAf' \\ \downarrow & D''A & \downarrow & MA(gf) & \downarrow & D'A & \downarrow \\ Ax & \longrightarrow & Ay & \longrightarrow & MAf & \longrightarrow & Ax \end{array}$$

in which the object in each square is homotopically equivalent to the mapping cone of the diagonal of the square. This leads to the *three-diamond diagram*:



which properly lives on the surface of a Moebius band; in each diamond the middle composite is the sum of the outside two, and at each vertex of kind



the composites of parallel arrows are 0 .

Diagrams of the three-diamond type were constructed by Wall [22]. It was Keith Hardie's interest in such diagrams which led me to reconsider my work [13] and to contemplate again a common generalization of that work and the filtered complex case [16]. The present paper represents my achievements in that direction. It also provides an opportunity to develop the theory of categories enriched in the closed category CAb , so called DG-*categories*.

Why DG-categories? Starting with a diagram A of type $\mathcal{C}$ we have constructed above a larger diagram $H_{\mathcal{D}}A$ of a certain type $\mathcal{D}$. The basic operation in this construction is that of *mapping cone*. To keep the creation of new objects to a minimum one considers graded objects and allows degree shifts, *suspension*, as operations too. At the semantic level the construction is that of $\mathcal{D}$ from $\mathcal{C}$. Mapping cones and suspensions are naturally occurring limits for DG-categories and are precisely what is needed to build $\mathcal{D}$ from $\mathcal{C}$. In fact, these limits are *absolute* so that, when they are possessed by a DG-category $\mathcal{A}$, the DG-functor category $[\mathcal{D},\mathcal{A}]$ is equivalent to $[\mathcal{C},\mathcal{A}]$.

We prove homotopy classification theorems for graded functors of the form

$$H_{\mathcal{D}} : H[\mathcal{C},CAb] \longrightarrow [H\mathcal{D},GAb]$$

induced by homology. This involves an analysis of the projective dimension of objects of $[H\mathcal{D},GAb]$. Most such objects (unless $\mathcal{C}$ is discrete) have infinite projective dimension; however, the interesting objects are those which are in the image of $H_{\mathcal{D}}$ and these have projective dimension 1. The objects of projective dimension 1 are precisely the objects F with certain sequences in the diagram F exact; these sequences are precisely the ones which in the case where $F = H_{\mathcal{D}}A$ *are* exact by virtue of their coming from mapping cone sequences via homology. In order to prove this it is necessary to characterize the projective objects of $[H\mathcal{C},GAb]$ as those F which take certain sequences to exact sequences and have projective values; for such F we are able to apply a Fourier-Moebius-inversion argument to solve the equation

$$F \cong \textstyle\sum_D H\mathcal{D}(D,-) \otimes P_D$$

for projective objects P_D of GAb . (We are reminded of André Joyal's lectures on a structural generalization of Rota's work on quantitative Moebius inversion.) This provides the main theorem of [15] as a particular case.

I am grateful to Murray Adelman for many helpful conversations on this material; in particular, for his insistence that there was an analogy with Fourier theory.

§1. Differential graded categories

A *complex* A in an additive category G is a diagram

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

in which $d_n d_{n+1} = 0$ for all integers n. For complexes A,B in G, a complex $CG(A,B)$ in the category Ab of abelian groups is defined by:

$$CG(A,B)_n = \Pi_{r\in Z} G(A_r, B_{r+n}), \quad d_n(f)_r = d_{r+n}f_r - (-1)^n f_{r-1}d_r .$$

There is an additive category Z_oCG whose objects are complexes in G and whose arrows $f : A \longrightarrow B$ are elements of $CG(A,B)_o$ with $d_o(f) = 0$. Objects of G are identified with objects A of Z_oCG for which $A_n = 0$ for all $n \neq 0$.

The *suspension* SA of a complex A is the complex for which $(SA)_n = A_{n-1}$ and d_n for SA is $-d_{n-1}$ for A. The inverse operation S^{-1} is *desuspension*.

When G has finite (co)products, each arrow $f : A \longrightarrow B$ in Z_oCG has a *mapping cone* Mf which is the complex given by

$$(Mf)_n = B_n \oplus A_{n-1}, \quad d_n = \begin{pmatrix} d_n & f_{n-1} \\ 0 & -d_{n-1} \end{pmatrix} .$$

Notice that the mapping cone of $0 \longrightarrow B$ is B and the mapping cone of $A \longrightarrow 0$ is SA.

There is a symmetric monoidal closed category CAb described as follows. The underlying category is the category Z_oCAb of complexes in Ab, the internal hom is CAb(A,B), and the tensor product $A \otimes B$ is defined by:

$$(A\otimes B)_n = \sum_{p+q=n} A_p \otimes B_q, \quad d(a\otimes b) = da\otimes b + (-1)^p a\otimes db$$

for $a \in A_p$, $b \in B_q$ (where we have omitted the subscripts from the arrows d_n as we shall continue to do when no confusion is likely).

A DG-*category* (or *differential graded category*) is a category with homs enriched in the base CAb ; that is, a CAb-category. The general theory of enriched categories is by now well developed [11, 10 ,20], however, this special case has its own peculiarities as one expects.

A DG-category A has objects, and, for each pair of objects A,B , a complex $A(A,B)$ in Ab . Elements of $A(A,B)_n$ are called *protomaps of degree* n *from* A *to* B . Protomaps $f : A \longrightarrow B$, $g : B \longrightarrow C$ of degree m,n , respectively, compose to give a protomap $gf : A \longrightarrow C$ of degree $m+n$ with:

$$d(gf) = d(g)f + (-1)^n gd(f) .$$

A protomap f with $d(f) = 0$ is called a *map*. Each object A has an identity $1_A : A \longrightarrow A$ for composition which is a map of degree 0 . Composition is associative. (Note that the protomaps themselves are the arrows for an additive category and so we may speak of *invertible protomaps*.)

A DG-*functor* $F : A \longrightarrow X$ is precisely a CAb-functor. It assigns to each object A of A an object FA of X and to each protomap $f : A \longrightarrow B$ of degree n in A a protomap $Ff : Ff \longrightarrow FB$ of degree n in X such that $Fd(f) = d(Ff)$, $F1_A = 1_{FA}$, $F(gf) = (Fg)(Ff)$.

For DG-functors $F,G : A \longrightarrow X$, a *protonatural transformation* (p.n.t.) $\alpha : F \longrightarrow G$ *of degree* n is a family of protomaps $\alpha_A : FA \longrightarrow GA$ of degree n in X such that

$$\alpha_B \cdot Ff = (-1)^{pn} Gf \cdot \alpha_A$$

for all protomaps $f : A \longrightarrow B$ of degree p in A . A DG-*natural transformation* is a p.n.t. α for which each α_A is a map of degree 0 .

There is a (meta-) DG-category $[A,X]$ whose objects are the DG-functors from A to X, whose protomaps of degree n are p.n.t.'s of degree n, and whose differentials are given by $d(\alpha)_A = d(\alpha_A)$.

Each DG-category A has an underlying additive category $Z_0 A$ with the same objects and with the maps of degree 0 as arrows. Each additive category G can be regarded as a DG-category with all hom-complexes 0 except perhaps in degree 0.

Each additive category G yields a DG-category CG whose objects are complexes in G and whose hom-complexes are $CG(A,B)$ as defined above.

When dealing with categories enriched in a closed category V, the useful notion of limit ("indexed limit") is described in Kelly's book [10; Ch.3]. When V is the category Set of sets this notion looks more general than the notion of limit usually considered for ordinary categories ("conical limits"). The reason that conical limits "suffice" for ordinary categories is that the internal hom and tensor product in Set satisfy:

$$\mathrm{Set}(X,Y) \cong \prod_X Y ,$$

$$X \times Y \cong \sum_X Y .$$

For a general V, it is necessary to regard $V(X,Y)$, $X \otimes Y$ as limits, colimits in V in the V-enriched sense; so the correct notion of limit should incorporate both conical limits and cotensoring.

