7 ~~3~~ Sep 2010

Separable enriched categories and absolute ends

An algebra A over a field is separable when there is an element $e = \sum_{i=1}^{n} a_i \otimes b_i \in A \otimes A$ such that

$$\sum_{i=1}^{n} a_i b_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} (r a_i) \otimes b_i = \sum_{i=1}^{n} a_i \otimes (b_i r)$$

for all $a \in A$. It follows that A is finitely generated and projective as a left A-, right A-bimodule. There are many references.

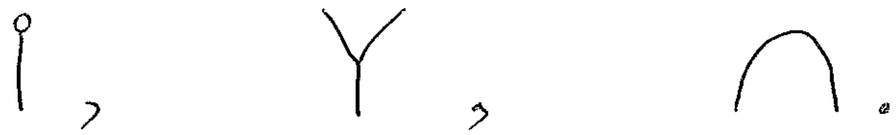
An obvious generalization of this notion is as follows. Let $\mathcal{V}$ denote a monoidal category. A monoid A in $\mathcal{V}$ is separable when it is equipped with a morphism

$$\rho : I \longrightarrow A \otimes A$$

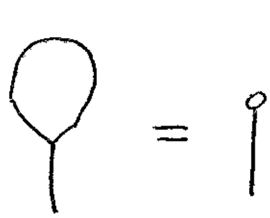
such that

$$\mu \circ \rho = \eta \quad \text{and} \quad (\mu \otimes 1)(1 \otimes \rho) = (1 \otimes \mu)(\rho \otimes 1).$$

Here I is the tensor unit object in $\mathcal{V}$ and $\eta : I \to A$, $\mu : A \otimes A \to A$ are the unit and multiplication for the monoid A. Using the string notation of [GTC I], we denote η, μ, ρ, respectively, as follows:



The axioms for separability then become

 and 

We wish to generalize this even further, still working with a monoidal category $\mathcal{V}$. A $\mathcal{V}$-category $\mathcal{A}$ is separable when it is non-empty and equipped with a family of morphisms

$$\rho : I \longrightarrow \mathcal{A}(B,A) \otimes \mathcal{A}(A,B), \quad A, B \in \mathcal{A},$$

such that

$$(1) \qquad \begin{tikzcd} I \arrow[r, "\rho"] \arrow[dr, "\eta"'] & \mathcal{A}(B,A) \otimes \mathcal{A}(A,B) \arrow[d, "\mu"] \\ & \mathcal{A}(A,A) \end{tikzcd} \qquad \text{and}$$

$$(2) \qquad \begin{tikzcd} \mathcal{A}(A,C) \arrow[r, "1 \otimes \rho"] \arrow[d, "\rho \otimes 1"'] & \mathcal{A}(A,C) \otimes \mathcal{A}(B,A) \otimes \mathcal{A}(A,B) \arrow[d, "\mu \otimes 1"] \\ \mathcal{A}(B,C) \otimes \mathcal{A}(C,B) \otimes \mathcal{A}(A,C) \arrow[r, "1 \otimes \mu"'] & \mathcal{A}(B,C) \otimes \mathcal{A}(A,B). \end{tikzcd}$$

Now η and μ denote identities and compositions in the $\mathcal{V}$-category $\mathcal{A}$. The diagonal of the square (2) will be denoted by

$$\delta : \mathcal{A}(A,C) \longrightarrow \mathcal{A}(B,C) \otimes \mathcal{A}(A,B);$$

it satisfies the Frobenius identity:

$$\begin{tikzcd}
\mathcal{A}(C,D) \otimes \mathcal{A}(A,C) \arrow[r, "1 \otimes \delta"] \arrow[d, "\delta \otimes 1"'] & \mathcal{A}(C,D) \otimes \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \arrow[d, "\mu \otimes 1"] \\
\mathcal{A}(B,D) \otimes \mathcal{A}(C,B) \otimes \mathcal{A}(A,C) \arrow[r, "1 \otimes \mu"'] & \mathcal{A}(B,D) \otimes \mathcal{A}(A,B).
\end{tikzcd} \tag{3}$$

We recover ρ from δ via $\rho = \delta \circ \eta$ where $C = A$.

We can adapt string notation to work in this situation. For the morphisms

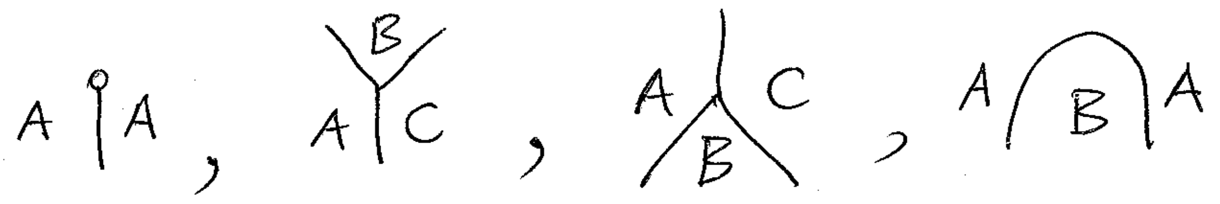
$$\eta : I \longrightarrow \mathcal{A}(A,A),$$

$$\mu : \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,C),$$

$$\delta : \mathcal{A}(A,C) \longrightarrow \mathcal{A}(B,C) \otimes \mathcal{A}(A,B), \text{ and}$$

