<u>Higher dimensional nerves</u>

April–May 1982

§1. <u>Higher dimensional categories</u>

<u>A partial binary operation</u> $\circ$ <u>on a set</u> A is a partial map from $A \times A$ to A which we write between the arguments. An element u of A is called a $\circ$-<u>unit</u> when $u \circ a = a$, $b \circ u = b$ for all a, b for which the left-hand sides are defined.

Call a partial ~~binary~~ operation $\circ$ on A <u>a composition on</u> A when it satisfies the axioms:

c1. for all $a \in A$, there exists a unique $\circ$-unit u such that $a \circ u$ is defined, and a unique $\circ$-unit v such that $v \circ a$ is defined;

c2. for all $a, b \in A$, $a \circ b$ is defined if and only if there exists a $\circ$-unit u such that $a \circ u$ and $u \circ b$ are defined;

c3. for all $a, b, c \in A$, $(a \circ b) \circ c = a \circ (b \circ c)$ whenever the left-hand side is defined.

A set together with a composition is called a category. In the notation of (c1) we write $a : u \to v$. The reader will easily see that this definition agrees with the usual definitions of category when one identifies the elements of the set with <u>arrows</u> and the units with <u>objects</u>.

Suppose $\circ$, $*$ are two compositions on a set A.

Call $*$ <u>subordinate to</u> $\circ$, and write $*$ <u>sub</u> $\circ$, when the following axioms hold:

S1. every $*$-unit is a $\circ$-unit;

S2. if u, v are $\circ$-units then $u * v$ is a $\circ$-unit provided it is defined;

S3. $(a \circ b) * (a' \circ b') = (a * a') \circ (b * b')$ provided the left-hand side is defined.

Following the convention of Roberts [ ], we omit brackets when we mean that $*$ should be evaluated before $\circ$; so the expression $a \circ b * c$ means $a \circ (b * c)$. It is noted in [ ] that $\circ$ is subordinate to itself if and only if $\circ$ is <u>abelian</u> in the sense that $a \circ b = b \circ a$ whenever either side is defined.

A 2-<u>category</u> is a set A with two compositions $*, \circ$ such that $*$ is subordinate to $\circ$. This relates to the usual definition by taking $*$-units to be <u>objects</u>, $\circ$-units to be <u>arrows</u> (or 1-<u>cells</u>), and elements of A to be 2-<u>cells</u>. The notation

$$\begin{tikzcd} u \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow x"] & v \end{tikzcd}$$

can be used to mean $a : u \to v$, $b : u \to v$ in the category $(A, *)$, and $x : a \to b$ in the category $(A, \circ)$.

An ω-*category* is a set A together with a composition $\underset{n}{\circ}$ on A for each natural number $n \geq 0$, where $\underset{m}{\circ}$ is subordinate to $\underset{n}{\circ}$ whenever $m < n$.

A composition on A is *discrete* when every element of A is a unit.

An r-*category* is an ω-category A such that $\underset{r}{\circ}$ is discrete. It follows from §1 that $\underset{n}{\circ}$ is discrete for all $n > r$.

If A, B have partial binary operations on them, a function $f : A \to B$ is called a *homomorphism* when f takes units to units and $f(a \circ a') = (fa) \circ (fa')$ whenever the left-hand side is defined. If A, B are categories, a homomorphism is called a *functor*.

If A, B are ω-categories, an ω-*functor* $f : A \to B$ is a function which is a homomorphism for respective compositions.

Let ω-Cat denote the category of ω-categories and ω-functors. For each r, we have a full subcategory r-Cat of ω-Cat consisting of the r-categories. In particular, for $r=1$, we write Cat instead of r-Cat.

§2. Complicial sets

The category whose objects are the ordered sets $[n] = \{0, 1, \ldots, n\}$ and whose arrows are order-preserving functions will be denoted by Δ. For each n there are $n+1$ monics

$$\partial_0 \geq \partial_1 \geq \cdots \geq \partial_n : [n-1] \longrightarrow [n]$$

such that the image of ∂_i does not contain i; there are n epics

$$\sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{n-1} : [n] \longrightarrow [n-1].$$

A <u>simplicial set</u> is a functor $X : \Delta^{op} \longrightarrow \mathrm{Set}$. The set $X([n])$ is denoted by X_n and its elements called <u>elements of X of dimension</u> n. We write $d_i : X_n \longrightarrow X_{n-1}$ for the i-<u>th face</u> function $X(\partial_i)$; we write $s_i : X_{n-1} \longrightarrow X_n$ for the i-<u>th degeneracy</u> function $X(\sigma_i)$. Call $x \in X_n$ <u>degenerate</u> when it is in the image of some s_i.

Let **Smpl** denote the category $[\Delta^{op}, \mathrm{Set}]$ of simplicial sets and simplicial maps (= natural transformations).

A <u>simplicial set with neutrality</u> is a simplicial set X together with a distinguished subset of X_n for each $n \geq 1$ such that each degenerate element is in the distinguished subset of its dimension.

Elements in these distinguished subsets are called <u>neutral elements of</u> X. It will be convenient to agree that there are no neutral elements of dimension 0 and that the neutral elements of dimension 1 are precisely the degenerate elements.

Let NSmpl denote the category of simplicial sets with neutrality, and simplicial maps which preserve neutrality.

The functor $\Delta(-,[n]): \Delta^{op} \longrightarrow \mathrm{Set}$ represented by $[n]$ is denoted by $\Delta[n]$ and called the <u>standard n-simplex</u>. By the Yoneda lemma, a simplicial map $\Delta[n] \to X$ amounts to an element of X_n.

For $k \in [n]$, <u>the k-horn in</u> $\Delta[n]$ is the subobject $\Lambda^k[n]$ of $\Delta[n]$ consisting of those $\alpha: [m] \to [n]$ whose image does not contain $[n] - \{k\}$. A simplicial map $\Lambda^k[n] \to X$ amounts to n elements $x_i \in X_{n-1}$, $i \neq k$, satisfying the identities:

$$d_i x_j = d_{j-1} x_i \quad \text{for } i<j,\ i \neq k,\ j \neq k;$$

this is called a <u>k-horn in</u> X.

For $0 < k < n$, we shall define an object $N_k \Delta[n]$ of NSmpl whose underlying simplicial set is $\Delta[n]$. For $m > 1$, the elements $\alpha: [m] \to [n]$ apart from degenerates, which we require to be neutral are those α whose images contain $k-1, k, k+1$.

Write $N_k \Lambda^k[n]$ for the (regular) subobject of $N_k \Delta[n]$ which has underlying simplicial set $\Lambda^k[n]$ and has as neutral elements those of $N_k \Delta[n]$ which are also in $\Lambda^k[n]$.

Let $N'_k \Delta[n]$ denote the object of $NSmpl$ obtained from $N_k \Delta[n]$ by adding $\partial_{k-1}, \partial_{k+1} : [n-1] \to [n]$ as neutral elements.

Let $N''_k \Delta[n]$ denote the object of $NSmpl$ obtained from $N'_k \Delta[n]$ by adding $\partial_k : [n-1] \to [n]$ as a neutral element.

Recall that an object C is said to be <u>orthogonal</u> to an arrow $j : A \to B$ in a category when, for each arrow $f : A \to C$, there exists a unique arrow $k : B \to C$ with $kj = f$.

<u>A complicial set</u> is an object C of $NSmpl$ which is orthogonal to the inclusions:

(a) $N_k \Lambda^k[n] \longrightarrow N_k \Delta[n]$;

(b) $N'_k \Delta[n] \longrightarrow N''_k \Delta[n]$; [illegible] ABOUT REPLACING WITH $N''_k \Lambda^k[n] \longrightarrow N''_k \Delta[n$

for all $n > 0$ and all $0 < k < n$.

Let $Cmpl$ denote the full subcategory of $NSmpl$

consisting of the complicial sets. It follows from the work of Gabriel-Ulmer [ ] that the inclusion of Cmpl in NSmpl has a left adjoint and that Cmpl is locally presentable.

A k-horn $x_i \in C_{n-1}$, $i \neq k$, in $C \in$ NSmpl will be called <u>complicial</u> when $(C\mu)x_i$ is neutral for all monics $\mu : [m] \longrightarrow [n-1]$ and all i such that the image of $\partial_i \mu$ contains $k-1, k, k+1$.

Conditions (a), (b) above can be reformulated as follows:

(a) if $x_i \in C_{n-1}$, $i \neq k$, is a complicial k-horn in C then there exists a unique neutral $x \in C_n$ with $d_i x = x_i$ for $i \neq k$;

(b) if, in (a), x_{k-1}, x_{k+1} are neutral then $d_k x$ is neutral.

Let Cmpl_r denote the full subcategory of Cmpl consisting of those complicial sets whose elements of dimension $> r$ are all neutral.

§3. The nerve of a category

Ordered sets are categories: transitivity gives a composition law on the order relation. In this way Δ is a full subcategory of Cat. Each category A yields a simplicial set

$$\mathrm{Cat}(-, A) : \Delta^{op} \longrightarrow \mathrm{Set}$$

called the <u>nerve</u> of A and denoted by νA. The elements of νA of dimension n are composable strings of n arrows in A. Clearly nerve becomes a functor

$$\nu : \mathrm{Cat} \longrightarrow \mathrm{Smpl}$$

which is fully faithful.