This correct notion for $V = Ab$ appears in Freyd's book [6 ; Ch.5, Exercise I] under the name "symbolic hom" (limit) and "tensor product" (colimit) of functors. That tensoring as well as conical colimits should be considered colimits in the enriched setting was maintained in the author's thesis [14] for the case V = CAb ; furthermore, *suspension* was given as an example. This point of view for general V was developed in Day-Kelly [4] although the notion of indexed limit does not appear there (nor in [14]). In the early 1970's Street-Walters generalized Freyd's tensor products to Yoneda structures and hence included enriched categories (although this work was not published until [21]). Special kinds of Yoneda structures appeared in [17] where the notion was made explicit; in Street [17; Section 6] the case V = Cat was explained in detail and examples were given showing that many constructions in a 2-category were limits in this sense. Auderset [1] and Borceux-Kelly [2] defined indexed limits (but not under that name) for general V and developed some of the general theory (in the former case, mainly for V = Cat). Unfortunately the name "indexed limit" came from the author's paper [18]; this paper showed that lax limits were indexed limits for V = Cat and examined the combinatorial problem of determining which indexed limits existed in any finitely complete 2-category. I believe the term "indexed limit" is not used in that paper in isolation from the category-valued 2-functor which is its index. My purpose was to emphasize that the limits should not merely be indexed by categories (or 2-categories) but by V-valued V-functors (or better, V-modules). All limits are indexed by something! The correct term (if one other than "limit" is needed) should indicate the *nature* of the indexing type. In my Milan lecture (November, 1981), I emphasized the weighting nature of the indexing V-functor and suggested the term *weighted limit*. I am not happy with the term "mean tensor product" used in [2] since this de-emphasizes the asymmetric roles played by the two arguments.

In the present paper we are concerned with the case V = CAb . Suppose $J : A \longrightarrow$ CAb is a DG-functor. A J-*weighted limit* for a DG-functor $F : A \longrightarrow X$ is an object lim(J,F) of X together with a DG-natural isomorphism

$$X(X,\lim(J,F)) \cong [A,CAb](J,X(X,F)).$$

The isomorphism is determined by its value at the identity for $X = \lim(J,F)$: this gives a DG-natural transformation $\lambda : J \longrightarrow X(\lim(J,F),F)$ such that, for all p.n.t.s $\alpha : J \longrightarrow X(X,F)$, there exists a unique protomap $f : X \longrightarrow \lim(J,F)$ with $\alpha_A(x) = (-1)^{np}\lambda_A(x)f$ for all $A \in A$, $p \in Z$, $x \in (JA)_p$, and n is the degree of α.

Examples. 1) Conical limits. For any ordinary category L, let A be the free DG-category on L so that a functor $L \longrightarrow Z_oX$ amounts to a DG-functor $A \longrightarrow X$. Take $J : A \longrightarrow CAb$ to be the DG-functor corresponding to the functor $L \longrightarrow CAb$ which is constant at $Z \in Ab \subset CAb$. The J-weighted limit of a DG-functor $F : A \longrightarrow X$ is called the *conical limit* in X of the corresponding functor $L \longrightarrow Z_oX$. If L is discrete then the conical limit is the *product* in X; if L is $\cdot \rightrightarrows \cdot$ then the conical limit is the *equalizer* in X; and so on. Finite products are also called *direct sums* since they are finite coproducts too (as for additive categories).

2) Cotensors. Let I be the free DG-category on the discrete 1 with one object 0. A DG-functor $K : I \longrightarrow X$ amounts to an object K of X. The *cotensor* $J \pitchfork K$ in X of $J \in CAb$ with K is the limit of $K : I \longrightarrow X$ weighted by $J : I \longrightarrow CAb$; it satisfies $X(X, J \pitchfork K) \cong CAb(J, X(X,K))$.

3) Suspension. The *suspension* SK of K in X is the cotensor $S^{-1}Z \pitchfork K$ in X (where $S^{-1}Z$ has Z in degree -1 and 0 elsewhere); it satisfies:

$$X(X,SK) \cong SX(X,K).$$

The *desuspension* $S^{-1}K$ of K is the cotensor $SZ \pitchfork K$ in X. When both exist one has:

$$X(S^mX,S^nK) \cong S^{n-m}X(X,K)$$

for all $m,n \in Z$, and one calls X a *stable* DG-category.

Notice that $K' \cong SK$ if and only if there exists a protoisomorphism $K' \longrightarrow K$ of degree -1.

4) Mapping Cone. Let C denote the mapping cone of the identity of $S^{-1}Z$; so $C_n = 0$ for $n \neq 0,-1$, and $d : C_o \longrightarrow C_{-1}$ is the identity of Z. There is a map $j : S^{-1}Z \longrightarrow C$ of degree 0 which is the identity in degree-1. Let A be the free DG-category on the ordered set 2. A DG-functor $F : A \longrightarrow X$ amounts to a map $f : A \longrightarrow B$ of degree 0 in X. Let $J : A \longrightarrow CAb$ be the DG-functor corresponding to the map $j : S^{-1}Z \longrightarrow C$. The *mapping cone* Mf of f in X is $\lim(J,F)$ where F corresponds to f; it satisfies:

$$X(X,Mf) \cong MX(X,f) .$$

The identity of Mf corresponds under this isomorphism to a map $p_f : Mf \longrightarrow A$ of degree -1 and a protomap $q_f : Mf \longrightarrow B$ of degree 0 satisfying $d(q_f) + fp_f = 0$. Furthermore, for all protomaps $a : X \longrightarrow A$, $b : X \longrightarrow B$ with $\deg a = \deg b + 1$, there exists a unique protomap $u : X \longrightarrow Mf$ such that $p_f u = a$, $q_f u = b$. In particular, $0 : B \longrightarrow A$, $1_B : B \longrightarrow B$ yield a map $j_f : B \longrightarrow Mf$ of degree 0 such that $p_f j_f = 0$, $q_f j_f = 1_B$; and, $1_A : A \longrightarrow A$, $0 : A \longrightarrow B$ yield a protomap $i_f : A \longrightarrow Mf$ of degree 1 such that $p_f i_f = 1_A$, $q_f i_f = 0$.

$$B \underset{q_f}{\overset{j_f}{\rightleftarrows}} Mf \underset{i_f}{\overset{p_f}{\rightleftarrows}} A$$

The equations $i_f p_f + j_f q_f = 1_{Mf}$, $f = q_f d(i_f)$, $fp_f = -d(q_f)$, $j_f f = d(i_f)$ can be deduced.

Conversely, given any diagram

$$B \underset{q}{\overset{j}{\rightleftarrows}} C \underset{i}{\overset{p}{\rightleftarrows}} A$$

in X in which the solid arrows are maps and the dotted arrows are protomaps, the degrees of $i,j,p,q,$ are $1,0,-1,0,$ respectively, and the equations

$$pi = 1_A, \quad qj = 1_B, \quad ip + jq = 1_C$$

are satisfied, then C is the mapping cone of $qd(i)$. The existence of 0 in X (which is implied if X is non-empty and idempotents split) allows us to construct the suspension SA as the mapping cone of $A \longrightarrow 0$.

5) Colimits. Mapping cones, suspensions, desuspensions, finite products and splittings of idempotents are not typical limits at all. They are *absolute* in the sense that they are preserved by all DG-functors. In particular, they are preserved by the contravariant representables and are weighted *co*limits. (Cauchy complete DG-categories admit all these limits.) □

For each DG-category A, there is a DG-category CA described as follows (this will generalize the construction given earlier for the case where A is merely additive). The objects are complexes in the additive category $Z_0 A$. The hom-complexes are given by:

$$CA(A,B)_n = \Pi_{r,s} A(A_r, B_s)_{n+r-s}$$

$$d(f)_{r,s} = df_{r,s+1} + (-1)^{n+1} f_{r-1,s} d + (-1)^s d(f_{r,s}).$$

An object A of CA is called *bounded* when there exists a natural number n such that $A_m = 0$ for $|m| > n$. When A has 0, we identify objects of A with complexes which are 0 in all degrees except perhaps in degree 0. This gives a fully faithful DG-functor $I : A \longrightarrow CA$.

Each DG-functor $F : A^{op} \longrightarrow CAb$ extends to a DG-functor $\hat{F} : (CA)^{op} \longrightarrow CAb$ via the formulas:

$$(\hat{F}A)_n = \Pi_r (FA_r)_{n+r}, \quad d(a)_r = d(a_r) + (Fd)a_{r-1} .$$

The DG-natural isomorphism $\hat{F}I^{op} \cong F$ induces a canonical DG-natural transformation from $\hat{F}$ to the right Kan extension of F along $I^{op}: A^{op} \longrightarrow (CA)^{op}$; the component at $A \in CA$ is a map of degree 0 in CAb as follows:

$$\hat{F}A \longrightarrow [A^{op}, CAb](CA(I-,A),F) .$$

This map is monic for all A; *if* A *is bounded then it is an isomorphism* (an extension of Yoneda's lemma reminiscent of [19; §5]). It follows that the full sub-DG-category C_bA of CA consisting of the bounded complexes is equivalent to the full sub-DG-category of $[A^{op}, CAb]$ consisting of the objects $CA(I-,A)$ with A bounded (this uses the fact that $\hat{F}A = CA(A,B)$ when $F = CA(I-,B)$). In fact, C_bA is a free completion in the following sense.