$$\rho : I \longrightarrow \mathcal{A}(B,A) \otimes \mathcal{A}(A,B),$$

we respectively write

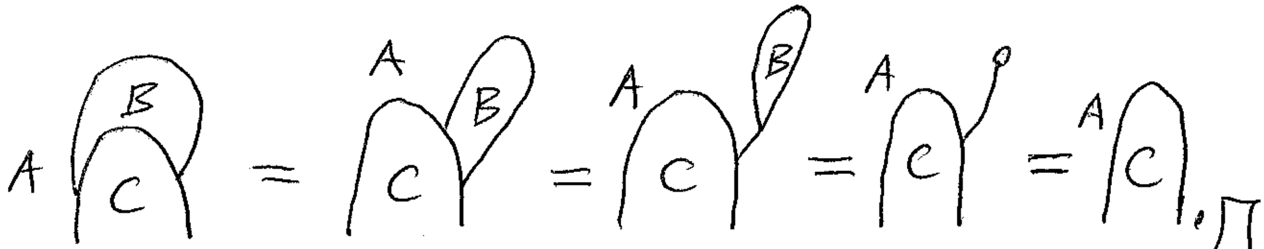


<u>Lemma 1</u> <u>For a separable</u> $\mathcal{V}$-<u>category</u> $\mathcal{A}$, <u>the following diagram commutes.</u>

$$\begin{tikzcd} I \arrow[r, "\rho"] \arrow[drr, "\rho"'] & \mathcal{A}(B,A)\otimes\mathcal{A}(A,B) \arrow[r, "1\otimes\rho\otimes 1"] & \mathcal{A}(B,A)\otimes\mathcal{A}(C,B)\otimes\mathcal{A}(B,C)\otimes\mathcal{A}(A,B) \arrow[d, "\mu\otimes\mu"] \\ & & \mathcal{A}(C,A)\otimes\mathcal{A}(A,C) \end{tikzcd}$$

In diagrams,

<u>Proof</u>

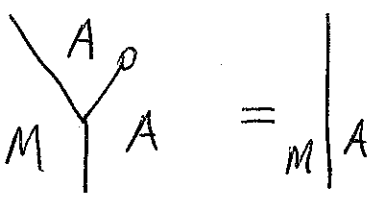


For a $\mathcal{V}$-category $\mathcal{A}$, a <u>right</u> $\mathcal{A}$-<u>module</u> M consists of a family of objects MA, $A\in\mathcal{A}$, in $\mathcal{V}$ equipped with morphisms

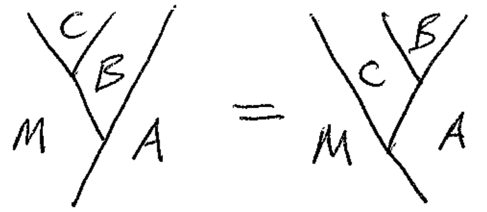
$$\mu_r : MB\otimes\mathcal{A}(A,B)\longrightarrow MA$$

satisfying the conditions for an action. For μ_r we write

and the action conditions become



and



<u>Left</u> $\mathcal{A}$ - <u>module</u> N is defined symmetrically:

$\mu_\ell : \mathcal{A}(A,B) \otimes NA \longrightarrow NB,$ 

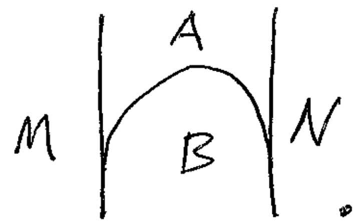
Suppose M and N are right and left $\mathcal{A}$-modules, respectively. Define morphisms

$$e_{A,B} : MA \otimes NA \longrightarrow MB \otimes NB$$

to be the composite

$$\begin{tikzcd} MA \otimes NA \arrow[r, "1 \otimes \rho \otimes 1"] & MA \otimes \mathcal{A}(B,A) \otimes \mathcal{A}(A,B) \otimes NA \arrow[d, "\mu_r \otimes \mu_\ell"] \\ & MB \otimes NB \end{tikzcd}$$

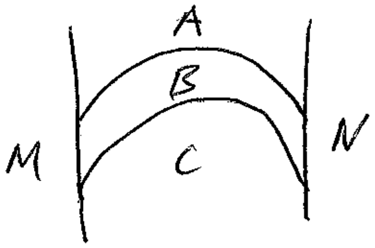
when $\mathcal{A}$ is separable. Using strings, we draw e_{AB} as



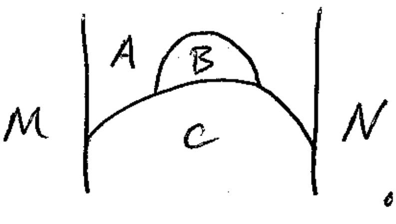
<u>Proposition 2</u> <u>The following diagram commutes.</u>

$$\begin{tikzcd} MA \otimes NA \arrow[rr, "e_{AC}"] \arrow[dr, "e_{AB}"'] & & MC \otimes NC \\ & MB \otimes NB \arrow[ur, "e_{BC}"'] & \end{tikzcd}$$

<u>Proof</u> The lower path in the triangle is



which, by the action conditions, is



Lemma 1 now gives the result. □

The <u>chaotic</u> category on a set K is denoted by K_{ch} : its objects are the elements of K and its homs $K_{ch}(i,j)$ are all singletons. A <u>clique</u> [GTCI] in a category $\mathcal{X}$ is a functor $K_{ch} \to \mathcal{X}$ from a non-empty chaotic category.

We write $Q\mathcal{X}$ for the idempotent splitting completion of the ordinary category $\mathcal{X}$. Objects are pairs (X, e) where $e : X \to X$ is an idempotent $e^2 = e$ in $\mathcal{X}$. Morphisms $f : (X, e) \longrightarrow (X', e')$ are morphisms $f : X \to X'$ in $\mathcal{X}$ satisfying $f = fe = e'f$. We identify $X \in \mathcal{X}$ with $(X, 1_X) \in Q\mathcal{X}$.

__Corollary 3__ *Each e_{AA} is an idempotent on $MA \otimes NA$ and the diagram*

$$e_{AB} : (MA \otimes NA, e_{AA}) \longrightarrow (MB \otimes NB, e_{BB}),$$
$$A, B \in \mathcal{A},$$

is a clique in $Q\mathcal{V}_0$. $\square$

For a clique $X : K_{ch} \longrightarrow \mathcal{X}$, since every object of K_{ch} is both initial and terminal, every value $X(i)$ of X is both a limit and colimit for the functor X.