It is easy to see that a simplicial set C is isomorphic to the nerve of some category if and only if the following squares are all pullbacks

$$\begin{array}{ccc} C_n & \xrightarrow{d_0} & C_{n-1} \\ {\scriptstyle d_n}\downarrow & & \downarrow{\scriptstyle d_{n-1}} \\ C_{n-1} & \xrightarrow[d_0]{} & C_{n-2} \end{array} \qquad n > 1$$

Note that Cmpl_1 is a full subcategory of Smpl since all elements of dimension > 1 of an object of Cmpl_1 are neutral.

Proposition 1. A simplicial set C is (isomorphic to) the nerve of a category if and only if C is in $Cmpl_1$.

Proof. If $C \in Cmpl_1$ we must show that the squares above are all pullbacks for $n > 1$. The case $n = 2$ is precisely property (a) with $k = 1$. Assume the result true for $n-1$ and take $x_0, x_n \in C_{n-1}$ with $d_0 x_n = d_{n-1} x_0$. From the calculation

$$d_0 d_i x_n = d_{i-1} d_0 x_n = d_{i-1} d_{n-1} x_0 = d_{n-2} d_{i-1} x_0,$$

the inductive hypothesis gives a unique x_i satisfying

$$d_0 x_i = d_{i-1} x_0 \text{ and } d_{n-1} x_i = d_i x_n.$$

Notice that for $i < j$:

$$\begin{aligned} d_0 d_i x_j &= d_{i-1} d_0 x_j \\ &= d_{i-1} d_{j-1} x_0 = d_{j-2} d_{i-1} x_0 \\ &= d_{j-2} d_0 x_i = d_0 d_{j-1} x_i, \end{aligned} \qquad \begin{aligned} d_{n-2} d_i x_j &= d_i d_{n-1} x_j \\ &= d_i d_j x_n = d_{j-1} d_i x_n \\ &= d_{j-1} d_{n-1} x_i = d_{n-2} d_{j-1} x_i. \end{aligned}$$

So, again by induction, $d_i x_j = d_{j-1} x_i$. By applying (a) to the, say, 1-horn $x_0, x_2, \ldots, x_n$, there exists a unique $x \in C_n$ with $d_i x = x_i$ for $i \neq 1$. By uniqueness of x_1, we also have $d_1 x = x_1$. So, in fact, x is unique with $d_0 x = x_0$, $d_n x = x_n$. This proves the squares are pullbacks.

Suppose C is isomorphic to the nerve of some category. We must show that the pullback conditions imply property (a) (since (b) is trivial). Given x_i, $i \neq k$, a complicial k-horn, note that $d_0 x_n = d_{n-1} x_0$.

So there is a unique $x \in C_n$ with $d_0 x = x_0$, $d_n x = x_n$. Then

$$d_0 d_i x = d_{i-1} d_0 x = d_{i-1} x_0 = d_0 x_i,$$
$$d_{n-1} d_i x = d_i d_n x = d_i x_n = d_{n-1} x_i.$$

So $d_i x = x_i$. $\square$

Corollary 2. $\nu : \mathrm{Cat} \simeq \mathrm{Cmpl}_1$. $\square$

If $C \in \mathrm{Cmpl}_1$, a category A with $\nu A \cong C$ is described as follows:

the set of arrows A is C_1,

$x, y \in C_1$ are composable when $d_1 x = d_0 y$ in which case $y \circ x$ is $d_1 z$ where $z \in C_2$ with $d_0 z = x$, $d_1 z$

§4. Alternative characterization of complicial sets.

For $0 < k < n$, we shall call an arrow $\mu : [r] \to [n]$ in Δ a k-monic when its image contains $k-1, k, k+1$.

Lemma 3. *An arrow $\mu : [r] \to [n]$ in Δ is a k-monic if and only if one of the following conditions holds:*

(i) μ *is the identity function;*

(ii) $\mu = \partial_i \nu$ *where* $i < k-1$ *and* ν *is a* $(k-1)$-*monic*

(iii) $\mu = \partial_i \nu$ *where* $i > k+1$ *and* ν *is a* k-*monic.*

Proof. If μ is not the identity and monic then $\mu = \partial_i \nu$ for some monic $\nu : [r] \to [n-1]$. If μ is a k-monic then $i \neq k-1, k, k+1$. If $i < k-1$ then μ is a k-monic precisely when ν is a $(k-1)$-monic. If $i > k+1$, μ is a k-monic precisely when ν is a k-monic. □

For any simplicial set C with neutrality, put:

$$P_n^k = \{ (x, y) \in C_{n-1} \times C_{n-1} \mid d_k x = d_{k-1} y \}, \text{ and}$$

$$D_n^k = \{ x \in C_n \mid (C\mu)x \text{ is neutral for all } k\text{-monics } \mu : [m] \to [n] \}.$$

It follows from the Lemma 3 that:

$$D_{n+1}^k = \{ x \in C_{n+1} \mid x \text{ is neutral},\ d_i x \in D_n^{k-1} \text{ for } i < k-1, \text{ and } d_i x \in D_n^k \text{ for } i > k+1 \}.$$

Theorem 3. *A simplicial set C with neutrality is a complicial set if and only if, for all $0 < k < n$, there is a function*

$$\varphi^k : P^k_n \longrightarrow C_n$$

satisfying the following conditions:

(0) $\varphi^k(x,y)$ *is neutral;*

(i) $d_{k-1}\varphi^k(x,y) = x, \quad d_{k+1}\varphi^k(x,y) = y;$

(ii)
$$d_i\varphi^k(x,y) = \begin{cases} \varphi^{k-1}(d_i x, d_i y) & \text{for } i < k-1, \\ \varphi^k(d_{i-1}x, d_{i-1}y) & \text{for } i > k+1; \end{cases}$$

(iii) *if x, y are neutral then so is* $d_k\varphi^k(x,y)$;

(iv) *if $w \in D^k_n$ then* $w = \varphi^k(d_{k-1}w, d_{k+1}w)$.

Furthermore, under these conditions, each φ^k is an isomorphism onto D^k_n and satisfies the further property:

(v) $d_k\varphi^k(d_k\varphi^k(x,y), z) = d_k\varphi^k(x, d_k\varphi^k(y,z))$.

Proof. Suppose C is a complicial set. Using induction on n, we shall define isomorphisms $\varphi^k : P^k_n \to D^k_n$ satisfying (i) – (iii).

For $n = 2$, $\varphi^1(x,y)$ satisfying (i) is uniquely determined by property (a) for C. So φ^1 is an isomorphism

~~and~~ (ii) ~~...~~ is vacuous and (iii) follows from (b).

Take $(x, y) \in P_n^k$. Our inductive hypothesis allows us to consider:

$$z_i = \begin{cases} \varphi^{k-1}(d_i x, d_i y) & \text{for } i < k-1 \\ x & \text{for } i = k-1 \\ y & \text{for } i = k+1 \\ \varphi^k(d_{i-1} x, d_{i-1} y) & \text{for } i > k+1. \end{cases}$$

The following calculations show that this defines a k-horn.

For $i < j < k-1$,

$$d_i z_j = d_i \varphi^{k-1}(d_j x, d_j y) = \varphi^{k-2}(d_i d_j x, d_i d_j y)$$
$$= \varphi^{k-2}(d_{j-1} d_i x, d_{j-1} d_i y) = d_{j-1} \varphi^{k-1}(d_i x, d_i y) = d_{j-1} z_i.$$

For $i < j = k-1$,

$$d_i z_j = d_i x = d_{k-2} \varphi^{k-1}(d_i x, d_i y) = d_{j-1} z_i.$$

For $i < j = k+1$, and $i < k-1$,

$$d_i z_j = d_i y = d_k \varphi^{k-1}(d_i x, d_i y) = d_{j-1} z_i.$$

For $i = k-1 < j = k+1$,

$$d_i z_j = d_{k-1} y = d_k x = d_{j-1} z_i.$$

For $i = k+1 < j$,

$$d_i z_j = d_{k+1} \varphi^k(d_{j-1} x, d_{j-1} y) = d_{j-1} y = d_{j-1} z_i.$$

For $k+1 < i < j$,

$$d_i z_j = d_i \varphi^k(d_{j-1} x, d_{j-1} y) = \varphi^k(d_{i-1} d_{j-1} x, d_{i-1} d_{j-1} y)$$

$= \varphi^k(d_{j-2}d_{i-1}x, d_{j-2}d_{i-1}y) = d_{j-1}\varphi^k(d_{i-1}x, d_{i-1}y)$
$= d_{j-1}z_i.$

We shall now show that this k-horn is complicial. Consider $\mu : [m] \to [n-1]$ monic and $0 \le i \le n$ such that $\partial_i \mu$ is a k-monic. By Lemma 3, either $i < k-1$ and μ is a $(k-1)$-monic, or $i > k+1$ and μ is a k-monic. In the former case, $(C\mu)z_i = (C\mu)\varphi^{k-1}(d_i x, d_i y)$ which is neutral since induction gives $\varphi^{k-1}(d_i x, d_i y) \in D^{k-1}_{n-1}$. In the latter case, $(C\mu)z_i = (C\mu)\varphi^k(d_{i-1}x, d_{i-1}y)$, which is neutral since $\varphi^k(d_{i-1}x, d_{i-1}y) \in D^k_{n-1}$.