Proposition 1. *If* A *is a DG-category with direct sums then* C_bA *is stable and has direct sums and mapping cones. For all stable DG-categories* X *with direct sums and mapping cones, composition with the inclusion* $A \longrightarrow C_bA$ *yields an equivalence of DG-categories:*

$$[C_bA, X] \cong [A, X]$$

Proof. The objects of $[A^{op}, CAb]$ of the form $CA(I-,A)$, with A bounded, are closed under desuspension and mapping cone. Furthermore, every such object can be obtained from representables in $[A^{op}, CAb]$ by iterated desuspensions and mapping cones. □

(In fact, the *cauchy completion* QA[11, 10, 20] of a DG-category A with direct sums is the idempotent completion of C_bA.)

§2. Homology and homological functors.

A *graded object* in an additive category G is a complex A for which each $d_n = 0$. Write GG for the sub-DG-category of CG consisting of the graded objects. Each hom-complex of GG is a graded abelian group. So GAb is a closed category. A G-*category* (or *graded category*) is a GAb-category; that is, a DG-category for which all the hom-complexes are graded abelian groups. In a G-category each protomap is a map.

A G-category X is called *abelian* when it is stable and $Z_o X$ is an abelian category. A sequence

$$A' \xrightarrow{\quad u \quad} A \xrightarrow{\quad v \quad} A''$$

of maps u,v of degree m,n in X is called *exact* when the corresponding sequence

$$S^m A' \longrightarrow A \longrightarrow S^{-n} A''$$

in $Z_o X$ is exact.

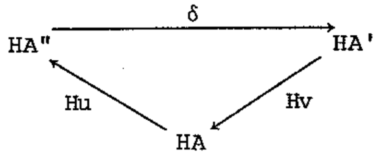
Suppose G is an abelian category. Then GG is an abelian G-category. A complex A in G can be identified with an object A of GG enriched with a map $d : A \longrightarrow A$ of degree -1 and $dd = 0$. From such a complex A, we obtain objects dA, ZA, HA which appear in the exact sequences

$$0 \longrightarrow ZA \xrightarrow{i} A \xrightarrow{\eta} dA \longrightarrow 0$$
$$0 \longrightarrow dA \xrightarrow{j} ZA \xrightarrow{\zeta} HA \longrightarrow 0$$

in GG where $d = ij\eta$ and i,j,η,ζ have degrees $0,0,-1,0$, respectively. The object HA is called the *homology of* A. It is well known that each sequence

$$0 \longrightarrow A' \xrightarrow{u} A \xrightarrow{v} A'' \longrightarrow 0$$

of maps of complexes which is exact in GG yields an exact triangle

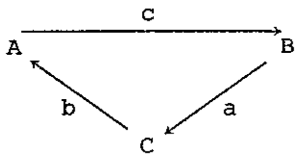


in GG where the sum of the degrees of the maps Hu, Hv, δ is -1.

If A is a complex in Ab, elements of A_p are called elements of A of degree p. Elements a of ZA are called *cycles* and $\zeta(a) \in HA$ is called the *homology class* of a. We write $a \simeq b$ when a,b are cycles with the same homology class.

Each DG-category $\mathcal{A}$ yields G-categories $Z\mathcal{A}$, $H\mathcal{A}$ with the same objects as $\mathcal{A}$, where the maps of $Z\mathcal{A}$ are the maps of $\mathcal{A}$, and where the maps of $H\mathcal{A}$ are the homology classes of maps of $\mathcal{A}$. Maps $f,g : A \longrightarrow B$ in $\mathcal{A}$ are called *homotopic* when $f \simeq g$; that is, when f,g have the same homology class in $(H\mathcal{A})(A,B) = H(\mathcal{A}(A,B))$. A G-functor $T : Z\mathcal{A} \longrightarrow \mathcal{X}$ which takes homotopic maps to equal maps amounts to a G-functor $H\mathcal{A} \longrightarrow \mathcal{X}$ also denoted by T. Notice that stability in $\mathcal{A}$ is carried over to $H\mathcal{A}$ since $HS \cong SH$; however, mapping cones in $\mathcal{A}$ do *not* yield them in $H\mathcal{A}$.

A *triangle* in a DG-category $\mathcal{A}$ is a diagram (a,b,c):



of maps a,b,c of degrees ℓ,m,n, respectively, such that $\ell + m + n = -1$. (We use ℓ',m',n' for the degrees of a',b',c' for a triangle (a',b',c'), and so on.) A *protosplitting* (u,v,w) of the triangle consists of protomaps u,v,w satisfying the relations

$$cb = d(u), \qquad ac = d(v), \qquad ba = d(w),$$

$(-1)^{\ell}wc + (-1)^{n}bv \simeq 1_A$, $(-1)^{m}ua + (-1)^{\ell}cw \simeq 1_B$, $(-1)^{n}vb + (-1)^{m}au \simeq 1_C$.
A triangle is called *protosplit* when it admits a protosplitting; it is called *split* when the u,v,w can be chosen to be maps. Clearly DG-functors preserve protosplit and split triangles.

If $f : A \longrightarrow B$ is a map of degree 0 which admits a mapping cone Mf then (j_f, p_f, f) is called the *mapping-cone triangle of* f; it has a protosplitting $(-q_f, i_f, 0)$.

Proposition 2. *Suppose* (a,b,c) *is a protosplit triangle. Then:*

(a) (c,a,b) *is a protosplit triangle;*

(b) *if* $a' \simeq a$ *then* (a',b,c) *is a protosplit triangle;*

(c) *for all objects* K *the triangle* (a,b,c) *in* ZA *is taken by* $HA(K,-) : ZA \longrightarrow GAb$ *to an exact triangle of graded abelian groups;*

(d) *if* (a',b',c') *is a protosplit triangle and* f,g *are maps with* $c'f \simeq gc$ *then there exists a map* h *with* $b'h \simeq fb$, $ha \simeq (-1)^{\sigma+m+m'}a'g$ *where* σ *is the degree of* g;

(e) *if in* (d) *both* f,g *become invertible in* HA *then so does* h;

(f) *if* (a',b',c') *is a triangle and* f,g,h *are maps of degree* ρ,σ,τ *which become invertible in* HA *and satisfy* $c'f \simeq (-1)^{\alpha}gc$, $a'g \simeq (-1)^{\beta}ha$, $b'h \simeq (-1)^{\gamma}fb$, *where* $\rho+\sigma+\tau+\alpha+\beta+\gamma$ *is even, then* (a',b',c') *is protosplit.*

Proof.

(a) The protosplitting relations have cyclic symmetry.

(b) If $a' = a + d(t)$ then $(u, v+tc, w+(-1)^{m}bt)$ is a protosplitting for (a',b,c) where (u,v,w) is one for (a,b,c).

(c) By (a), exactness has only to be verified at $HA(K,B)$. On the one hand, $ac = d(v) \simeq 0$. On the other hand, if $g : K \longrightarrow B$ is a map with $ag = d(t)$ then

$f = (-1)^{\ell} wg + (-1)^{n} bt$ is a map with $cf \simeq g$.

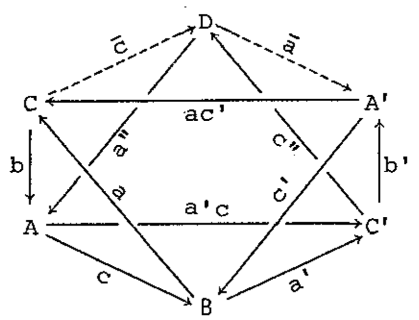
(d) If $c'f = gc + d(s)$ and (u',v',w') is a protosplitting for (a',b',c') then $h = (-1)^{\sigma+m'} a'gu + (-1)^{n'} v'fb + (-1)^{m'} a'sb$ can be directly verified to have the desired properties.

(e) To see that h becomes invertible in $H\mathcal{A}$, we must see that $H\mathcal{A}(K,h)$ is invertible for all K. This follows from (c) and the "five lemma".