__Corollary 4__ *If idempotents split in $\mathcal{V}_0$ then the diagram*

$$(4) \qquad MA \otimes NA \xrightarrow{e_{AB}} MB \otimes NB, \qquad A, B \in \mathcal{A},$$

has an absolute colimit (and limit) in $\mathcal{V}_0$. $\square$

To be specific, we have a cocone

$$\begin{tikzcd} MA \otimes NA \arrow[rr, "e_{AB}"] \arrow[dr, "h_A"'] & & MB \otimes NB \arrow[dl, "h_B"] \\ & T & \end{tikzcd}$$

and morphisms $k_A : T \longrightarrow MA \otimes NA$ satisfying

$$e_{AB} = k_B h_A \quad \text{and} \quad h_A k_A = 1_T.$$

From these equations it can be deduced that $(h_A)_{A\in\mathcal{A}}$ is a colimit which, consequently, is absolute. Also $(k_A)_{A\in\mathcal{A}}$ is a limit cone.

{There is no Proposition 5.}

~~that the cocone is a colimit. Hence such colimits are preserved by all functors. □~~

Recall that the <u>tensor product</u> (or <u>composite</u>) of a right $\mathcal{A}$-module M and a left $\mathcal{A}$-module N is defined to be the colimit

$$(5)\qquad \begin{tikzcd} MB \otimes \mathcal{A}(A,B) \otimes NA \arrow[r, "\mu_r \otimes 1"] \arrow[d, "1 \otimes \mu_l"'] & MA \otimes NA \arrow[d, dashed, "p_A"] \\ MB \otimes NB \arrow[r, dashed, "p_B"'] & M \otimes_{\mathcal{A}} N \end{tikzcd}$$

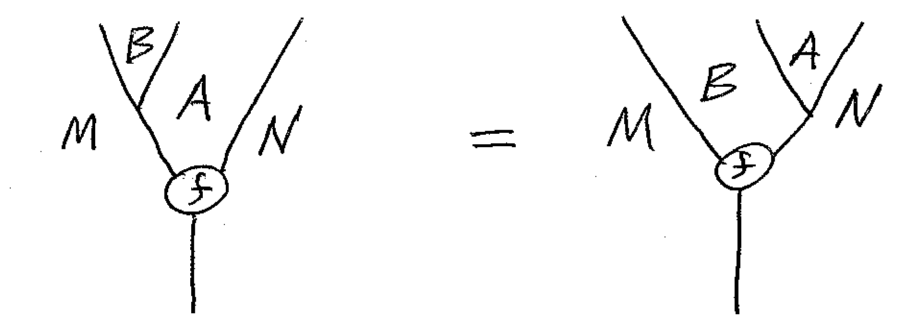
in $\mathcal{V}$, when it exists.

<u>Proposition 6</u> <u>A family of morphisms</u>

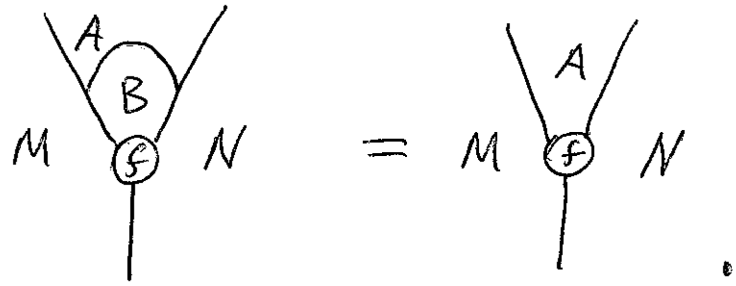
$$f_A : MA \otimes NA \longrightarrow X, \quad A \in \mathcal{A},$$

<u>is a cocone for the solid diagram of (5) if and only if it is a cocone for (4).</u>

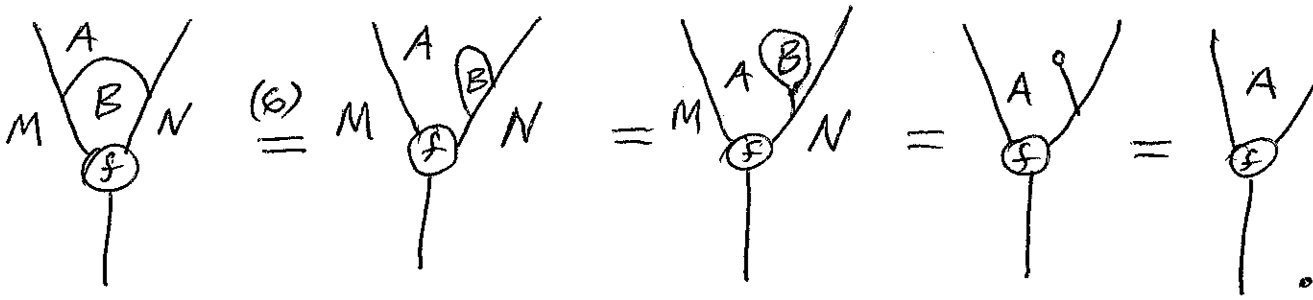
<u>Proof</u> A cocone for (5) means

(6) 