By property (a) for C, there exists a unique neutral $\varphi^k(x,y) \in C_n$ with $d_i\varphi^k(x,y) = z_i$ for $i \neq k$. Thus (i), (ii) are satisfied, and (iii) follows from (b) for C and induction. The uniqueness property of $\varphi^k(x,y)$ precisely yields that φ^k is an isomorphism.

Properties (0), (iv) follow since $\varphi^k : P^k_n \cong D^k_n$.

Conversely, suppose $\varphi^k : P^k_n \to C_n$ exist satisfying (0)–(iv). Suppose $z_i \in C_{n-1}$, $i \neq k$, form a complicial k-horn. Put $z = \varphi^k(z_{k-1}, z_{k+1})$ which is neutral by (0). By (i), $d_{k-1}z = z_{k-1}$, $d_{k+1}z = z_{k+1}$. For $i < k-1$, $z_i \in D^{k-1}_{n-1}$ and, by (ii), $d_i z = \varphi^{k-1}(d_i z_{k-1}, d_i z_k)$
$= \varphi^{k-1}(d_{k-2}z_i, d_k z_i) = z_i$ by (iv). For $i > k+1$,

$z_i \in D^k_{n-1}$ and, by (ii), $d_i z = \varphi^k(d_{i+1} z_{k-1}, d_{i-1} z_{k+1}) = \varphi^k(d_{k+1} z_i, d_{k+1} z_i) = z_i$ by (iv). The uniqueness of z follows from (iv). So we have property (a) for C. Property (b) is a consequence of (iii).

Condition (v) will be a consequence of Theorem 5 below. □

<u>Lemma 4</u>. (a) *If $x \in D^k_{n+1}$ and $d_{k-1}x \in D^k_n$ then $x \in D^{k+1}_{n+1}$.*

(b) *If $x \in D^k_{n+1}$ and $d_k x$, $d_{k+1} x \in D^{k-1}_n$ then $x \in D^{k-1}_{n+1}$.*

(c) *If C is a complicial set then $x \in D^k_{n+1}$, $d_{k-1}x$, $d_{k+1}x \in D^{k-}_n$ imply $d_k x \in D^{k-1}_n$.*

(d) *If C is a complicial set then $x \in D^k_{n+1}$, $d_{k-1}x$, $d_{k+1}x \in D^k_n$ imply $d_k x \in D^k_n$.*

<u>Proof</u>. The cases $n = 2$ all become clear when one notes that D^1_2 is the set of neutral elements of C_2, D^1_3 is the set of neutral $x \in C_3$ with $d_3 x$ neutral, and D^2_3 is the set of neutral $x \in C_3$ with $d_0 x$ neutral.

(a) Assume (a) for $n-1$. Take $x \in D^k_{n+1}$ with $d_{k-1} x \in D^k_n$. To show $x \in D^{k+1}_{n+1}$, since x is neutral, we must see that

(i) $i < k$ implies $d_i x \in D^k_n$; and

(ii) $i > k+2$ implies $d_i x \in D^{k+1}_n$!

To prove (i) note that the case $i = k-1$ is given so we may assume $i < k-1$. Observe then that $d_i x \in D^{k-1}_n$ (since $x \in D^k_{n+1}$) and $d_{k-2} d_i x = d_i d_{k-1} x \in D^{k-1}_{n-1}$ (since $d_{k-1} x \in D^k_n$), so by induction $d_i x \in D^k_n$. To prove (ii), suppose $i > k+2$ and observe that $d_i x \in D^k_n$ (since $x \in D^k_{n+1}$) and $d_{k-1} d_i x = d_{i-1} d_{k-1} x \in D^k_{n-1}$ (since $d_{k-1} x \in D^k_n$), so by inductio

$d_i x \in D_n^{k+1}$ as required.

(b) Assume (b) for $n-1$. Take $x \in D_{n+1}^{k}$ with $d_k x, d_{k+1} x \in D_n^{k-1}$. To prove $x \in D_{n+1}^{k-1}$ we must see:

(i) $i < k-2$ implies $d_i x \in D_n^{k-2}$; and

(ii) $i > k$ implies $d_i x \in D_n^{k-1}$.

If $i < k-2$ then $d_i x \in D_n^{k-1}$ (since $x \in D_{n+1}^{k}$), $d_{k-1} d_i x = d_i d_k x \in D_{n-1}^{k-2}$ (since $d_k x \in D_n^{k-1}$), and $d_k d_i x = d_i d_{k+1} x \in D_{n-1}^{k-2}$ (since $d_{k+1} x \in D_n^{k-1}$); so by induction $d_i x \in D_n^{k-2}$, this gives (i). If $i > k+1$ then $d_i x \in D_n^{k}$ (since $x \in D_{n+1}^{k}$) $d_k d_i x = d_{i-1} d_k x \in D_{n-1}^{k-1}$ (since $d_k x \in D_n^{k-1}$), and $d_{k+1} d_i x = d_{i-1} d_{k+1} x \in D_{n-1}^{k-1}$ (since $d_{k+1} x \in D_n^{k-1}$); so by induction, $d_i x \in D_n^{k-1}$. If $i = k+1$ then $d_i x \in D_n^{k-1}$ is given. So (ii) does hold.

(c) Assume (c) for $n-1$. Take $x \in D_{n+1}^{k}$ with $d_{k-1} x, d_{k+1} x \in D_n^{k-1}$. Using Theorem 3, we have that $d_k x$ is neutral, s in order to show $d_k x \in D_n^{k-1}$ we must show:

(i) $i < k-2$ implies $d_i d_k x \in D_{n-1}^{k-2}$; and

(ii) $i > k$ implies $d_i d_k x \in D_{n-1}^{k-1}$.

If $i < k-2$ then $d_i d_k x = d_{k-1} d_i x$; now $d_i x \in D_n^{k-1}$ (since $x \in D_{n+1}^{k}$), $d_{k-2} d_i x = d_i d_{k-1} x \in D_{n-1}^{k-2}$, and $d_k d_i x = d_i d_{k+1} x \in D_{n-1}^{k-2}$ (since $d_{k-1} x, d_{k+1} x \in D_n^{k-1}$); so by induction, $d_{k-1} d_i x \in D^{k-2}_{n-1}$ as required for (i). To prove (ii) suppose $i > k$ and we must show $d_i d_k x = d_k d_{i+1} x \in D^{k-1}_{n-1}$. But $d_{i+1} x \in D^k_n$ (since $x \in D^k_{n+1}$), $d_{k-1} d_{i+1} x = d_i d_{k-1} x$ and $d_{k+1} d_{i+1} x = d_i d_{k+1} x$ both lie in D^{k-1}_{n-1} (since $d_{k-1}x, d_{k+1}x \in D^{k-1}_n$); so by induction $d_k d_{i+1} x \in D^{k-1}_{n-1}$, as required.

(d) Take $x \in D^k_{n+1}$ with $d_{k-1}x, d_{k+1}x \in D^k_n$. By Theorem $d_k x$ is neutral. If $i < k-1$ then $d_i x \in D^{k-1}_n$, $d_{k-2} d_i x = d_i d_{k-1} x \in D^{k-1}_{n-1}$, $d_k d_i x = d_i d_{k+1} x \in D^{k-1}_{n-1}$, so by induction $d_{k-1} d_i x \in D^{k-1}_{n-1}$; so $d_i d_k x \in D^{k-1}_{n-1}$. If $i > k+1$ then $d_{i+1} x \in D^k_n$, $d_{k-1} d_{i+1} x = d_i d_{k-1} x \in D^k_{n-1}$, $d_{k+1} d_{i+1} x = d_i d_{k+1} x \in D^k_{n-1}$, so by induction $d_i d_k x = d_k d_{i+1} x \in D^k_{n-1}$. Thus $d_k x \in D^k_n$. $\square$

For $C \in NSmpl$ and $0 < k \leq n$, there is a simplicial set $C(n,k)$ defined as follows:

$$C(n,k)_0 = C_{n-1}, \qquad C(n,k)_1 = C_n,$$

$$C(n,k)_m = \{ x \in D^k_{m+n-1} \mid d_{k-1}x, \dots, d_{k+m-1}x \in C(n,k)_{m-1} \}$$

for $m \geq 2$,

the face functions $C(n,k)_m \to C(n,k)_{m-1}$ are $d_{k-1}, \dots, d_{m+k-1}$ of C,
the degeneracies $C(n,k)_{m-1} \to C(n,k)_m$ are $s_{k-1}, \dots, s_{m+k-2}$ of C.

Notice that $C(n,k)_2 = D^k_{n+1}$ and, from Lemma 4(a), we have:

$$C(n,k)_m \subset \bigcap_{l=0}^{m-2} D^{k+l}_{m+n-1} \quad \text{for } m \geq 2.$$

__Theorem 5.__ __If C is a complicial set then, for all $0 < k \leq n$ the simplicial set $C(n,k)$ is the nerve of a category whose set of arrows is C_n and whose composition $\underset{k}{*}$ is given by:__ $y \underset{k}{*} x = d_k \varphi^k(x,y)$.