(f) If f,g,h are invertible in $\mathcal{A}$ and the homotopy relations in (f) are equalities then a protosplitting for (a',b',c') is $((-1)^{\sigma+\gamma+\beta} guh^{-1}, (-1)^{\tau+\alpha+\gamma} hvf^{-1}, (-1)^{\rho+\beta+\alpha} fwg^{-1})$. Clearly the general result is unaffected by completing; so we may assume the DG-category is stable and has mapping cones. Using the earlier parts of this Proposition, we can then reduce the problem to proving (a',b',c') protosplit if there exist maps h,k of degree 0 such that $kh = 1 + d(s)$, $hk = 1 + d(t)$, $ph = b'$, $kj = a'$, where (j,p,c') is a mapping cone triangle. But then $(-qh, ki, -ptj)$ is a protosplitting of (a',b',c'). □

The following is the ancestor of *Mayer-Vietoris sequences* and *Verdier's octohedral property* [7 ; p.21].

Proposition 3. *In a stable DG-category with direct sums, suppose* (a,b,c), (a',b',c'), (a",a'c,c") *are protosplitting triangles.*



Then there exist maps $\bar{a},\bar{c}$ *such that* $a''\bar{c} \simeq b$, $\bar{a}c'' \simeq b'$, *and the following triangles are protosplit*

$$A' \xrightarrow{ac'} C \xrightarrow{\bar{c}} D \xrightarrow{\bar{a}} A'$$

$$D \xrightarrow{ca''} B \xrightarrow{\begin{pmatrix} a \\ (-1)^{\ell}\kappa^{-1}a' \end{pmatrix}} C \oplus S^{\ell-\ell'}C' \xrightarrow{((-1)^{\ell''}\bar{c},\ -(-1)^{n'+n''}c''\kappa)} D$$

$$B \xrightarrow{(-1)^{n''}c''a'} D \xrightarrow{\begin{pmatrix} -\tau a'' \\ \bar{a} \end{pmatrix}} S^{n-n'}A \oplus A' \xrightarrow{(c\tau^{-1},c')} B$$

where $\kappa : S^{\ell-\ell'}C' \longrightarrow C'$, $\tau : A \longrightarrow S^{n-n'}A$ *are the canonical invertible maps.*

Proof. We may assume mapping cones exist and use Proposition 2 to replace (a,b,c), (a',b',c'), $(c'',a'',a'c)$, by mapping-cone triangles (j,p,f), (j',p',g), (j'',p'',gf). Put $j''gq + i''p$, $k = j'q'' + i'fp''$. Then the triangles

$$Mf \xrightarrow{h} M(gf) \xrightarrow{k} Mg \xrightarrow{jp'} Mf\,,$$

$$M(gf) \xrightarrow{fp''} B \xrightarrow{\begin{pmatrix} -j \\ g \end{pmatrix}} Mf \oplus C \xrightarrow{(h,j'')} M(gf)\,,$$

$$B \xrightarrow{j''g} M(gf) \xrightarrow{\begin{pmatrix} -\tau p'' \\ k \end{pmatrix}} SA \oplus Mg \xrightarrow{(f\tau^{-1},p')} B$$

have protosplittings $(-j''q',\ i'q,\ ip'')$, $\left((-q,0),\begin{pmatrix} -ip'' \\ -q'' \end{pmatrix},0\right)$, $\left((i''\tau^{-1},\ -j''q'),\begin{pmatrix} 0 \\ i' \end{pmatrix},0\right)$, respectively. □

Protosplit triangles (a,b,c) in a G-category are all split, of course. More can be said. Idempotents on the objects A,B,C are obtained via $e_1 = (-1)^{\ell}wc$, $e_2 = (-1)^{m}ua$, $e_3 = (-1)^{n}vb$; if these split, the triangle is isomorphic to a triangle of the form:

$$A_1 \oplus A_2 \xrightarrow{\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}} B_1 \oplus B_2 \xrightarrow{\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}} C_1 \oplus C_2 \xrightarrow{\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}} A_1 \oplus A_2$$

where α,β,γ are invertible maps with the same degrees as a,b,c. If the G-category is abelian these triangles are amongst the exact triangles, but are certainly not all of them.

We have already remarked that DG-functors take protosplit triangles to protosplit triangles. However, a G-functor $HA \longrightarrow X$ generally does not take protosplit triangles in A to exact triangles in X (let alone to split ones). This suggests the following definition.

Suppose A is a stable DG-category with finite direct sums and suppose X is an abelian G-category. A *homological functor* $T : HA \longrightarrow X$ is a G-functor which takes protosplit triangles in A to exact triangles in X. If "exact" can be replaced by "split" then T is called *split homological.* If A does not have the specified limits then T is homological when its extension (unique up to isomorphism) $T' : HA' \longrightarrow X$ is homological, where A' is the completion of A with respect to those limits.

Examples. (1) *For any abelian category* G, *homology* $H : HCG \longrightarrow GG$ *is a homological functor.* (Since CG is stable and admits mapping cones, it suffices to see that H takes mapping-cone triangles to exact triangles; this follows from the fact that the sequence

$$0 \longrightarrow B \xrightarrow{j_f} Mf \xrightarrow{p_f} A \longrightarrow 0$$

is exact in GG.)

(2) *Representable functors* $HA(K,-) : HA \longrightarrow GAb$ *are homological.* (Proposition 2(c)).

(3) *If* $T : HA \longrightarrow X$ *is a homological functor and* $\mathcal{D}$ *is any small DG-category then the composite*

$$T_{\mathcal{D}} : H[\mathcal{D},A] \longrightarrow [H\mathcal{D},HA] \xrightarrow{[1,T]} [H\mathcal{D},X]$$

is a homological functor. (If X is abelian then so is

$[H\mathcal{D},\mathcal{A}]$ with exactness given pointwise. The protosplitness relations in $[\mathcal{D},\mathcal{A}]$ are pointwise such relations in $\mathcal{A}$.)

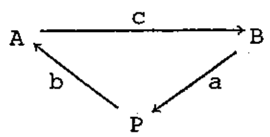
The results in the remainder of this section are proved in [14], [16].

Suppose $T : H\mathcal{A} \longrightarrow \mathcal{X}$ is a homological functor. An object X of $\mathcal{X}$ is called *projective* when $\mathcal{X}(X,-)$ preserves exactness. An object P of $\mathcal{A}$ is called T-*projective* when TP is projective and the map

$$T : H\mathcal{A}(P,A) \longrightarrow \mathcal{X}(TP,TA)$$

is an isomorphism for all objects A of $\mathcal{A}$.

For any object A of $\mathcal{A}$ and natural number r, the statement $T\text{-dim}A \le r$ is defined inductively. Take $T\text{-dim}A = 0$ to mean A is T-projective. For $r > 0$, $T\text{-dim}A \le r$ means there eixsts a protosplit triangle



in $\mathcal{A}$ such that P is T-projective, $T\text{-dim}B \le r-1$, and $TC = 0$.

Classification Theorem. *For objects* A,B *in* $\mathcal{A}$, *if* $T\text{-dim}A \le 1$ *then there is a G-natural short exact sequence*

$$0 \longrightarrow \text{Ext}^1_{\mathcal{X}}(STA,TB) \longrightarrow H\mathcal{A}(A,B) \xrightarrow{T} \mathcal{X}(TA,TB) \longrightarrow 0$$

of graded abelian groups. □

For larger T-dimension, there is a spectral sequence with second term involving $\text{Ext}^p_{\mathcal{X}}(S^pTA,TB)$ and converging to $H\mathcal{A}(A,B)$. If every object of $\mathcal{A}$ has finite T-dim then it follows that $T : H\mathcal{A} \to \mathcal{X}$ *reflects isomorphisms.*

In order to apply these results to a particular homological functor $T : H\mathcal{A} \longrightarrow \mathcal{X}$ one needs to determine which objects in $\mathcal{A}$ have T-dim $\leq r$. This requires one to first determine the T-projectives in $\mathcal{A}$ which may require an examination of the projective objects in $\mathcal{X}$. Our applications will be to homological functors obtained from homology as in Example (3) above. So, as a starting point, it is useful to see what H-dim $\leq r$ means (assuming we know the projectives in our basic abelian category $\mathcal{G}$).

An object C of a DG-category $\mathcal{A}$ is called *contractible* when its identity map is homotopic to 0. This means $C \cong 0$ in $H\mathcal{A}$.