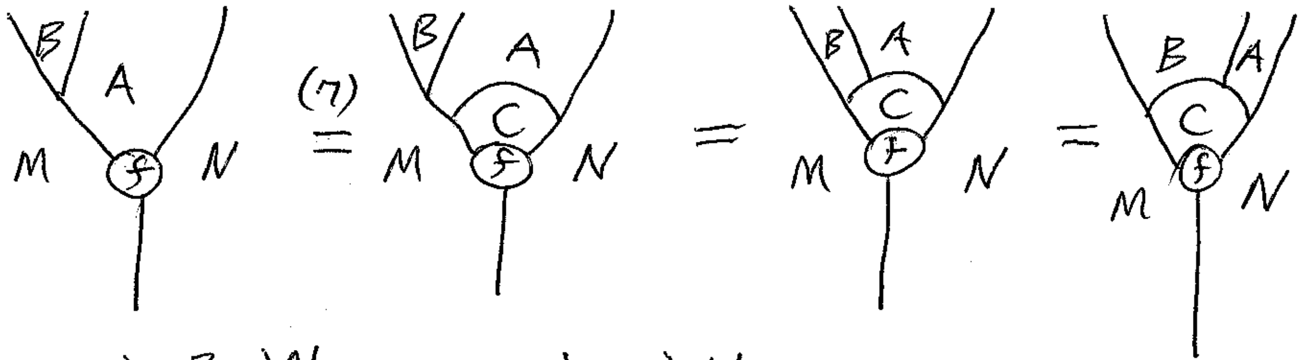
while a cocone for (4) means

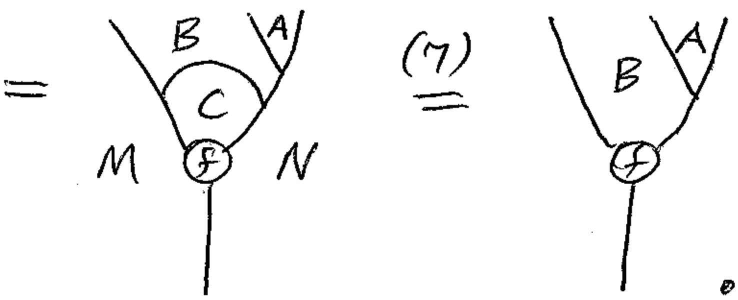
(7) 

Assuming (6), we have



Assuming (7), we have





. □

<u>Corollary 7</u> <u>If idempotents split in $\mathcal{V}$ and $\mathcal{A}$ is a separable $\mathcal{V}$-category then</u> $M \underset{\mathcal{A}}{\otimes} N$ <u>exists.</u>

A monoidal functor $F : \mathcal{V} \longrightarrow \mathcal{W}$ will be called <u>separable</u> when there is a natural family of morphisms

$$\delta_{U,V} : F(U \otimes V) \longrightarrow FU \otimes FV$$

such that

$$(8) \qquad \begin{tikzcd} F(U\otimes V) \arrow[r, "\delta_{U,V}"] \arrow[dr, "1"'] & FU \otimes FV \arrow[d, "\varphi_{U,V}"] \\ & F(U\otimes V), \end{tikzcd}$$

$$(9) \qquad \begin{tikzcd} F(U\otimes V)\otimes FW \arrow[r, "\delta_{U,V}\otimes 1"] \arrow[d, "\varphi_{U\otimes V, W}"'] & FU\otimes FV \otimes FW \arrow[d, "1\otimes\varphi_{V,W}"] \\ F(U\otimes V\otimes W) \arrow[r, "\delta_{U, V\otimes W}"'] & FU \otimes F(V\otimes W) \end{tikzcd}$$

$$(10) \qquad \begin{tikzcd} FU\otimes F(V\otimes W) \arrow[r, "1\otimes\delta_{V,W}"] \arrow[d, "\varphi_{U, V\otimes W}"'] & FU\otimes FV\otimes FW \arrow[d, "\varphi_{U,V}\otimes 1"] \\ F(U\otimes V\otimes W) \arrow[r, "\delta_{U\otimes V, W}"'] & F(U\otimes V)\otimes FW \end{tikzcd}$$

where $\varphi_{U,V}$ and $\varphi_0 : I \longrightarrow FI$ provide the monoidal structure on F, and we have suppressed associativity constraints.

Eilenberg–Kelly [CC] showed that a monoidal functor $F : \mathcal{V} \to \mathcal{W}$ takes a $\mathcal{V}$-category $\mathcal{A}$ to a $\mathcal{W}$-category $F_*\mathcal{A}$ with the same objects and

$$(F_*\mathcal{A})(A, B) = F(\mathcal{A}(A, B)).$$

Proposition 8 *If $F : \mathcal{V} \to \mathcal{W}$ is a separable monoidal functor and $\mathcal{A}$ is a separable $\mathcal{V}$-category then $F_*\mathcal{A}$ is a separable $\mathcal{W}$-category.*

Proof Define ρ for $F_*\mathcal{A}$ to be the composite

$$I \xrightarrow{\varphi_0} FI \xrightarrow{F\rho} F(\mathcal{A}(B,A) \otimes \mathcal{A}(A,B)) \xrightarrow{\delta} F\mathcal{A}(B,A) \otimes F\mathcal{A}(A,B).$$

To save space we write (AB) for $\mathcal{A}(A, B)$. The following proves (1) for this ρ:

$$\begin{tikzcd}
I \arrow[r, "\varphi_0"] \arrow[ddrrr, "\eta"'] & FI \arrow[r, "F\rho"] \arrow[ddrr, "F\eta"] & F((BA)\otimes(AB)) \arrow[r, "\delta"] \arrow[dr, "1"'] \arrow[dr, phantom, "(1)", shift right=6ex] & F(BA)\otimes F(AB) \arrow[d, "\varphi"] \arrow[d, phantom, "(8)", shift right=6ex] \\
 & & & F((BA)\otimes(AB)) \arrow[d, "F\mu"] \\
 & & & F(AA)
\end{tikzcd}$$