__Proof.__ We begin by proving the following statement by induction on m:

__if__ $w \in D^{k+l}_{m+n}$, $d_{k+l-1}w$, $d_{k+l+1}w \in C(n,k)_m$ __for some__ $0 \leq l \leq m-$
__then__ $w \in C(n,k)_{m+1}$.

For $m=1$ we must have $l=0$ and the statement is clear.

Assume the result for $m-1$ and take $w \in D^{k+l}_{m+n}$ with $d_{k+l-1}w$, $d_{k+l+1}w \in C(n,k)_m$. We shall show that, for all $0 \leq r \leq m-1$, we have $w \in D^{k+r}_{m+n}$ and $d_{k+r-1}w$, $d_{k+r+1}w \in C(n,k)_m$. This must be done by induction on r in two steps: one with $r \leq l$ and the other with $l \leq r$. The starting point for both these inductions is $r = l$ in which case the result is given. Suppose $r < l$ and we already know $w \in D^{k+r+1}_{m+n}$ with $d_{k+r}w$, $d_{k+r+2}w \in C(n,k)_m$. By Lemma (4)(b) & (c), we have $w \in D^{k+r}_{m+n}$. To prove

$d_{k+r-1}w \in C(n,k)_m$ we use our inductive hypothesis for $m-1$ which gives the result provided $d_{k+r-1}w \in D^{k+r}_{m+n-1}$, $d_{k+r-1}d_{k+r-1}w \in C(n,k)_{m-1}$ and $d_{k+r+1}d_{k+r-1}w \in C(n,k)_{m-1}$. The first of these follows since $w \in D^{k+r+1}_{m+n}$; the second follows from $d_{k+r-1}d_{k+r-1}w = d_{k+r-1}d_{k+r}w$ and $d_{k+r}w \in C(n,k)_m$; the third follows from $d_{k+r+1}d_{k+r-1}w = d_{k+r-1}d_{k+r+2}w$ and $d_{k+r+2}w \in C(n,k)_m$. To prove $d_{k+r+1}w \in C(n,k)$ we again use ~~induction~~ the inductive hypothesis: so we must see that $d_{k+r+1}w \in D^{k+r}_{m+n-1}$, $d_{k+r-1}d_{k+r+1}w \in C(n,k)_{m-1}$ and $d_{k+r+1}d_{k+r+1}w \in C(n,k)_{m-1}$. The first follows from Lemma 4 (c) since $w \in D^{k+r+1}_{m+n}$, $d_{k+r}w$, $d_{k+r+2}w \in D^{k+r}_{m+n-1}$. The second follows from $d_{k+r-1}d_{k+r+1}w = d_{k+r}d_{k+r-1}w$ since we already have proved $d_{k+r-1}w \in C(n,k)_m$. The third follows from $d_{k+r+1}d_{k+r+1}w = d_{k+r+1}d_{k+r+2}w$ and $d_{k+r+2}w \in C(n,k)_m$.

Now suppose $l < r$ and we already have $w \in D^{k+r-1}_{m+n}$ with $d_{k+r-2}w$, $d_{k+r}w \in C(n,k)_m$. By Lemma 4 (a), we have $w \in D^{k+r}_{m+n}$. To prove $d_{k+r+1}w \in C(n,k)_m$ we must show that $d_{k+r+1}w \in D^{k+r-1}_{m+n-1}$, $d_{k+r-2}d_{k+r+1}w \in C(n,k)_{m-1}$, $d_{k+r}d_{k+r+1}w \in C(n,k)_{m-1}$. The first is true since $w \in D^{k+r-1}_{m+n}$. The second and third follow from

$d_{k+r-2}\, d_{k+r+1} w = d_{k+r}\, d_{k+r-2} w$, $d_{k+r}\, d_{k+r+1} w = d_{k+r}\, d_{k+r} w$ and $d_{k+r-2} w, d_{k+r} w \in C(n,k)_m$. To prove $d_{k+r-1} w \in C(n,k)_m$ we must show that $d_{k+r-1} w \in D^{k+r-1}_{m+n-1}$, $d_{k+r-2}\, d_{k+r-1} w \in C(n,k)_{m-1}$ and $d_{k+r}\, d_{k+r-1} w \in C(n,k)_{m-1}$. The first follows from Lemm 4(d) since $w \in D^{k+r-1}_{m+n}$, $d_{k+r-2} w, d_{k+r} w \in C(n,k)_m$. The second follows from $d_{k+r-2}\, d_{k+r-1} w = d_{k+r-2}\, d_{k+r-2} w$ and $d_{k+r-2} w \in C(n,k)$. The third follows from $d_{k+r}\, d_{k+r-1} w = d_{k+r-1}\, d_{k+r+1} w$ since we already have proved $d_{k+r+1} w \in C(n,k)_m$.

Thus we have completed our two step induction on r. In particular, we can take $r = 0$ and obtain $w \in D^{k}_{m+n}$. Allowing r to run from 0 to $m-1$, we also get $d_{k-1} w, \ldots, d_{k+m} w \in C(n,k)_m$. Hence $w \in C(n,k)_{m+1}$. This completes the induction on m.

Now we turn to a proof of the theorem. Consider $C(n,k) \in NSmpl$ by taking all elements of degree > 1 to be neutral (as they are also in the sense of C). To see that $C(n,k)$ is the nerve of a category we must see, by Proposition 1, that $C(n,k)$ is a complicial set ~~By~~ For this we shall show that Theorem 3 applies to $C(n,k)$.

Suppose $x, y \in C(n,k)_{m-1}$ with $d_{k+l} x = d_{k+l-1} y$.

Then $w = \varphi^{k+l}(x,y) \in D^{k+l}_{m+n-1}$ with $d_{k+l-1}w = x$, $d_{k+l+1}w = y \in C(n,k)_{m-1}$. So, by the first part of this proof, $w \in C(n,k)_m$. So we have functions:

$$\varphi^{k+l} : P^{k+l}_{m+n-2} \cap \left(C(n,k)_{m-1} \times C(n,k)_{m-1}\right) \longrightarrow C(n,k)_m,$$
$$\text{for } 0 \leq l \leq m-2,$$

as required for Theorem 3 applied to $C(n,k)$. Properties (0) – (iii) of Theorem 3 are clear since they hold in C. For any $w \in C(n,k)_m$ we have $w \in D^{k+l}_{m+n-1}$ for all $l = 0, \ldots, m-2$. So $w = \varphi^{k+l}(d_{k+l-1}w, d_{k+l+1}w)$. This proves property (iv) of Theorem 3.

The final assertion of Theorem 5 is now clear as is property (v) of Theorem 3. $\square$

<u>Corollary 6. A simplicial set C with neutrality is a complicial set if and only if, for all $0 < k \leq n$, the diagram</u>

$$\begin{tikzcd} D^{k}_{n+1} \arrow[r, shift left=2, "d_{k-1}"] \arrow[r, "d_k" description] \arrow[r, shift right=2, "d_{k+1}"'] & C_n \arrow[r, shift left, "d_{k-1}"] \arrow[r, shift right, "d_k"'] & C_{n-1} \arrow[l, bend left, "s_{k-1}"] \end{tikzcd}$$

<u>can be completed to the nerve of a category (whose set of arrows is C_n and composition is denoted $\underset{k}{*}$) such that the following conditions hold:</u>

(i) <u>composites of neutral elements are neutral;</u>

(ii) <u>for $i < k-1$, $d_i : (C_n, \underset{k}{*}) \longrightarrow (C_{n-1}, \underset{k-1}{*})$ is a functor;</u>

(iii) <u>for</u> $i > k+1$, $d_{i-1} : (C_n, \underset{k}{*}) \longrightarrow (C_{n-1}, \underset{k}{*})$ <u>is a functor</u>. $\square$

§5. The 2-category underlying a complicial set.

Lemma 7. The following identities hold in a complicial set C:

$$w \underset{k}{*} (u \underset{k+1}{*} v) = (\varphi^k(d_{k+1}v, d_{k+1}w) \underset{k}{*} u) \underset{k+1}{*} (w \underset{k}{*} v),$$

$$(y \underset{k-1}{*} z) \underset{k}{*} x = (y \underset{k}{*} x) \underset{k-1}{*} (z \underset{k}{*} \varphi^{k-1}(d_{k-2}x, d_{k-2}y)).$$

In particular, if $d_{k+1}w$ is a unit for $\underset{k}{*}$ and $d_{k-2}x$ is a unit for $\underset{k-1}{*}$ then these reduce to the identities:

$$w \underset{k}{*} (u \underset{k+1}{*} v) = u \underset{k+1}{*} (w \underset{k}{*} v),$$

$$(y \underset{k-1}{*} z) \underset{k}{*} x = (y \underset{k}{*} x) \underset{k-1}{*} z.$$

by induction on n

Proof. Put $a = \varphi^k(\varphi^{k+1}(v,u), \varphi^k(v,w))$. By Theorem 3(ii $d_k a$ is neutral. Now $d_{k-1} d_k a = d_{k-1} d_{k-1} a = d_{k-1}\varphi^{k+1}(v,u) = \varphi^k(d_{k-1}v, d_{k-1}u) \in D^k_n$; while, for $i < k-1$, $d_i d_k a \in D^k_n$; and, for $i > k+2$, $d_i d_k a \in D^{k+1}_n$. So $d_k a \in D^{k+1}_{n+1}$. But $d_k d_k a = d_k d_{k+1} a = d_k \varphi^k(v,w)$ and $d_{k+2} d_k a = d_k d_{k+3} a = d_k \varphi^k(u, \varphi^k(d_{k+1}v, d_{k+1}w))$. Thus $d_k a = \varphi^{k+1}(d_k \varphi^k(v,w), d_k \varphi^k(u, \varphi^k(d_{k+1}v, d_{k+1}w)))$. So

$d_{k+1}\varphi^{k+1}(d_k\varphi^k(v,w), d_k\varphi^k(u, \varphi^k(d_{k+1}v, d_{k+1}w))) = d_{k+1} d_k a = d_k d_{k+2} a = d_k \varphi^k(d_{k+1}\varphi^{k+1}(v,u), w)$ which is the first equation.