Proposition 4. *Suppose* $\mathcal{G}$ *is an abelian category and* $H : HC\mathcal{G} \longrightarrow G\mathcal{G}$ *is the homology functor.*

(a) *An object* A *of* $C\mathcal{G}$ *is* H-*projective if and only if* $A \cong C \oplus P$ *in* $Z_0 C\mathcal{G}$ *where* P *is a projective object of* $G\mathcal{G}$ *and* C *is a contractible object of* $C\mathcal{G}$.

(b) *If* A *is a complex of projective objects in* $\mathcal{G}$ *such that* ZA, HA *have projective dimension* $\leq r$ *in* $G\mathcal{G}$ *then* H-dimA $\leq r$. □

For example, when $\mathcal{G}$ = Ab each complex A of free abelian groups has H-dimA ≤ 1; if also each $H_n A$ is free then A is H-projective. The Classification Theorem in this case gives a familiar result [3 ; p.71] derived from Künneth and implying the "Universal Coefficients Theorem".

§3. Diagrams of interlocking sequences.

Proposition 5. *Suppose* $\mathcal{X}$ *is a cocomplete abelian* $\mathcal{G}$-*category with enough projectives. Suppose* $\mathcal{D}$ *is a DG-category such that each* $H\mathcal{D}(D',D)$ *is a free graded abelian group. If* F *is a projective object of the abelian* $\mathcal{G}$-*category* $[H\mathcal{D},\mathcal{X}]$ *then* $F : H\mathcal{D} \longrightarrow \mathcal{X}$ *is split homological and each value* FD *of* F *is a projective object of* $\mathcal{X}$.

Proof. The projective objects F of $[H\mathcal{D},\mathcal{X}]$ are retracts of coproducts of objects of the form $H\mathcal{D}(D',-)\otimes X = F'$ where X is a projective object of $\mathcal{X}$ (see [5] for example). The properties in question are respected by coproducts and retracts; so it suffices to check for $F = F'$. Since $H\mathcal{D}(D',D)$ is free, FD is a coproduct of copies of X and so is projective. By Proposition 2(c), $H\mathcal{D}(D',-)$ is homological; it is split homological since a sub-graded-abelian group of a free graded abelian group is free. Since $-\otimes X$ preserves split triangles, F is split homological. □

Our concern is with DG-categories $\mathcal{D}$ for which the projectives F in $[H\mathcal{D},\mathcal{X}]$ are *precisely* the G-functors $F : H\mathcal{D} \longrightarrow \mathcal{X}$ which are split homological and have projective values.

For the purposes of the next definition; let $\mathcal{D}'$ denote the completion of $\mathcal{D}$ with respect to suspension, desuspension and finite direct sums; as usual, this can be done in $[\mathcal{D}^{op},CAb]$.

A small DG-category $\mathcal{D}$ is called *interlocking* when it possesses:

(a) a distinguished set of maps, called *short*;

(b) for each object A, a map $\ell_A : A \longrightarrow A^*$;

(c) for each pair of objects A,B, a subset Λ_{AB} of $H\mathcal{D}(A,B)$ which consists of composites of shorts and which freely generates $H\mathcal{D}(A,B)$ as a graded abelian group;

satisfying the following axioms:

I1. for any chain

$$\cdots \longrightarrow A_3 \xrightarrow{u_3} A_2 \xrightarrow{u_2} A_1 \xrightarrow{u_1} A_0$$

of short maps u_i, there exists an n such that $u_1 u_2 \ldots u_n \simeq 0$;

I2. for each object A of $\mathcal{D}$, there is a protosplit triangle

$$A \xrightarrow{\ell_A} A^* \longrightarrow C \xrightarrow{s} A$$

in $\mathcal{D}'$ where $C = \sum_{i=1}^{m} S^{n_i} B_i$ and s is induced by short maps $u_i : B_i \longrightarrow A$;

I3. if $u \in \Lambda_{AB}$ is a composite of n short maps then there exists a map $u^{\perp} : B \longrightarrow A^*$ such that, if $v \in \Lambda_{DB}$ is a composite of n or more short maps,

$$u^{\perp} v \simeq \begin{cases} \ell_A & \text{for } u \simeq v\,, \\ 0 & \text{otherwise.} \end{cases}$$

(It is a consequence of these axioms that each Λ_{AB} is finite.)

Theorem 6. *Suppose $\mathcal{D}$ is an interlocking DG-category and X is an abelian G-category with enough projectives. The following conditions on a G-functor $F : H\mathcal{D} \longrightarrow X$ are equivalent:*

(a) F *is a projective object of* $[H\mathcal{D}, X]$;

(b) F *is split homological with projective values;*

(c) *there exist projective objects* P_A *of* X *and a G-natural isomorphism*

$$F \cong \sum_{A \in \mathcal{D}} H\mathcal{D}(A,-) \otimes P_A\,.$$

Proof. Proposition 5 gives (a) $\Rightarrow$ (b), and (c) $\Rightarrow$ (a) is trival, so (b) $\Rightarrow$ (c) remains. Factor $F\ell_A : FA \longrightarrow FA^*$ as $F\ell_A = m_A p_A$ where $p_A : FA \longrightarrow P_A$ is epic and $m_A : P_A \longrightarrow FA^*$ is monic. Since F is split homological, it follows from I2 that $p_A i_A = 1$ for some $i_A : P_A \longrightarrow FA$. Since FA is projective, P_A is too. The coretractions i_A yield G-naturals $H\mathcal{D}(A,-) \longrightarrow X(P_A, F-)$ via Yoneda's lemma. The corresponding G-naturals $H\mathcal{D}(A,-) \otimes P_A \longrightarrow F$ induce a G-natural

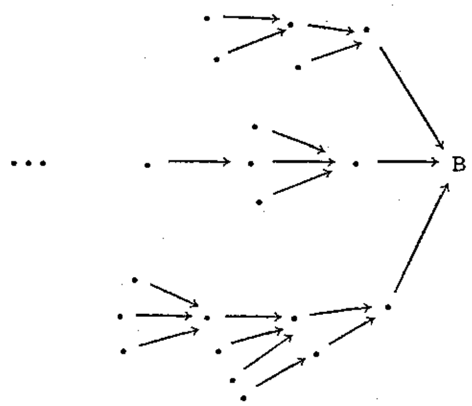
$$\theta : \sum_A H\mathcal{D}(A,-) \otimes P_A \longrightarrow F$$

which we shall show is an isomorphism by showing that each

$$X(X,\theta_B) : X(X, \sum_A H\mathcal{D}(A,B) \otimes P_A) \longrightarrow X(X,FB)$$

is an isomorphism for given B,X.

By I2, for each object A of $\mathcal{D}$, we obtain a finite set of short maps $u_i : B_i \longrightarrow A$ into A; by I3 with $n=0$, any composite $v : D \longrightarrow A$ of shorts has $\ell_A v \simeq 0$, and so, by I2, $v \simeq \sum u_i w_i$ for some maps $w_i : D \longrightarrow B_i$ (Proposition 2(c)). Starting with B and repeating this construction, we obtain a tree of short maps



where we stop at a given vertex when the composite down to B is homotopic to 0. Since composites of shorts generate $H\mathcal{D}(A,B)$, it follows from I1 that every map $v : A \longrightarrow B$ has the form $v \simeq \sum v_i$ where each v_i is a composite of shorts occurring in the tree. Thus

$$\sum_A H\mathcal{D}(A,B) \otimes P_A = \sum_A \Lambda_{AB} \cdot P_A$$

is a finite direct sum.

Call $a : X \longrightarrow FA$ *primitive* when $i_A p_A a = a$. We shall show that each $b : X \longrightarrow FB$ has a unique expression in the form $b = \sum_u (Fu) a_u$ where A varies over vertices of the tree for

B, $u : A \longrightarrow B$ varies over Λ_{AB}, and $a_u : X \longrightarrow FA$ is primitive.

Given $b : X \longrightarrow FB$, put $a_1 = i_B p_B b$, which is primitive. Then $(F\ell_B)(b - a_1) = m_B p_B (1 - i_B p_B) b = 0$. Since F is homological, we obtain $b - a_1 = \sum_u (Fu) b_u$ where u runs over the shorts into B in the tree for B. Repeat for each b_u to obtain $b_u - a_u = \sum_v (Fv) b_{u,v}$ with a_u primitive and v shorts at the second level in the tree for B. So

$$b = a + \sum (Fu) a_u + \sum_{u,v} F(vu) b_{u,v} .$$

Continue until the tree is used up. This proves the existence of such an expression for b.