The following proves (2) for $F_*\mathcal{A}$:

$$
\begin{tikzcd}
F(AC) \arrow[rr, "1\otimes\varphi_0"] \arrow[d, "\varphi_0\otimes 1"'] \arrow[drr, "1"] & & F(AC)\otimes FI \arrow[rr, "1\otimes F\rho"] \arrow[d, "\varphi"] \arrow[drr, phantom, "\text{nat.}"] & & F(AC)\otimes F((BA)\otimes(AB)) \arrow[dd, "\varphi"'] \arrow[dr, "1\otimes\delta"] & \\
FI\otimes F(AC) \arrow[rr, "\varphi"] \arrow[dd, "F\rho\otimes 1"'] \arrow[dr, phantom, "\text{nat.}"] & & F(AC) \arrow[drr, "F(1\otimes\rho)"] \arrow[dddr, "F(\rho\otimes 1)"] \arrow[ddr, phantom, "(2)"] & & & F(AC)\otimes F(BA)\otimes F(AB) \arrow[dd, "\varphi_2\otimes 1"] \arrow[dl, phantom, "(10)"] \\
& & & & F((AC)\otimes(BA)\otimes(AB)) \arrow[dr, "\delta"] \arrow[dd, bend left, "F(\mu\otimes 1)"] & \\
F((BC)\otimes(CB))\otimes F(AC) \arrow[dd, "\delta\otimes 1"'] \arrow[drrr, "\varphi"] \arrow[dr, phantom, "(9)"] & & & & & F((AC)\otimes(BA))\otimes F(AB) \arrow[dd, "F\mu\otimes 1"] \arrow[dl, phantom, "\text{nat.}"] \\
& & & F((BC)\otimes(CB)\otimes(AC)) \arrow[r, "F(1\otimes\mu)"] \arrow[d, "\delta"] \arrow[dr, phantom, "\text{nat.}"] & F((BC)\otimes(AB)) \arrow[dr, "\delta"] & \\
F(BC)\otimes F(CB)\otimes F(AC) \arrow[rrr, "1\otimes\varphi_2"'] & & & F(BC)\otimes F((CB)\otimes(AC)) \arrow[rr, "1\otimes F\mu"'] & & F(BC)\otimes F(AB)
\end{tikzcd}
$$

$\square$

Each right $\mathcal{A}$-module M determines a right $F_*\mathcal{A}$-module F_*M defined by

$$(F_*M)A = FMA$$

$$\mu_r^F : FMB \otimes F\mathcal{A}(A,B) \xrightarrow{\varphi} F(MB\otimes\mathcal{A}(A,B)) \xrightarrow{F\mu_r} FMA,$$

and similarly for left $\mathcal{A}$-modules. If $\mathcal{A}$ is separable, by Proposition 8 we have the family

$$FMA\otimes FNA \xrightarrow{e^F_{AB}} FMB\otimes FNB$$

of Proposition 2 for the $\mathcal{W}$-category $F_*\mathcal{A}$ and modules F_*M and F_*N.

<u>Proposition 9</u> <u>*The following diagram commutes*</u>

$$(11)\qquad \begin{tikzcd} FMA\otimes FNA \arrow[r, "e^F_{AB}"] \arrow[d, "\varphi"'] & FMB\otimes FNB \arrow[d, "\varphi"] \\ F(MA\otimes NA) \arrow[r, "Fe_{AB}"'] & F(MB\otimes NB). \end{tikzcd}$$

<u>Proof</u>

$$\begin{tikzcd}[column sep=small]
 & & FMA\otimes F((BA)\otimes(AB))\otimes FNA \arrow[dr, "1\otimes\delta\otimes 1"] \arrow[dd, "1"] & \\
 & FMA\otimes FI\otimes FNA \arrow[ur, "1\otimes F\rho\otimes 1"] \arrow[dr, "1\otimes F\rho\otimes 1"] \arrow[dddl, "\varphi_3"'] & & FMA\otimes F(BA)\otimes F(AB)\otimes FNA \arrow[dl, "1\otimes\varphi\otimes 1"'] \arrow[dd, "\varphi\otimes\varphi"] \\
FMA\otimes FNA \arrow[ur, "1\otimes\varphi_0\otimes 1"] \arrow[dd, "\varphi"'] & & FMA\otimes F((BA)\otimes(AB))\otimes FNA \arrow[dd, "\varphi_3"] & \\
 & & & F(MA\otimes(BA))\otimes F((AB)\otimes NA) \arrow[dl, "\varphi"'] \arrow[dd, "F\mu_r\otimes F\mu_\ell"] \\
F(MA\otimes NA) \arrow[rr, "F(1\otimes\rho\otimes 1)"] \arrow[ddrr, "Fe"'] & & F(MA\otimes(BA)\otimes(AB)\otimes NA) \arrow[dd, "F(\mu_r\otimes\mu_\ell)"'] & \\
 & & & FMB\otimes FNB \arrow[dl, "\varphi"] \\
 & & F(MB\otimes NB) &
\end{tikzcd}$$

$\square$

It follows from Corollary 4 and Proposition 9 that φ induces a morphism

$$\bar{\varphi} : F_*M \otimes_{F_*\mathcal{A}} F_*N \longrightarrow F(M\otimes_{\mathcal{A}} N)$$

on the (absolute) colimits of the rows of (11). Notice that, <u>*if*</u> *$F : \mathcal{V}\to\mathcal{W}$* <u>*is strong monoidal*</u> *then $\bar{\varphi}$* <u>*is invertible.*</u>

Suppose N and L are both left $\mathcal{A}$-modules. A functor $\hom_{\mathcal{A}}(N, L) : \mathcal{V}^{op} \longrightarrow \mathrm{Set}$ is defined pointwise as the limit

$$
\begin{tikzcd}
\hom_{\mathcal{A}}(N,L)(X) \arrow[r, dashed, "p_A"] \arrow[ddd, dashed, "p_B"'] & \mathcal{V}(NA\otimes X, LA) \arrow[d, "\mathcal{A}(A,B)\otimes -"] \\
 & \mathcal{V}(\mathcal{A}(A,B)\otimes NA\otimes X, \mathcal{A}(A,B)\otimes LA) \arrow[dd, "\mathcal{V}(1,\mu_\ell)"] \\
 & \\
\mathcal{V}(NB\otimes X, LB) \arrow[r, "\mathcal{V}(\mu_\ell\otimes 1, 1)"'] & \mathcal{V}(\mathcal{A}(A,B)\otimes NA\otimes X, LB).
\end{tikzcd}
$$

A (right) <u>hom for</u> N <u>and</u> L <u>over</u> $\mathcal{A}$ is a representing object $\underset{\sim}{H}_{\mathcal{A}}(N, L)$ for $\hom_{\mathcal{A}}(N, L)$, when it exists.