The second equation follows similarly starting with $b = \varphi^{k+1}(\varphi^k(x,y), \varphi^{k-1}(z,y))$.

If $d_{k+1}w$ is a unit for $\underset{k}{*}$ then $\varphi^k(d_{k+1}v, d_{k+1}w)$ is a unit for $\underset{k}{*}$. If $d_{k-2}x$ is a unit for $\underset{k-1}{*}$ then $\varphi^{k-1}(d_{k-2}x, d_{k-2}y)$ is a unit for $\underset{k}{*}$. So the right-hand sides simplify as asserted in these cases. □

Suppose C is a complicial set. There is a <u>shift function</u> $sh : C_2 \longrightarrow C_2$ given by $sh\, x = d_2\varphi^2(x, s_0 d_2 x)$. This function is idempotent and its image is the set of x in C_2 with $d_2 x$ degenerate. Notice that:

$$sh\,\varphi^1(a,b) = s_0 d_1 \varphi^1(a,b), \text{ and}$$

$$sh\,(y \underset{1}{*} x) = y \underset{1}{*} sh\, x \text{ if } d_2 y \text{ is degenerate.}$$

The latter uses Lemma 7 (first particular case).

Let $K_n = \mathrm{Smpl}(\dot{\Delta}[n], C)$. More explicitly, K_n is the subset of $C_{n-1} \times \dots \times C_{n-1}$ ($n+1$ factors) consisting of those $(x_0, x_1, \dots, x_n)$ satisfying $d_i x_j = d_{j-1} x_i$ for $i < j$. This is the <u>simplicial kernel</u> of $d_0, \dots, d_{n-1} : C_{n-1} \longrightarrow C_{n-2}$.
There is a canonical function $d : C_n \longrightarrow K_n$ given by $dy = (d_0 y, \dots, d_n y)$.

<u>Lemma 8.</u> *Let A denote the image of $sh : C_2 \to C_2$ for a complicial set C.*

(i) *The following square is a pullback:*

$$\begin{tikzcd} C_2 \arrow[r, "d"] \arrow[d, "sh"'] & K_2 \arrow[d] \\ A \arrow[r] & C_1 \times C_1 , \end{tikzcd}$$

where the bottom function takes x to $(d_0 x, d_1 x)$ and the right function takes (y_0, y_1, y_2) to $(y_2 \ast_1 y_0, y_1)$.

(ii) *The function $d : C_3 \to K_3$ restricted to the neutral elements in C_3 is the equalizer of the following two functions from K_3 to A:*

$$\begin{tikzcd} & D_3^1 \arrow[r, "d_1"] & C_2 \arrow[dr, "sh"] & \\ K_3 \arrow[ur, "k_1"] \arrow[dr, "k_2"'] & & & A \\ & D_3^2 \arrow[r, "d_2"'] & C_2 \arrow[ur, "sh"'] & \end{tikzcd}$$

where $k_1(y_0, y_1, y_2, y_3) = \varphi^1(y_0, y_2)$, $k_2(y_0, y_1, y_2, y_3) = \varphi^2(y_1, y_3)$.

<u>Proof.</u> (i) Take $x \in A$, $(y_0, y_1, y_2) \in K_2$ with $d_0 x = y_2 \ast_1 y_0$, $d_1 x = y_1$. Put $z = x \ast_1 \varphi^1(y_0, y_2)$. Then:

$d_0 z = d_0 d_0 \varphi^1(\varphi^1(y_0, y_2), x) = d_0 \varphi^1(y_0, y_2) = y_0$,

$d_1 z = d_1 d_2 \varphi^1(\varphi^1(y_0, y_2), x) = d_1 x = y_1$,

$d_2 z = d_1 d_3 \varphi^1(\varphi^1(y_0, y_2), x) = d_1\varphi^1(y_2, d_2 x) = y_2$ since $x \in A$.
Also $sh\, z = sh(x \underset{1}{*} \varphi^1(y_0, y_2)) = x \underset{1}{*} sh\,\varphi^1(y_0, y_2)$
$= x \underset{1}{*} s_0 d_1 \varphi^1(y_0, y_2) = x$.

It remains to show that z is unique. Suppose $sh\, z' = x$ and $d_i z' = y_i$. Now $\varphi^2(z', s_0 y_2) \in D_3^1$ since it and $d_3\varphi^2(z', s_0 y_2) = s_0 y_2$ are neutral. Hence

$$\begin{aligned}\varphi^2(z', s_0 y_2) &= \varphi^1\left(d_0\varphi^2(z', s_0 y_2), d_2\varphi^2(z', s_0 y_2)\right)\\ &= \varphi^1(\varphi^1(y_0, y_2), sh\, z').\\ &= \varphi^1(\varphi^1(y_0, y_2), x).\end{aligned}$$

So $z' = d_1\varphi^2(z', s_0 y_2) = d_1\varphi^1(\varphi^1(y_0, y_2), x) = z$.

(ii) Take $(y_0, y_1, y_2, y_3) \in K_3$. Consider $a = \varphi^2(\varphi^2(y_1, y_3), \varphi^1(sh\, y_0, y_2))$ which is defined if and only if $d_2\varphi^2(y_1, y_3) = d_1\varphi^1(sh\, y_0, y_2)$. Then $d_2 a$ is a neutral element of C_3 with $d_i d_2 a = y_i$.

Conversely, suppose w is a neutral element of C_3 with $d_i w = y_i$. Put $b = \varphi^3(w, s_1 y_3)$. It is easily checked that $b \in D_4^2$. So $b = \varphi^2(d_1 b, d_3 b)$. But $d_1 b = \varphi^2(y_1, y_3)$ and it can be seen that $d_3 b \in D_3^1$; so $d_3 b = \varphi^1(d_0 d_3 b, d_2 d_3 b) = \varphi^1(d_2 d_0 b, d_2 d_2 b) = \varphi^1(sh\, y_0, y_2)$
So $b = a$.

We have shown that there is a unique neutral w in C_3 with $d_i w = y_i$ if and only if $d_2\varphi^2(y_1, y_3) =$

$d_1\varphi^1(sh\, y_0, y_2)$. The corresponding faces of the ~~each~~ sides of the last equation are equal, so by (i) above, the last equation holds if and only if it does after sh is applied. So there is a unique neutral $w \in C_3$ with $d_i w = y_i$ if and only if $sh(y_3 \underset{2}{\ast} y_1) = sh(y_2 \underset{1}{\ast} sh\, y_0)$ $= sh(y_2 \underset{1}{\ast} y_0)$. □

Corollary 9. If $x, y, z \in C_2$ with $d_2 x$, $d_0 y$ degenerate then

$$sh(y \underset{2}{\ast} (z \underset{1}{\ast} x)) = sh((y \underset{2}{\ast} z) \underset{1}{\ast} x)$$

Proof. Put $c = \varphi^2(\varphi^1(x,z), \varphi^2(z,y))$. Then $d_2 c$ is neutral,

$d_0 d_2 c = d_1 d_0 c = d_1 \varphi^1(x, \varphi^1(d_0 z, d_0 y)) = x$ since $d_0 y$ is degener

$d_1 d_2 c = d_1 d_1 c = d_1 \varphi^1(x, z)$,

$d_2 d_2 c = d_2 d_3 c = d_2 \varphi^2(z, y)$, and

$d_3 d_2 c = d_2 d_4 c = d_2 \varphi^2(\varphi^1(d_2 x, d_2 z), y) = y$ since $d_2 x$ is degenerate.

The result follows on evaluating the two functions in Lemma 8 (ii) at the faces of $d_2 c$. □

Theorem 10. For each complicial set C, there is a 2-category defined as follows:

the set of 2-cells is $A = \{x \in C_2 \mid d_2 x \text{ is degenerate}\}$;

the vertical composite $y \circ x$ is defined when $d_1 x = d_0 y$ and is given by $y \circ x = y \underset{1}{\ast} x$;

the horizontal composite $y * x$ is defined when $d_1 d_1 x = d_0 d_0 y$ and is given by

$$y * x = y \underset{2}{\ast} (\varphi^1(d_1 x, d_1 y) \underset{1}{\ast} x).$$

The underlying category of this 2-category is isomorphic via $s_0 : C_1 \longrightarrow A$ to the category $(C_1, \underset{1}{*})$ whose nerve is $C(1,1)$.