For uniqueness we show that, for all natural numbers n, if $\sum_v (Fv) a_v = 0$ where $v \in \Lambda_{DB}$ varies over composites of $\geq n$ shorts and each a_v is primitive, then $a_u = 0$ for each u which is a composite of n shorts. To see this, use I3 to obtain:

$$(F\ell_A) a_u = \sum_v F(u^{\perp} v) a_v = F(u^{\perp}) \sum_v (Fv) a_v = 0 .$$

So $p_A a_u = 0$ since m_A is monic. So $a_u = i_A p_A a_u = 0$ since a_u is primitive.

Define $x : X \longrightarrow \sum_A \Lambda_{AB} \cdot P_A$ to be the map whose composite with the u-th projection is $p_A a_u$. Then x is the unique map with $X(X, \theta_B) x = \sum_u (Fu) a_u$. □

Theorem 7. *In the situation of Theorem 6, assume* X *has finite projective dimension. Then an object* F *of* $[H\mathcal{D}, X]$ *has projective dimension* $\leq n$ *if and only if* F *is homological and each value of* F *has projective dimension* $\leq n$.

Proof. The case $n = 0$ amounts to the statement that the word "split" can be omitted in Theorem 6(b) under this extra condition on $\mathcal{X}$. Splitness was only used in Theorem 6 to show that each P_A was projective. However, if F is homological, I2 gives a long exact sequence

$$0 \longrightarrow P_A \longrightarrow FA^* \longrightarrow \dots \longrightarrow FA^* \longrightarrow FC \longrightarrow FA \longrightarrow P_A \longrightarrow 0,$$

longer than the projective dimension of $\mathcal{X}$. Since F has projective values, P_A is projective.

Suppose F is homological and each value of F has projective dimension $\leq n$. Since $[H\mathcal{D},\mathcal{X}]$ has enough projectives, there is an exact sequence

$$0 \longrightarrow F' \longrightarrow F_n \longrightarrow \dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F \longrightarrow 0$$

with each F_i projective. By evaluating at objects of $\mathcal{D}$, we see that each value of F' is projective. Since F and all F_i are homological, evaluating at protosplit triangles, we see that F' is homological. So F' is projective. So F has projective dimension $\leq n$.

Conversely, if F has projective dimension $\leq n$ we have an exact sequence as above with all F_i and F' projective. Evaluating again we see that F is homological and each value has projective dimension $\leq n$. □

Lemma 8. *Suppose* $T : H\mathcal{A} \longrightarrow \mathcal{X}$ *is a homological functor and* $A \xrightarrow{c} B \xrightarrow{a} C \xrightarrow{b} A$ *is a protosplit triangle in* $\mathcal{A}$ *with* A,B,C *T-projective. If the image of* $Ta : TB \longrightarrow TC$ *is isomorphic to* TA' *for some* $A' \in \mathcal{A}$ *then the triangle* $TA \xrightarrow{Tc} TB \xrightarrow{Ta} TC \xrightarrow{Tb} TA$ *is split.*

Proof. Apply $H\mathcal{A}(-,A')$ to the protosplit sequence and use T-projectiveness to obtain an exact sequence:

$$X(TC,TA') \longrightarrow X(TB,TA') \longrightarrow X(TA,TA') .$$

There is a monic $m : TA' \to TC$ with $Ta = me$ where e is epic. Then $me(Tc) = T(ac) = 0$, so $e(Tc) = 0$. So e is in the kernel of $X(Tc,TA')$. So e is in the image of $X(Ta,TA')$. So $e = f(Ta)$ for some f. So $e = fme$. So $fm = 1$. So TA' is a retract of TC. □

The above Lemma applies in particular to the case where T is the homology functor $H : HCG \to GG$ (since every object of GG is in the image of H; just give the object zero differential).

A category C will be said to *support* an interlocking (see the beginning of this section) DG-category $\mathcal{D}$ when:

(i) the free DG-category on C is a full sub-DG-category of $\mathcal{D}$;

(ii) each object of $\mathcal{D}$ is obtained from C by iterated suspensions, desuspensions and mapping cones;

(iii) for all $C \in C$, $D \in \mathcal{D}$, there is a protosplit triangle

$$D \longrightarrow \Lambda_{DC} \cdot C \longrightarrow N_{DC} \longrightarrow D$$

in $\mathcal{D}'$ where the first map is the canonical map into the direct sum of Λ_{DC} copies of C.

In this situation, a DG-functor $F : \mathcal{D} \longrightarrow A$ is determined up to isomorphism by its restriction $F : C \longrightarrow ZA$ to C when A is stable and admits mapping cones.

We now specialize to the case where $A = CG$ for an abelian category G with enough projectives, $H : HCG \longrightarrow GG$ is the homology G-functor, and

$$H_{\mathcal{D}} : H[\mathcal{D},CG] \longrightarrow [H\mathcal{D},GG]$$

is the G-functor induced as in Example (3) of Section 2. Let

$U : ZCG \longrightarrow GG$ denote the G-functor which forgets differentials.

A functor $F : C \longrightarrow ZCG$ (or its extension $F : \mathcal{D} \longrightarrow CG$) will be called *combinatorial* when there are projective objects Q_C of GG and a G-natural isomorphism:

$$UF \cong \sum_C C(C,-) \cdot Q_C .$$

It follows from Mitchell [12;Ch.9 §7] that, when C is a finite ordered set, F is combinatorial if and only if $UF \in [C,GG]$ is projective.

Theorem 9. *Suppose* C *is a finite ordered set which supports an interlocking DG-category* $\mathcal{D}$ *and suppose* G *is a cocomplete abelian category with enough projectives. Then, for a combinatorial object* F *of* $H[\mathcal{D},CG]$,

$$H_{\mathcal{D}} - \dim F \leq r \iff H\text{-}\dim FD \leq r \quad \textit{for all } D \in \mathcal{D} .$$

Proof. For any DG-functor $F : \mathcal{D} \longrightarrow CG$, we have that $H_{\mathcal{D}}(F)$ is homological since F preserves protosplit triangles and H is homological.

Suppose now that each FD is H-projective. By Lemma 8, $H_{\mathcal{D}}(F)$ is split homological and so projective by Theorem 6. To prove that each $H[\mathcal{D},CG](F,K) \longrightarrow [H\mathcal{D},GG](H_{\mathcal{D}}(F), H_{\mathcal{D}}(K))$ is an isomorphism we proceed as follows.

Suppose $C_o \in C$ is such that $FC \cong 0$ whenever $C < C_o$. Taking the components at $C \leq C_o$ of the G-natural isomorphism in the definition of "combinatorial", we see that $UFC_o \cong Q_{C_o}$ and $Q_C \cong 0$ for $C < C_o$. Hence the canonical map $C(C_o,-) \cdot FC_o \longrightarrow F$ becomes a split monic after applying U. Define F^o by the short exact sequence

$$0 \longrightarrow C(C_o,-) \cdot FC_o \longrightarrow F \longrightarrow F^o \longrightarrow 0$$

in $[C,ZCG]$. Then F^o is *combinatorial*, $F^oC \cong 0$ *for* $C \leq C_o$, and we have a protosplit triangle

$$\mathcal{D}(C_o,-)\cdot FC_o \longrightarrow F \longrightarrow F^o \longrightarrow \mathcal{D}(C_o,-)\cdot FC_o$$

in $[\mathcal{D},C\mathcal{G}]$. By Proposition 2(c), we obtain an exact triangle.

$$\longrightarrow HC\mathcal{G}(FC_o,KC_o) \longrightarrow H[\mathcal{D},C\mathcal{G}](F^o,K) \longrightarrow H[\mathcal{D},C\mathcal{G}](F,K) \longrightarrow$$

for all $K \in [\mathcal{D},C\mathcal{G}]$.

Let $I : C \to Z\mathcal{D}$ denote the inclusion. Since C supports $\mathcal{D}$ there is a protosplit triangle

$$\mathcal{D}(C_o,-)\otimes C_o \longrightarrow I \longrightarrow N^o \longrightarrow \mathcal{D}(C_o,-)\otimes C_o$$

in $[\mathcal{D},\mathcal{D}']$. Since F is a DG-functor, we obtain a protosplit triangle

$$\mathcal{D}(C_o,-)\otimes FC_o \longrightarrow F \longrightarrow FN^o \longrightarrow \mathcal{D}(C_o,-)\otimes FC_o$$

in $[\mathcal{D},C\mathcal{G}]$. By Proposition 2(d), (e), we have $F^o \cong FN^o$ in $H[\mathcal{D},C\mathcal{G}]$. Thus F^oD *is H-projective for all* $D \in \mathcal{D}$.