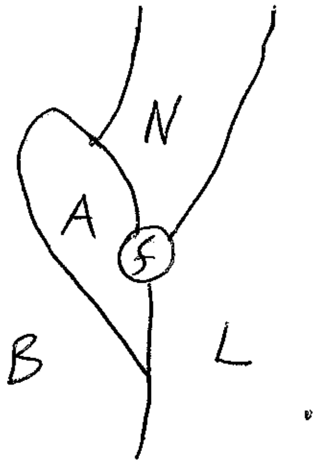
If $\mathcal{A}$ is separable we can define a natural family of functions

$$\xi_{A,B;X} : \mathcal{V}(NA\otimes X, LA) \longrightarrow \mathcal{V}(NB\otimes X, LB)$$

taking $f : NA\otimes X \longrightarrow LA$ to the composite

$$NB\otimes X \xrightarrow{\rho\otimes 1\otimes 1} \mathcal{A}(A,B)\otimes\mathcal{A}(B,A)\otimes NB\otimes X \xrightarrow{1\otimes\mu_\ell\otimes 1} \mathcal{A}(A,B)\otimes NA\otimes X$$
$$\xrightarrow{1\otimes f} \mathcal{A}(A,B)\otimes LA \xrightarrow{\mu_\ell} LB.$$

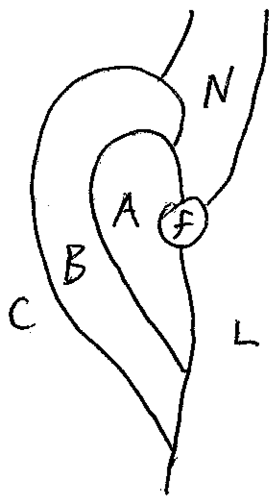
In string diagrams:

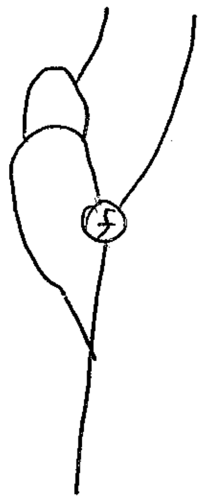
$\xi(f) =$ 

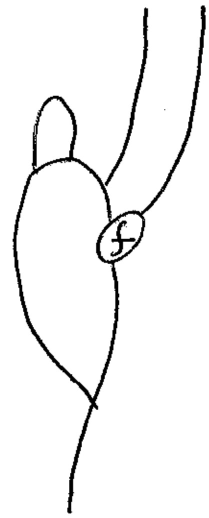
<u>Proposition 10</u> <u>*The following triangle commutes.*</u>

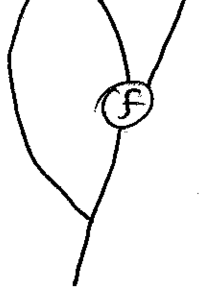
$$\begin{tikzcd}
\mathcal{V}(NA\otimes X, LA) \arrow[rr, "\xi_{ACX}"] \arrow[dr, "\xi_{ABX}"'] & & \mathcal{V}(NC\otimes X, LC) \\
& \mathcal{V}(NB\otimes X, LB) \arrow[ur, "\xi_{BCX}"'] &
\end{tikzcd}$$

<u>Proof</u> We use Lemma 1:



$=$ 

$=$ 

$\overset{\text{(Lem 1)}}{=}$ 

. $\square$

Proposition 11 The limit in Set of the diagram

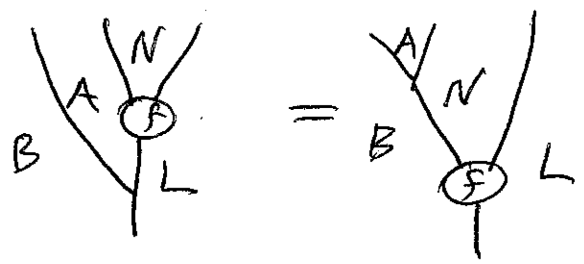
$$\mathcal{V}(NA\otimes X, LA) \xrightarrow{\xi_{ABX}} \mathcal{V}(NB\otimes X, LB), \quad A, B \in \mathcal{A},$$

is $\hom_{\mathcal{A}}(N, L)(X)$.

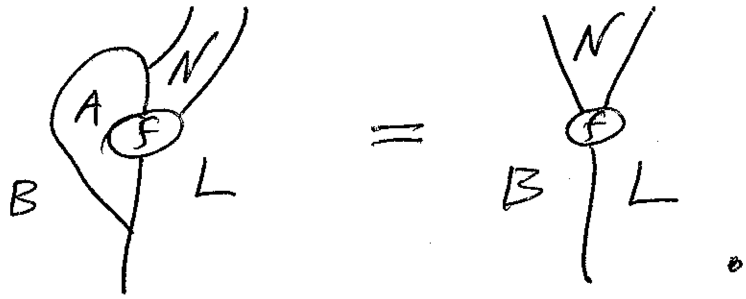
Proof We need to see that a family of morphisms