Proof. The following calculation shows that $\underset{1}{*}$ on C_2 restricts to A:

$$d_2(y \underset{1}{*} x) = d_2 d_1 \varphi'(x,y) = d_1 d_3 \varphi'(x,y) = d_1 \varphi'(d_2 x, d_2 y) = d_2 y \underset{1}{*} d_2 x.$$

Given the first sentence of the Theorem, the second follows from the calculation: for $a, b \in C_1$ with $d_1 a = d_0 b$, we have $s_0 b * s_0 a = s_0 b \underset{2}{*} (\varphi'(a,b) \underset{1}{*} s_0 a) = s_0 b \underset{2}{*} \varphi'(a,b)$
$= d_2 \varphi^2(\varphi'(a,b), s_0 b) = sh\, \varphi'(a,b) = s_0 d_1 \varphi'(a,b) = s_0(b \underset{1}{*} a)$.

Take $x, y \in A$ with $d_1 d_1 x = d_0 d_0 y$ and put $a = d_0 x$, $b = d_1 x$, $e = d_0 y$, $f = d_1 y$. We shall prove:

$$(s_0 f * x) \circ (y * s_0 a) = y * x = (y * s_0 b) \circ (s_0 e * x).$$

First note that $y * s_0 a = y \underset{2}{*} (\varphi'(a,f) \underset{1}{*} s_0 a) = y \underset{2}{*} \varphi'(a,f)$.
Also note that $\varphi'(b,f) \underset{1}{*} x = sh(\varphi'(b,f) \underset{1}{*} x) \underset{1}{*} \varphi'(a,f)$ since both sides of the equation have the same faces and
$sh(sh(\varphi'(b,f) \underset{1}{*} x) \underset{1}{*} \varphi'(a,f)) = sh(\varphi'(b,f) \underset{1}{*} x) \underset{1}{*} sh\,\varphi'(a,f)$
$= sh(\varphi'(b,f) \underset{1}{*} x) \underset{1}{*} s_0(f \underset{1}{*} a) = sh(\varphi'(b,f) \underset{1}{*} x)$ so
Lemma 8 (i) applies. Thus:
$(s_0 f * x) \circ (y * s_0 a) = (s_0 f \underset{2}{*} (\varphi'(b,f) \underset{1}{*} x)) \underset{1}{*} (y \underset{2}{*} \varphi'(a,f))$
$= y \underset{2}{*} ((s_0 f \underset{2}{*} (\varphi'(b,f) \underset{1}{*} x)) \underset{1}{*} \varphi'(a,f)) = y \underset{2}{*} (sh(\varphi'(b,f) \underset{1}{*} x) \underset{1}{*} \varphi'$
$= y \underset{2}{*} (\varphi'(b,f) \underset{1}{*} x) = y * x$, where Lemma 7 was used at the second step.

For the other equality, we have:

$(y * s_0 b) \circ (s_0 e * x) = (y *_2 \varphi'(b,f)) *_1 (s_0 e *_2 (\varphi'(b,e) *_1 x))$

$= (y *_2 \varphi'(b,f)) *_1 sh(\varphi'(b,e) *_1 x) = sh((y *_2 \varphi'(b,f)) *_1 (\varphi'(b,e) *_1 x)$

$= sh(((y *_2 \varphi'(b,f)) *_1 \varphi'(b,e)) *_1 x) = sh(((y *_2 \varphi'(b,f)) *_1 (s_1 e *_2 \varphi'(b,e))) *_1 x$

$= sh(((y *_1 s_1 e) *_2 \varphi'(b,f)) *_1 x) = sh((y *_1 s_1 e) *_2 (\varphi'(b,f) *_1 x))$

$= sh(y *_1 s_1 e) *_2 (\varphi'(b,f) *_1 x) = y *_2 (\varphi'(b,f) *_1 x) = y * x$,

where the sixth step uses Lemma 7 and the seventh uses Corollary 9.

Take $x \in A$ and $u, t \in C_1$ with $d_1 t = d_0 u$, $d_1 u = d_0 d_0 x$ and put $a = d_0 x$, $b = d_1 x$. Then

$$(x * s_0 u) * s_0 t = x * s_0(u *_1 t).$$

To see this, consider $z = \varphi^2(\varphi'(t, b *_1 u), \varphi'(u,b))$. Since $d_2 z$ is neutral we have $d_2 z = \varphi'(d_0 d_2 z, d_2 d_2 z) =$ $\varphi'(d_1 d_0 z, d_2 d_3 z) = \varphi'(d_1 \varphi'(t,u), d_2 \varphi'(u,b)) = \varphi'(u *_1 t, b)$. So $(x * s_0 u) * s_0 t = (x *_2 \varphi'(u,b)) *_2 \varphi'(t, b *_1 u) =$ $x *_2 (\varphi'(u,b) *_2 \varphi'(t, b *_1 u)) = x *_2 d_2 z = x *_2 \varphi'(u *_1 t, b)$ $= x * s_0(u *_1 t)$.

Take also $v \in C_1$ with $d_0 v = d_1 d_1 x$. Then

$$((s_0 v) * x) * s_0 u = (s_0 v) * (x * s_0 u).$$

To see this, we have:

$$(s_0 v) * (x * s_0 u) = s_0 v * (x \underset{2}{*} \varphi'(u, b))$$
$$= (s_0 v) \underset{2}{*} (\varphi'(b \underset{1}{*} u, v) \underset{1}{*} (x \underset{2}{*} \varphi'(u, b)))$$
$$= (s_0 v) \underset{2}{*} ((\varphi'(b, v) \underset{1}{*} x) \underset{2}{*} (\varphi'(b \underset{1}{*} u, v) \underset{1}{*} \varphi'(u, b)))$$
$$= (s_0 v) \underset{2}{*} ((\varphi'(b, v) \underset{1}{*} x) \underset{2}{*} \varphi'(u, v \underset{1}{*} b))$$
$$= ((s_0 v) \underset{2}{*} (\varphi'(b, v) \underset{1}{*} x)) \underset{2}{*} \varphi'(u, v \underset{1}{*} b)$$
$$= ((s_0 v) * x) * s_0 u .$$

Take also $w \in C_1$ with $d_0 w = d_1 v$. Then

$$(s_0 w) * ((s_0 v) * x) = s_0 (w \underset{1}{*} v) * x .$$

To see this, we have:

$$(s_0 w) * ((s_0 v) * x) = (s_0 w) \underset{2}{*} (\varphi'(v \underset{1}{*} b, w) \underset{1}{*} (s_0 v \underset{2}{*} (\varphi'(b, v) \underset{1}{*} x)))$$
$$= s_0 w \underset{2}{*} ((\varphi'(v, w) \underset{1}{*} s_0 v) \underset{2}{*} (\varphi'(v \underset{1}{*} b, w) \underset{1}{*} (\varphi'(b, v) \underset{1}{*} x)))$$
$$= s_0 w \underset{2}{*} \varphi'(v, w) \underset{2}{*} (\varphi'(v \underset{1}{*} b, w) \underset{1}{*} \varphi(b, v) \underset{1}{*} x)$$
$$= s_0 (w \underset{1}{*} v) \underset{2}{*} (\varphi'(b, w \underset{1}{*} v) \underset{1}{*} x)$$
$$= s_0 (w \underset{1}{*} v) * x .$$

Associativity of $*$ and the middle-four-interchang law S3 (of §1) are immediate consequences of the above properties. The units for $*$ are elements of the form $s_0 s_0 a$ where $a \in C_0$. The remaining axioms for a 2-category are easily deduced. $\square$

It is obvious that each arrow $f : C \to C'$ in Cmpl induces a 2-functor $A \longrightarrow A'$ which is the restriction of $f_2 : C_2 \to C'_2$ (since the 2-category structure is defined in terms of the complicial structure). Thus we have a functor

$$\mathrm{Cmpl} \longrightarrow 2\text{-}\mathrm{Cat}.$$

<u>Theorem 11</u>. <u>The restriction of the above functor $\mathrm{Cmpl} \to 2\text{-}\mathrm{Cat}$ to Cmpl_2 is an equivalence of categories</u>

$$\mathrm{Cmpl}_2 \simeq 2\text{-}\mathrm{Cat}.$$

<u>Proof</u>. Let $(A, *, \circ)$ be a 2-category. Define a simplicial set C as follows. Let C_0 denote the set of $*$-units, let C_1 denote the set of $\circ$-units, and let $d_0, d_1 : C_1 \to C_0$ denote the source and target functions for $*$-composition. Let K_2 denote the simplicial kernel of d_0, d_1. Let $C_2 = \{(a; x_0, x_1, x_2) \mid a \in A,\ (x_0, x_1, x_2) \in K_2,\ a : x_2 * x_0 \Rightarrow x_1\}$. Let $d_0, d_1, d_2 : C_2 \to C_1$ be the functions which take $(a; x_0, x_1, x_2)$ to x_0, x_1, x_2, respectively. Let K_3 denote the simplicial kernel of d_0, d_1, d_2. Let $p : C_2 \to A$ be the function which takes $(a; x_0, x_1, x_2)$ to a. Let $C_3 = \{(u_0, u_1, u_2, u_3) \in K_3 \mid pu_2 \circ d_2 u_2 * pu_0 = pu_1 \circ pu_3 * d_0 u_1\}$. The projections $K_3 \to C_2$ restrict to give $d_0, d_1, d_2, d_3 : C_3 \to C_2$. The remaining C_n $(n > 3)$ are taken to be the simplicial kernels K_n of the $d_0, \dots, d_{n-1} : C_{n-1} \to C_{n-2}$; so C is to be

the coskeleton of the truncated simplicial set defined up to dimension 3. The degeneracies are clear-cut: $s_0 : C_0 \to C_1$ is the inclusion; $s_0, s_1 : C_1 \to C_2$ are given by $s_0 x = u$, $s_1 x = v$ where pu, pv are o-units, $d_2 u, d_0 v$ are $*$-units, and $d_0 u = d_1 u = d_1 v = d_2 v = x$.