Next we shall prove that, for $\bar{F},\bar{K} : H\mathcal{D} \to G\mathcal{G}$, if $\bar{F}$ is projective and K is homological then the triangle

$$\longrightarrow G\mathcal{G}(\bar{F}C_o,\bar{K}C_o) \longrightarrow [H\mathcal{D},G\mathcal{G}](\bar{F}N^o,\bar{K}) \longrightarrow [H\mathcal{D},G\mathcal{G}](\bar{F},\bar{K}) \longrightarrow$$

is exact. For this it suffices to suppose $\bar{F} = H\mathcal{D}(D,-)\otimes Q$ where Q is projective in $G\mathcal{G}$. Since C supports $\mathcal{D}$, we have a protosplit triangle

$$D \longrightarrow \Lambda_{DC_o}\cdot C_o \longrightarrow D' \longrightarrow D ,$$

and so an exact triangle

$$\ldots \longrightarrow H\mathcal{D}(D,C_o)\otimes H\mathcal{D}(C_o,-) \longrightarrow H\mathcal{D}(D,-) \longrightarrow H\mathcal{D}'(D',-) \longrightarrow \ldots$$

Comparing this with $H\mathcal{D}(D,-)$ applied to the triangle defining N^o, we obtain $H\mathcal{D}'(D',-) \cong H\mathcal{D}'(D,N^o-)$ and so $\bar{F}N^o \cong H\mathcal{D}'(D',-)\otimes Q$. Since $\bar{K}$ is homological, the triangle

$$\bar{K}D \longrightarrow \Lambda_{DC_o} \cdot \bar{K}C_o \longrightarrow \bar{K}D' \longrightarrow \bar{K}D$$

is exact. Since Q is projective, this last triangle is taken to an exact triangle by $GG(Q,-)$; using Yoneda's lemma, we thus obtain an exact triangle

$$\longrightarrow GG(H\mathcal{D}(D,C_o) \otimes Q, \bar{K}C_o) \longrightarrow [H\mathcal{D}(D,-) \otimes Q, \bar{K}] \longrightarrow [H\mathcal{D}'(D',-) \otimes Q, \bar{K}] \longrightarrow$$

(where we have abbreviated $[H\mathcal{D}, GG](-,-)$ to $[-,-]$) . This proves the claim.

In the last paragraph take $\bar{F} = H_{\mathcal{D}}(F)$, $\bar{K} = H_{\mathcal{D}}(K)$ and apply a "5-lemma" argument to that exact triangle and the exact triangle of the third last paragraph. Since FC_o is H-projective, it follows that F^o *is* $H_{\mathcal{D}}$*-projective if and only if* F *is* $H_{\mathcal{D}}$*-projective.*

This gives us the basis for an inductive proof that F is $H_{\mathcal{D}}$-projective. The induction is on the *height* of the objects of $\mathcal{C}$. An object C has height k when k is the largest natural number n for which there is a chain $C_o < C_1 < \ldots < C_n = C$ in $\mathcal{C}$. If $C < C'$ then the height of C is less than that of C' . Apply the above argument with C_o of height 0 and replace F by F^o . Continue until there are no more objects of height 0 . Continue with the objects of height 1 . The process stops when we are left with an F which has $FC \cong 0$ for all $C \in \mathcal{C}$. Clearly this F *is* $H_{\mathcal{D}}$-projective. So the original F is too.

Suppose F is combinatorial, $r > 0$ and $H\text{-dim}FD \leq r$ for all $D \in \mathcal{D}$. For each $D \in \mathcal{D}$, choose an epic map $u_D : P_D \longrightarrow FD$ which induces an epic on boundaries and has P_D an H-projective complex of projective objects of $\mathcal{G}$. Then $F' = \sum_D \mathcal{D}(D,-) \otimes P_D$ is combinatorial and $H_{\mathcal{D}}$-projective (from the above). Also the map $u : F' \longrightarrow F$ induced by the u_D is epic with $H_{\mathcal{D}}(u)$ epic. Let F'' be the kernel of u in $[\mathcal{C}, ZC\mathcal{G}]$. Since F, F' are combinatorial, so too is F'' and we have a protosplit triangle

$$F'' \longrightarrow F' \xrightarrow{u} F \longrightarrow F''$$

in $[\mathcal{D},CG]$. Evaluating at D , we see that $H\text{-dim}F''D \le r-1$. By induction, $H_{\mathcal{D}}\text{-dim}F \le r$.

Thus we have proved $\Leftarrow$. To prove the converse, for $r = 0$ make use of the right adjoints to evaluation $[\mathcal{D},CG] \longrightarrow CG$ at D , and proceed to $r > 0$ by an easy induction. □

Corollary 10. *In the situation of* Theorem 9, *if the projective dimension of* G *is* r *then every combinatorial object* F *of* $H[\mathcal{D},CG]$ *has* $H_{\mathcal{D}}\text{-dim}F \le r$. □

Corollary 11. *In the situation of* Theorem 9, *if* G *has projective dimension* 1 (for example, $G = Ab$) *then, for all combinatorial* $F \in H[\mathcal{D},CG]$, *there is a G-natural short exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1(SH_{\mathcal{D}}F,H_{\mathcal{D}}K) \longrightarrow H[\mathcal{D},CG](F,K) \longrightarrow [H\mathcal{D},GG](H_{\mathcal{D}}F,H_{\mathcal{D}}K) \longrightarrow 0$$

of graded abelian groups. □

In fact, the short exact sequence of Corollary 11 splits so that we have:

$$H[\mathcal{D},CG](F,K) \cong [H\mathcal{D},GG](H_{\mathcal{D}}F,H_{\mathcal{D}}K) \oplus \mathrm{Ext}^1(SH_{\mathcal{D}}F,H_{\mathcal{D}}K),$$

although this isomorphism is not G-natural. The proof of this will not be given here.

§4. Examples.

A. The homology triangle

This is the simplest non-trivial case. It is included in the next example but, to fix ideas, is worthy of special mention. Even in this case the proof of Corollary 11 by direct calculation (some aspects of which appear in [8] and [9]) is not trivial.

We begin with the category $C = 2$ which has two objects 0,1 and one non-identity arrow $\tau : 0 \longrightarrow 1$. The free additive category $\mathrm{Add}C$ on C is also (as for any category C) the free DG-category on C (via the inclusion $Ab \longrightarrow CAb$). Notice that $\mathrm{Add}C$ has objects 0,1 and $\mathrm{Add}\,C(0,1)$ is the free abelian group generated by τ. We can identify $\mathrm{Add}C$ with the full sub-DG-category of $[(\mathrm{Add}C)^{op}, CAb]$ consisting of the objects

$$Z \longleftarrow 0 , \quad Z \xleftarrow{1} Z$$

so that $\tau : 0 \longrightarrow 1$ becomes the map

$$\begin{array}{ccc} Z & \longleftarrow & 0 \\ \downarrow & & \downarrow \\ Z & \longleftarrow & Z \end{array}$$

Let T denote the full sub-DG-category of $[(\mathrm{Add}C)^{op}, CAb]$ consisting of the objects

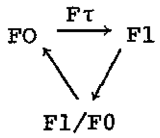
$$Z \longleftarrow 0 , \qquad Z \xleftarrow{1} Z , \qquad M1_Z \longleftarrow Z$$

So T contains $\mathrm{Add}C$ and a mapping cone for τ :

$$\begin{array}{ccc} 0 & \xrightarrow{\tau} & 1 \\ {\scriptstyle\sigma}\nwarrow & & \swarrow{\scriptstyle\rho} \\ & M\tau & \end{array}$$

we have "freely added a mapping cone to the free living map". The triangle above contains the short maps and the maps ℓ_A which are needed to show that T is interlocking. Clearly 2 supports the interlocking DG-category T.