$$f_A : NA\otimes X \longrightarrow LA, \qquad A \in \mathcal{A},$$

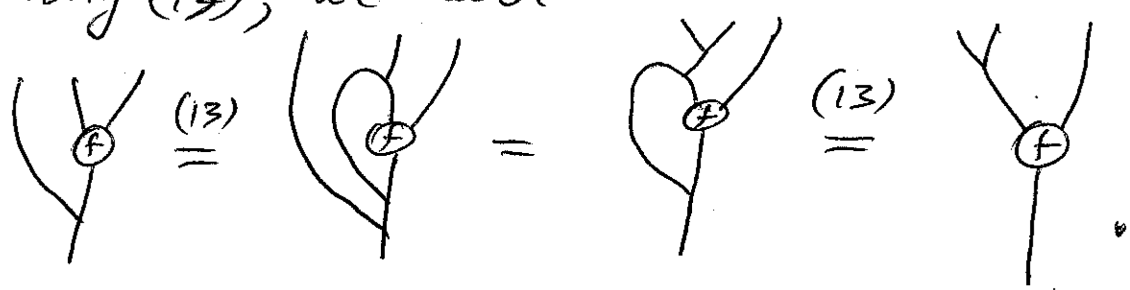
satisfies

(12) 

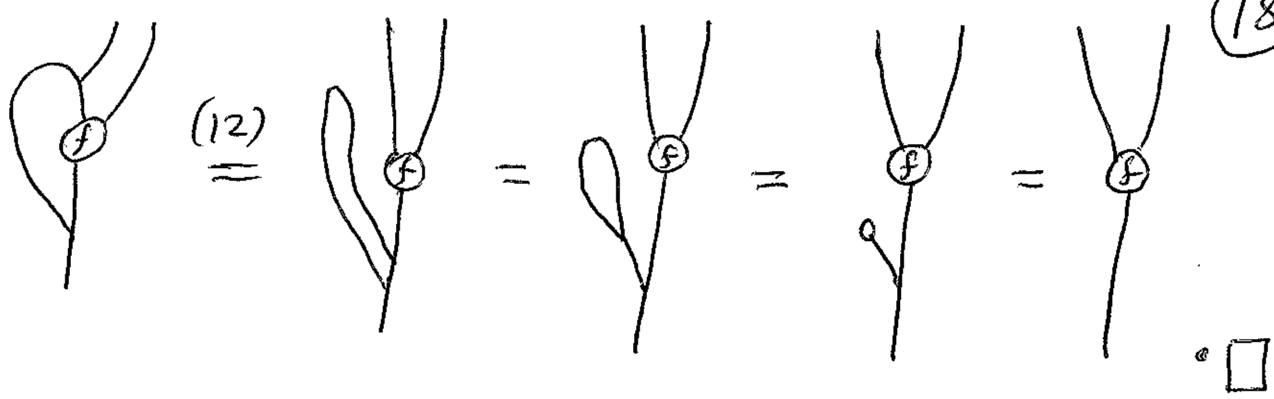
if and only it satisfies

(13) 

Assuming (13), we have



Assuming (12), we have



A right hom of object Y and Z in $\mathcal{V}$ will be denoted by Z^Y:

$$\mathcal{V}(X, Z^Y) \cong \mathcal{V}(Y \otimes X, Z).$$

<u>Corollary 12</u> <u>If idempotents split in $\mathcal{V}$ and $\mathcal{V}$ is right closed then, for a separable $\mathcal{V}$-category $\mathcal{A}$, all left $\mathcal{A}$-modules N and L have a right hom $H_{\mathcal{A}}(N, L)$.</u>

<u>Proof</u> The functions $\Xi_{A,B,X}$ transport to a natural family

$$\mathcal{V}(X, (LA)^{(NA)}) \longrightarrow \mathcal{V}(X, (LB)^{(NB)})$$

which, by Yoneda's Lemma, has the form $\mathcal{V}(1, t_{AB})$ for a family of morphisms

$$(14) \qquad t_{A,B} : (LA)^{(NA)} \longrightarrow (LB)^{(NB)}.$$

By Proposition 10 these form a clique in $Q\mathcal{V}_0 \simeq \mathcal{V}_0$. Hence the diagram (14) has an (absolute) limit $H_{\mathcal{A}}(N, L)$ which, by Proposition 11, represents the functor $\hom_{\mathcal{A}}(N, L)$. $\square$

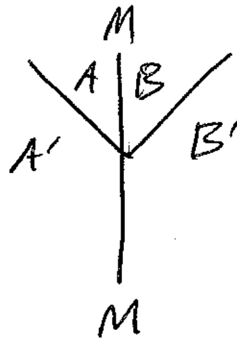
For any monoidal category $\mathcal{V}$, a <u>module</u> $M : \mathcal{A} \nrightarrow \mathcal{B}$ for $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ is a family of objects

$$M(B,A) \in \mathcal{V} \quad \text{for} \quad \in \mathcal{A}, \ B \in \mathcal{B}$$

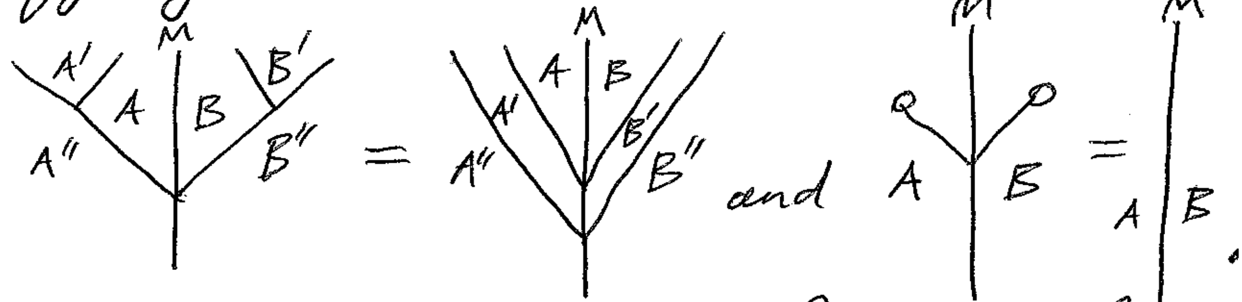
equipped with a family of <u>action</u> morphisms