Now we make C into an object of NSmpl by declaring $(a; x_0, x_1, x_2) \in C_2$ to be ~~a~~ neutral when a is a unit for o, and declaring all elements of C_n to be neutral for $n > 2$. It is clear that the assignment $A \mapsto C$ is the object function of a natural functor $\nu : 2\text{-Cat} \to \text{NSmpl}$. We must check that ν lands in Cmpl_2.

For each $0 < k < n$, there is a unique k-monic $\mu : [2] \to [n]$; so D_n^k is the set of $x \in C_n$ for which $p(C\mu)x$ is a o-unit for this particular μ.

We must verify conditions (a), (b) of §2. For $n = 2$ a complicial 1-horn $x_0, x_2 \in C_1$ gives $x = (x_2 * x_0; x_0, x_2 * x_0, x_2)$ as the unique neutral as required for (a) and (b).

Consider the case $n = 3$, $k = 1$. Take $x_0, x_2, x_3 \in C_2$ forming a complicial 1-horn. Then px_3 is a o-unit. Let $x_1 = (px_2 \circ d_2 x_2 * px_0; d_0 x_0, d_1 x_2, d_1 x_3)$. Then

$x = (x_0, x_1, x_2, x_3) \in C_3$ is the required element for (a) and (b).

Similarly for $n = 3$, $k = 2$, a complicial 2-horn $x_0, x_1, x_3 \in C_2$ has px_0 as a 0-unit. Let $x_2 = (px_1 \circ px_3 * d_0 x_1;\ d_1 x_0, d_1 x_1, d_2 x_3)$. Then $x = (x_0, x_1, x_2, x_3) \in C_3$ gives (a), (b).

Since $d_0, d_1, d_2, d_3 : C_3 \longrightarrow C_2$ are jointly monic and C is coskeletal after dimension 3, it follows that the arrows $d_i : C_n \longrightarrow C_{n-1}$, $i \neq k$, are jointly monic for $n > 3$. Thus any k-horn in C_{n-1} (whether or not it is complicial) comes from a unique element of C_n. So (a) holds and (b) is vacuous.

Thus $C \in Cmpl_2$ as asserted.

It is easily seen that the composite

$$2\text{-}Cat \longrightarrow Cmpl_2 \longrightarrow 2\text{-}Cat$$

is isomorphic to the identity.

To see that the composite

$$Cmpl_2 \longrightarrow 2\text{-}Cat \xrightarrow{\nu} Cmpl_2$$

is the identity, the only point remaining after Lemma 8 is that, for $C \in Cmpl_2$ and $(A, *, \circ) \in 2\text{-}Cat$ as in Theorem 10, the functions $d : C_n \longrightarrow K_n$ are isomorphisms for $n > 3$. In fact, we shall prove by induction that, for $n > 3$, the following hold:

(i) $d_0, d_1, d_2, d_3 : C_n \longrightarrow C_{n-1}$ are jointly monic;

(ii) for all $(y_0, y_1, \dots, y_n) \in K_n$, the equation
$d_2 \varphi^2(y_1, y_3) = d_1 \varphi^1(s_0 d_2 y_0 \underset{2}{*} y_0, y_2)$ holds;

(iii) $d : C_n \longrightarrow K_n$ is epic.

Notice that (i) holds even for $n = 3$ by Lemma 8(ii), so this begins the inductive proof of (i). To see that the equation of (ii) holds, notice first:

$$d_0 d_2 \varphi^2(y_1, y_3) = d_1 d_0 \varphi^2(y_1, y_3) = d_1 \varphi^1(d_0 y_1, d_0 y_3)$$
$$= d_1 \varphi^1(d_0 y_0, d_2 y_0) = d_0 d_2 \varphi^2(y_0, s_0 d_2 y_0) = d_0(s_0 d_2 y_0 \underset{2}{*} y_0)$$
$$= d_0 d_0 \varphi^1(s_0 d_2 y_0 \underset{2}{*} y_0, y_2) = d_0 d_1 \varphi^1(s_0 d_2 y_0 \underset{2}{*} y_0, y_2);$$

$$d_1 d_2 \varphi^2(y_1, y_3) = d_1 d_1 \varphi^2(y_1, y_3) = d_1 y_1 = d_1 y_2 = d_1 d_2 \varphi^1(s_0 d_2 y_0 \underset{2}{*} y_0,$$
$$y_2) = d_1 d_1 \varphi^1(s_0 d_2 y_0 \underset{2}{*} y_0, y_2);$$

$$d_2 d_2 \varphi^2(y_1, y_3) = d_2 d_3 \varphi^2(y_1, y_3) = d_2 y_3 = d_2 y_2 = d_1 \varphi^1(s_0 d_3 d_2 y_0, d_2 y_2)$$
$$= d_1 d_3 \varphi^1(s_0 d_2 y_0 \underset{2}{*} y_0, y_2) = d_2 d_1 \varphi^1(s_0 d_2 y_0 \underset{2}{*} y_0, y_2).$$

However, $d_3 d_2 \varphi^2(y_1, y_3) = d_2 \varphi^2(d_3 y_1, d_3 y_3) = d_2 \varphi^2(d_1 y_4, d_3 y_4)$ and $d_3 d_1 \varphi^1(s_0 d_2 y_0 \underset{2}{*} y_0, y_2) = d_1 \varphi^1(d_3(s_0 d_2 y_0 \underset{2}{*} y_0), d_3 y_2) =$ $d_1 \varphi^1(s_0 d_2 d_0 y_4 \underset{2}{*} d_0 y_4, d_2 y_4)$. These are equal when $n = 4$ (see proof of Lemma 8(ii)); for $n > 4$, they are equal by (ii) applied to $(d_0 y_4, d_1 y_4, \dots, d_{n-1} y_4) \in K_{n-1}$ (induction).

To prove (iii), take $(y_0, y_1, \dots, y_n) \in K_n$. By (ii) we can put $a = \Phi^2(\Phi^2(y_1, y_3), \Phi^1(s_0 d_2 y_0 \underset{2}{*} y_0, y_2))$ and easily check $d_0 a = y_0$, $d_1 a = y_1$, $d_2 a = y_2$, $d_3 a = y_3$. We also want $d_i a = y_i$ for $i > 3$. By (i) it suffices to check $d_j d_i a = d_j y_i$ for $j < 4$. But $d_j d_i a = d_{i-1} d_j a = d_{i-1} y_j = d_j y_i$. So $da = (y_0, y_1, \dots, y_n)$.

To prove (i) for $n > 3$, suppose $w \in C_n$. Let $b = \Phi^3(w, s_1 d_3 w)$ as in the proof of Lemma 8 (ii). The only 2-monic $\mu : [2] \longrightarrow [n+1]$ is $\mu = \partial_4 \partial_0 \partial_{n-1} \partial_{n-2} \dots \partial_4 \partial_3$; so $(C\mu) b = d_3 d_4 \dots d_{n-2} d_{n-1} d_0 d_4 b = d_3 d_4 \dots d_{n-2} d_{n-1} d_0 s_1 d_3 w$ $= d_3 d_4 \dots d_{n-2} d_{n-1} s_0 d_0 d_3 w = s_0 d_4 d_5 \dots d_n d_0 d_3 w$ which is degenerate. So $b \in D^2_{n+1}$. The only 1-monic $\nu : [2] \longrightarrow [n]$ is $\nu = \partial_n \dots \partial_4 \partial_3$; so $(C\nu) d_3 b =$ $d_3 d_4 \dots d_n d_3 b = d_3 d_4 \dots d_n d_{n+1} b = d_3 d_4 \dots d_n \Phi^3(d_n w, d_n s_1 d_3 w)$ $= d_3 d_4 \Phi^3(d_4 \dots d_n w, d_4 \dots d_n s_1 d_3 w) = d_3 d_4 \dots d_n s_1 d_3 w$ $= s_1 d_4 \dots d_{n+1} d_3 w$ which is degenerate. So $d_3 b \in D^1_n$. So $d_3 b = \Phi^1(d_0 d_3 b, d_2 d_3 b) = \Phi^1(d_2 d_0 b, d_2 d_2 b) =$ $\Phi^1(s_0 d_2 d_0 w \underset{2}{*} d_0 w, d_2 w)$. So $b = \Phi^2(d_1 b, d_3 b) =$ $\Phi^2(\Phi^2(d_1 w, d_3 w), \Phi^1(s_0 d_2 d_0 w \underset{2}{*} d_0 w, d_2 w))$. So $w = d_2 b$ is uniquely determined by $d_0 w, d_1 w, d_2 w, d_3 w$. $\square$

Corollary 12. If $C \in \mathrm{Cmpl}_2$ then $d_i : C_n \to C_{n-1}$, $i = 0, \dots, n$, form the simplicial kernel of $d_j : C_{n-1} \to C_{n-2}$, $j = 0, \dots, n-1$, for $n > 3$, and $d_0, d_1, d_2, d_3 : C_n \to C_{n-1}$ are jointly monic for $n \geq 3$. $\square$

§6. More 2-categories associated with a complicial set.