A DG-functor $F : T \longrightarrow CG$ is determined up to isomorphism by the map $F\tau : F0 \longrightarrow F1$. If $F\tau$ is a monic map and $F1/F0$ is a complex of projective objects of G then F becomes isomorphic to



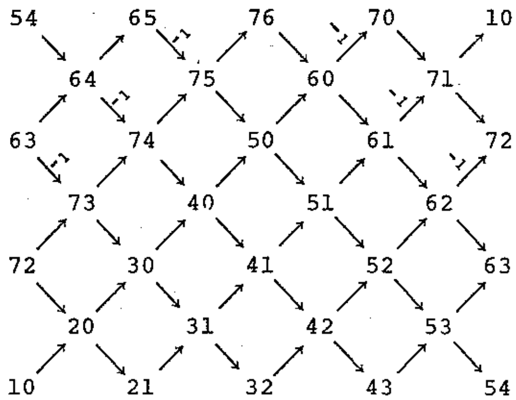
in H[T,CG]. So the homological functor

$$H_T : H[T,CG] \longrightarrow [HT,GG]$$

restricts to the G-category of short exact sequences of complexes of projective objects of G and homotopy classes of short exact sequence maps. To such a short exact sequence it assigns the homology triangle of graded objects of G. These short exact sequences yield combinatorial objects of H[T,CG].

B. Finitely filtered complexes

Take C to be the linearly ordered set $n = \{0,1,\ldots,n-1\}$. We can identify Add$C$ with the full sub-DG-category of $[(\mathrm{Add}C)^{op},\mathrm{CAb}]$ consisting of the representables. Let T_n denote the full sub-DG-category which contains n and, for all $0 \le q < p < n$, a mapping cone $\langle p,q \rangle$ for the map $q \longrightarrow p$. Write $\langle n,q \rangle$ in T_n for q in n. The short maps of T_7 (for example) are depicted below: the diagram belongs on a Moebius band. (The indicated maps have degree-1, the others degree 0).



It is easily seen that T_n is an interlocking DG-category supported by n. A filtered complex

$$A: \; A^0 \leq A^1 \leq \ldots \leq A^{n-1}$$

for which each A^p/A^q is a complex of projective objects in G yields a combinatorial object of $H[T_n, CG]$. Thus we obtain the results of Street [16; §4].

C. Three-diamond diagrams.

Take C to be the category 2×2:

$$\begin{array}{ccc} (0,0) & \xrightarrow{(0,\tau)} & (0,1) \\ {\scriptstyle(\tau,0)}\big\downarrow & & \big\downarrow{\scriptstyle(\tau,1)} \\ (1,0) & \xrightarrow[(1,\tau)]{} & (1,1) \end{array} \; .$$

Starting with the representables in $[(\mathrm{Add}C)^{op}, C\mathrm{Ab}]$ and iteratively adding mapping cones of certain maps, we obtain the three-diamond DG-category which is interlocking and supported by 2×2.

Another approach to this DG-category provides more insight. We already have (Example A) an interlocking DG-category T supported by 2. Write 2 for $M\tau$ in T. One might expect that $T \otimes T$ would be interlocking and supported by 2×2; this is *false*. In fact, $H(T \otimes T)$ is generated by:

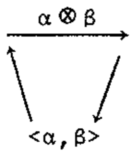
$$\begin{array}{ccccccc} (0,0) & \xrightarrow{(0,\tau)} & (0,1) & \xrightarrow{(0,\rho)} & (0,2) & \xrightarrow{(0,\sigma)} & (0,0) \\ {\scriptstyle(\tau,0)}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ (1,0) & \longrightarrow & (1,1) & \longrightarrow & (1,2) & \longrightarrow & (1,0) \\ {\scriptstyle(\rho,0)}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ (2,0) & \longrightarrow & (2,1) & \longrightarrow & (2,2) & \longrightarrow & (2,0) \\ {\scriptstyle(\sigma,0)}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ (0,0) & \longrightarrow & (0,1) & \longrightarrow & (0,2) & \longrightarrow & (0,0) \end{array} \; ,$$

and the G-functor $F : H(T \otimes T) \longrightarrow GAb$, determined by a free graded abelian group P and depicted by the diagram

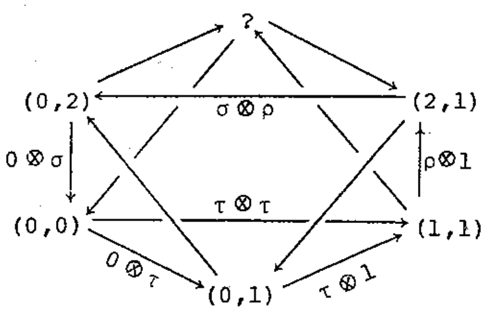
$$\begin{array}{ccccccc}
P & \xrightarrow{1} & P & \longrightarrow & 0 & \longrightarrow & P \\
{\scriptstyle 1}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\
P & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{1} & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow \\
0 & \longrightarrow & P & \xrightarrow{1} & P & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow \\
P & \xrightarrow[1]{} & P & \longrightarrow & 0 & \longrightarrow & P ,
\end{array}$$

is certainly split homological with projective values, yet F is *not* projective in $[H(T \otimes T), GAb]$ (since this last G-category is isomorphic to $[HT, [HT, GAb]]$ and $HT \longrightarrow [HT, GAb]$, corresponding to F , is not split homological). So $T \otimes T$ is not interlocking (or else Theorem 6 would be contradicted).

Let $T \hat{\otimes} T$ denote the DG-category obtained from $T \otimes T$ by freely adding a protosplit triangle



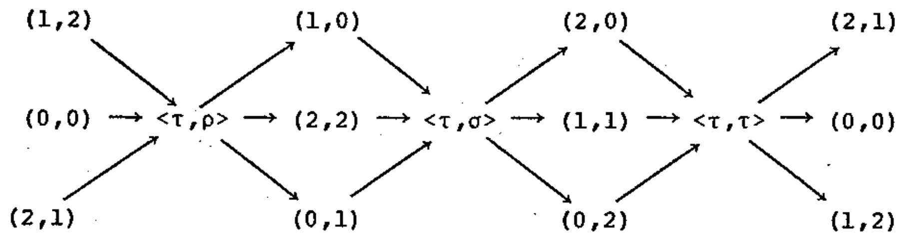
including the map $\alpha \otimes \beta$ in $T \otimes T$ for all pairs $\alpha, \beta \in \{\rho, \sigma, \tau\}$. This apparently adds 9 new objects to the original 9 in $T \otimes T$. In fact, only 3 new objects are needed because Proposition 3 yields octahedra such as:



showing that $\langle\tau,\tau\rangle \cong \langle\sigma,\rho\rangle$; indeed

$$\langle\tau,\tau\rangle \cong \langle\sigma,\rho\rangle \cong \langle\rho,\sigma\rangle,$$
$$\langle\tau,\rho\rangle \cong \langle\sigma,\sigma\rangle \cong \langle\rho,\tau\rangle,$$
$$\langle\tau,\sigma\rangle \cong \langle\sigma,\tau\rangle \cong \langle\rho,\rho\rangle.$$

The three-diamond DG-category is $T\hat{\otimes}T$; its short maps are depicted below.



The three diamonds appear on a Moebius band. Any composite of more than four shorts is 0.

The 9 protosplit triangles which include the maps $\alpha\otimes\beta$ for $\alpha,\beta \in \{\rho,\sigma,\tau\}$ together with the 3 protosplit triangles

$$\langle\tau,\rho\rangle \longrightarrow \langle\tau,\tau\rangle \longrightarrow (2,1)\oplus(0,0)\oplus(1,2) \longrightarrow \langle\tau,\rho\rangle$$
$$\langle\tau,\sigma\rangle \longrightarrow \langle\tau,\rho\rangle \longrightarrow (1,0)\oplus(2,2)\oplus(0,1) \longrightarrow \langle\tau,\sigma\rangle$$
$$\langle\tau,\tau\rangle \longrightarrow \langle\tau,\sigma\rangle \longrightarrow (2,0)\oplus(1,1)\oplus(0,2) \longrightarrow \langle\tau,\tau\rangle$$

provide the protosplit triangles required for axiom I2 in the definition of "interlocking" for $\mathcal{D} = T\hat{\otimes}T$. That $T\hat{\otimes}T$ is interlocking and supported by 2×2 is easily verified.

There are many protosplit triangles in the stabilized finite-direct-sum completion $(T\hat{\otimes}T)'$ of $T\hat{\otimes}T$. One may ask for finite collections of such triangles which guarantee the protosplitness of all the others. This is the semantic form of the problem considered by Wall [22]. By Theorem 6 the above-mentioned 12 triangles provide one such set. Another set consists of the first 9 of these and replaces the last 3 with the 6 occurring already in $T \otimes T$ (this collection is of special importance because it lives in $T \hat{\otimes} T$). Other collections involve Mayer-Vietoris triangles; see Wall [22].

A similar analysis can be applied to $C = 2 \times 2 \times \ldots \times 2$ from which Example B can be deduced.

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