Hall's Marriage Theorem

For sets A, B, each relation $R \subseteq A \times B$ can be identified with a union preserving function $\wp A \longrightarrow \wp B$ between the power sets taking $S \subseteq A$ to the subset

$$R(S) = \{ b \in B : (a, b) \in R \text{ for some } a \in S \}$$

of B. In particular, $R(\varnothing) = \varnothing$. Each function $f : A \longrightarrow B$ has a graph f_* which is the relation

$$\{ (a, f(a)) : a \in A \} \subseteq A \times B.$$

Theorem *If* $R \subseteq A \times B$ *is a relation between finite sets such that* $\#S \leq \#R(S)$ *for all* $S \subseteq A$ *then there exists an injective function* $f : A \longrightarrow B$ *with* $f_* \subseteq R$.

Proof (Halmos 1958?) If $A = \varnothing$ then the unique function $f : A \longrightarrow B$ will do.

The property of the relation R in the Theorem gives that $R(S)$ is non-empty for each singleton subset S. In particular, if A is a singleton $\{ a \}$, there is an element $b \in B$ with $(a, b) \in R$; so we can define f by $f(a) = b$.

We have proved the result when A has cardinality 0 or 1. The proof is completed by induction as follows. One of the two possibilities (a) or (b) must hold:

on #A

(a) there exists a subset $\varnothing \subset S \subset A$ with $\#S = \#R(S)$, or

(b) $\#S < \#R(S)$ for all $\varnothing \subset S \subset A$.

First consider case (a); so we have $T = R(S)$ with $\#T = \#S$. Let R' be the restriction of R to a relation between S and T. Clearly R' satisfies the hypothesis of the Theorem. Since $\#S < \#A$, we inductively obtain an injective (and hence bijective) function $g : S \longrightarrow T$ with $g_* \subseteq R'$. In fact, $R' = g_*$ for all $a \in S$; for, if $\#R(\{a\}) > 1$ for some $a \in S$ then

$$\#T = \#R(S) = \# \bigcup_{a \in S} R(\{a\}) > \#g(S) = \#T,$$

a contradiction. Let R'' be the restriction of R to a relation between the complements $\neg S, \neg T$ of S, T in A, B, respectively. We shall show that the relation $R'' \subseteq \neg S \times \neg T$ satisfies the hypothesis of the Theorem. Take $X \subseteq \neg S$ and let $Y = \{ a \in S : g(a) \in R(X) \}$ so that $R(X \cup Y) = R(X) \cup R(Y) = R''(X) \cup R(Y)$. Since X, Y are disjoint and $R''(X), R(Y)$ are disjoint, we have

$$\#X + \#Y = \#(X \cup Y) \leq \#R(X \cup Y) = \#R''(X) + \#R(Y).$$

But $Y \subseteq S$, so $\#Y = \#R(Y)$ and can be cancelled from the previous inequality to give $\#X \leq \#R''(X)$, as required. Since $\#\neg S < \#A$, induction provides an injective function $h : \neg S \longrightarrow \neg T$ with $h_* \subseteq R''$. Define a function $f : A \longrightarrow B$ to agree with g on S and to agree with h on $\neg S$. Since g, h are injective and have disjoint images, f is injective. Also $f_* \subseteq R$ follows from $g_* \subseteq R'$, $h_* \subseteq R''$.

It remains to deal with case (b). Select any $a \in A$ and $b \in R(\{a\})$. Put $C = A \cap \neg\{a\}$, $D = B \cap \neg\{b\}$. Let $R^{-} \subseteq C \times D$ be the restriction of R to a relation between C, D. If $\varnothing \subset X \subseteq C$ then (b) implies that $\#X \leq \#R(X) - 1 \leq \#R^{-}(X)$. So R^{-} satisfies the hypothesis of the Theorem. Since $\#C = \#A - 1$, induction produces an injective function $f : C \longrightarrow D$ with $f_* \subseteq R^{-}$. Extend f to $f : A \longrightarrow B$ by defining $f(a) = b$. This f is as required.$_{\text{qed}}$