For $C \in NSmpl$ and $0 < k \leq n$, there is a simplicial set $C[n,k]$ defined as follows:

$$C[n,k]_0 = C_{n-1}, \qquad C[n,k]_1 = C_n,$$

$$C[n,k]_2 = \{ x \in C_{n+1} \mid d_i x \in D_n^{k-1} \text{ for } i < k-1,\ d_i x \in D_n^{k} \text{ for } i > k+1 \},$$

$$C[n,k]_m = \{ x \in D^k_{m+n-1} \mid d_{k-1}x, \ldots, d_{k+m-1}x \in C[n,k]_{m-1} \text{ for } m > 2,$$

the faces of elements of dimension m are given by $d_{k-1}, \ldots, d_{m+k-1}$ of C the degeneracies on elements of dimension m are $s_{k-1}, \ldots, s_{m+k-2}$ of C

Notice that $C(n,k)$ (see §4) is a subsimplicial se of $C[n,k]$.

Declare $x \in C[n,k]_m$, $m \geq 2$, to be neutral when it is a neutral element of C of dimension $m+n-1$. Thus $C[n,k] \in NSmpl$ and all elements of dimension > 2 are neutral.

<u>Theorem 13</u>. <u>If C is a complicial set then, for all $0 < k \leq n$, $C[n,k]$ is an object of $Cmpl_2$.</u>

<u>Proof</u>. As in the proof of Theorem 5, we begin by proving

the following statement by induction on m:

<u>if</u> $w \in D^{k+l}_{m+n}$, $d_{k+l-1}w, d_{k+l+1}w \in C[n,k]_m$ <u>for</u> <u>some</u> $0 \leq l \leq m-1$ <u>then</u> $w \in C[n,k]_{m+1}$.

The statement is clear for $m=1$, $l=0$.

Consider the case $m=2$, $l=0$. If $w \in D^k_{n+2}$ and $d_{k-1}w, d_{k+1}w \in C[n,k]_2$, we must see that $d_k w, d_{k+2}w \in C[n,k]_2$. Since $w \in D^k_{n+2}$, we have $d_{k+2}w \in D^k_{n+1} \subset C[n,k]_2$. So it remains to see that $d_k w \in C[n,k]_2$. ~~For $i<k-1$, $d_i d_{k-1}w, d_i d_{k+1} w \in D^{k-1}_n$; so we have~~ For $i<k-1$, $d_i w \in D^{k-1}_{n+1}$ and $d_{k-2}d_i w = d_i d_{k-1} w$, $d_{k-1}d_i w = d_i d_{k+1} w \in D^{k-1}_n$; so, by Lemma 4(d), $d_i d_k w = d_{k-1} d_i w \in D^{k-1}_n$. For $i > k+1$, $d_{i+1} w \in D^k_{n+1}$ and $d_{k-1}d_{i+1}w = d_i d_{k-1} w$, $d_{k+1} d_{i+1} w = d_i d_{k+1} w \in D^k_n$; so, by Lemma 4(d), $d_i d_k w = d_k d_{i+1} w \in D^k_n$.

Consider the case $m=2$, $l=1$. If $w \in D^{k+1}_{n+2}$ and $d_k w, d_{k+2} w \in C[n,k]_2$, we must see that $d_{k-1}w, d_{k+1}w \in C[n,k]_2$. Since $w \in D^{k+1}_{n+2}$, we have $d_{k-1}w \in D^k_{n+1} \subset C[n,k]_2$. So it remains to see that $d_{k+1}w \in C[n,k]_2$. For $i<k-1$, $d_i w \in D^k_{n+1}$ and $d_{k-1}d_i w = d_i d_k w$, $d_{k+1} d_i w = d_i d_{k+2} w \in D^k_n$; so, by Lemma 4(d), $d_i d_{k+1} w = d_k d_i w \in D^k_n$. For $i > k+1$, $d_{i+1} w \in D^{k+1}_{n+1}$ and $d_k d_{i+1} w = d_i d_k w \in D^k_n$, $d_{k+2} d_{i+1} w = d_i d_{k+2} w \in D^k_n$; so, by Lemma 4(c), $d_i d_{k+1} w = d_{k+1} d_{i+1} w \in D^k_n$.

With this start for the induction, the proof of the statement is word for word the same as the proof of the statement at the beginning of the proof of Theorem 5 with the only exception being that $C(n,k)$ should be replaced by $C[n,k]$.

Now we turn to a proof of the theorem. We shall show that Theorem 3 applies to $C[n,k]$.

Suppose $x, y \in C[n,k]_{m-1}$ with $d_{k+l}x = d_{k+l-1}y$. Then $w = \varphi^{k+l}(x,y) \in D^{k+l}_{m+n-1}$ with $d_{k+l-1}w = x$, $d_{k+l+1}w = y \in C[n,k]_{m-1}$. By the first part of this proof, $w \in C[n,k]_m$. So we have functions:

$$\varphi^{k+l} : P^{k+l}_{m+n-2} \cap (C[n,k]_{m-1} \times C[n,k]_{m-1}) \longrightarrow C[n,k]_m,$$
$$0 \leq l \leq m-2,$$

as required for Theorem 3 applied to $C[n,k]$. Properties (0)–(iii) of Theorem 3 are clear since they hold in C.

Property (iv) amounts to the following statement:

<u>if</u> $w \in C[n,k]_m$, $0 \leq l \leq m-2$, <u>and</u> $(C[n,k]\mu)w$ <u>is neutral for the unique</u> $(l+1)$-<u>monic</u> $\mu : [2] \to [m]$ <u>then</u> $w \in D^{k+l}_{m+n-1}$.

We again must resort to induction. For $m=2$ and for $m \geq 2$, $l=0$ the statement is clear. If $m=3$, $l=1$, the unique 2-monic $\mu : [2] \to [3]$ is ∂_0, and $(C[n,k]\mu)w = d_{k-1}w$. If $w \in C[n,k]_3$ and $d_{k-1}w$ is neutral then $d_{k-1}w \in D^k_{n+1}$; so $w \in D^{k+1}_{n+2}$ by Lemma 4(a). This proves the statement for $m=3$, $l=1$.

Assume the statement for $m-1 \geq 3$ and take $w \in C[n,k]_m$ with $(C[n,k]\mu)w$ neutral for the $(l+1)$-monic $\mu : [2] \to [m]$. Since $w \in D^{k}_{m+n-1}$, it is neutral. To prove $w \in D^{k+l}_{m+n-1}$ we must prove that $d_i w \in D^{k+l-1}_{m+n-2}$ for $i < k+l-1$, and that $d_i w \in D^{k+l}_{m+n-2}$ for $i > k+l+1$. For $i < k+l-1$, $(C[n,k]\mu)w = (C[n,k]\nu)d_i w$ where $\nu : [2] \to [m-1]$ is the unique l-moni[c] so $d_i w \in D^{k+l-1}_{m+n-2}$ by induction provided $d_i w \in C[n,k]_{m-1}$. For $i > k+l+1$, $(C[n,k]\mu)w = (C[n,k]\nu)d_i w$ where ν is the unique $(l+1)$-monic; so $d_i w \in D^{k+l}_{m+n-2}$ by induction provided $d_i w \in C[n,k]_{m-1}$.

In fact, if $x \in C[n,k]_m$ then all $d_i x \in C[n,k]_{m-1}$ so the provisos in the above proof can be eliminated. For $m = 3$, if we have $x \in C[n,k]_m$ then certainly $d_{k-1}x, d_k x, d_{k+1}x, d_{k+2}x \in C[n,k]_2$. For $j > k+2$, $x \in D^{k}_{n+2}$ implies $d_i x \in D^{k}_{n+1} \subset C[n,k]_2$. For $i < k-1$, $d_i x \in D^{k-1}_{n+1}$ and $d_{k-2} d_j x = d_j d_{k-1} x \in D^{k-1}_{n}$ (since $x \in D^{k}_{n+2}$, $d_{k-1}x \in D^{k}_{n+1}$); so, by Lemma 4(a), $d_i x \in D^{k}_{n+1} \subset C[n,k]_2$. Now assume the result for $m-1 \geq 3$ and take $x \in C[n,k]_m$. Then we must see $d_i x \in C[n,k]_{m-1}$ for $i < k-1$ and $i > k+m-1$. As in the $m=3$ case we deduce that for these i, $d_i x \in D^{k}_{m+n-2}$. For $i < k-1$, $d_{k-1} d_i x = d_i d_k x$, $d_k d_i x = d_i d_{k+1} x, \ldots, d_{k+m-2} d_i x = d_i d_{k+m-1} x \in C[n,k]_{m-2}$ by induction. For $i > k+m-1$, $d_{k-1} d_i x = d_{i-1} d_{k-1} x, \ldots, d_{k+m-2} d_i x = d_{i-1} d_{k+m-2} x \in C[n,k]_{m-2}$ by induction. So $d_i x \in C[n,k]_{m-1}$ as required.

$\square$