$$\mu : \mathcal{A}(A,A') \otimes M(B,A) \otimes \mathcal{B}(B',B) \longrightarrow M(B',A')$$

denoted by



satisfying



A <u>morphism</u> $f : M \longrightarrow N : \mathcal{A} \nrightarrow \mathcal{B}$ of modules is a family of morphisms

$$f_{BA} : M(B,A) \longrightarrow N(B,A), \qquad A \in \mathcal{A}, B \in \mathcal{B},$$

preserving the actions. It is easy to see that when we have modules

$$\mathcal{A} \overset{M}{\nrightarrow} \mathcal{B} \overset{N}{\nrightarrow} \mathcal{C},$$

if $M \otimes_{\mathcal{B}} N$ exists (see (5)), actions are induced in the obvious way to obtain a module

$$M \underset{\mathcal{B}}{\otimes} N : \mathcal{A} \nrightarrow \mathcal{C} .$$

<u>Proposition 13</u> <u>There is a bicategory $\mathcal{V}$-Mod_{sep} when idempotents split in $\mathcal{V}$. The objects are separable $\mathcal{V}$-categories. The morphisms are modules. The 2-cells are module morphisms. Composition of modules</u> $M : \mathcal{A} \nrightarrow \mathcal{B}$, $N : \mathcal{B} \to \mathcal{C}$ <u>is defined by</u>

$$N \circ M = M \otimes_{\mathcal{B}} N : \mathcal{A} \nrightarrow \mathcal{C}.$$

<u>The identity module of $\mathcal{A}$ is the family of homs $\mathcal{A}(A', A)$ with composition as action. Associativity constraints are induced by those of $\mathcal{V}$. Unit constraints are induced by actions.</u>

<u>Proof</u> Use Corollary 7. $\square$

<u>Proposition 14</u> <u>In Proposition 13, if $\mathcal{V}$ is also right closed, all right extensions exist in the bicategory</u> $\mathcal{V}$-Mod_{sep}.

<u>Proof</u> Use Corollary 12. $\square$

For ordinary categories, separability is not very interesting however.

<u>Proposition 15</u> <u>For $\mathcal{V} = \underset{\sim}{\mathrm{Set}}$, a $\mathcal{V}$-category $\mathcal{A}$ is separable if and only if it is equivalent to $\underset{\sim}{1}$.</u>

<u>Proof</u> The function $\rho : 1 \to \mathcal{A}(B, A) \times \mathcal{A}(A, B)$ picks out a pair of morphisms

$$a_B^A : B \to A \quad \text{and} \quad b_A^B : A \to B$$

such that $a_B^A \, b_A^B = 1$ and, for all $f : A \to C$,

$$f \, a_B^A = a_B^C \quad \text{and} \quad b_A^B = b_C^B f .$$

It follows that

$$f = a_B^C \, b_A^B$$

for any choice of B. So each $\mathcal{A}(A, C)$ is a singleton. $\square$

What we have looked at so far has required no braiding on $\mathcal{V}$, and no completeness or cocompleteness apart from splitting idempotents. Now we go to the other extreme and assume $\mathcal{V}$ is symmetric closed monoidal, complete and cocomplete. Then we can take advantage of the enriched category theory of [Kelly's book]. In particular, we have the $\mathcal{V}$-functor category $[\mathcal{A}^{op}\otimes\mathcal{A}, \mathcal{V}]$ for each small $\mathcal{V}$-category $\mathcal{A}$ and we know that it is the free cocompletion of $\mathcal{A}^{op}\otimes\mathcal{A}$.

An object P of a $\mathcal{V}$-category $\mathcal{X}$ is called <u>small projective</u> when $\mathcal{X}(P,-) : \mathcal{X} \to \mathcal{V}$ preserves small colimits.

~~Bo~~ceux-Vitale [Azumaya categories] define a $\mathcal{V}$-category $\mathcal{A}$ to be separable when the hom-$\mathcal{V}$-functor $\mathcal{A}(-,-)$ is a small projective object of $[\mathcal{A}^{op}\otimes\mathcal{A}, \mathcal{V}]$. We shall say $\mathcal{A}$ is BV-<u>separable</u> in this case to distinguish it from our earlier notion.

Proposition 16 For $\mathcal{V}$ the monoidal category of abelian groups, a one-object $\mathcal{V}$-category $\mathcal{A}$ is separable if and only if it is BV-separable.

Proof Proposition 1.1 of [BV: Az.cat.]. $\square$

Proposition 17 The $\mathcal{V}$-functor

$$[\mathcal{A}^{op}\otimes\mathcal{A}, \mathcal{V}](\mathcal{A}(-,-), -) : [\mathcal{A}^{op}\otimes\mathcal{A}, \mathcal{V}] \longrightarrow \mathcal{V}$$

is isomorphic to

$$\int_A \; : [\mathcal{A}^{op}\otimes\mathcal{A}, \mathcal{V}] \longrightarrow \mathcal{V}$$

which takes each $M : \mathcal{A}^{op}\otimes\mathcal{A} \to \mathcal{V}$ to its end $\int_A M(A,A)$.

Proof We have the calculation:

$$[\mathcal{A}^{op}\otimes\mathcal{A}, \mathcal{V}](\mathcal{A}(-,-), M) = \int_{A,B} [\mathcal{A}(A,B), M(A,B)]$$

$$\cong \int_A M(A,A)$$

which uses the "end form" of the Yoneda Lemma. $\square$

<u>Corollary 18</u> <u>A</u> $\mathcal{V}$-category $\mathcal{A}$ is BV-separable if and only if $\int_A : [\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{V}] \to \mathcal{V}$ preserves small colimits.

The composite of $\int_A$ with the Yoneda embedding is the $\mathcal{V}$-functor

$$W_{\mathcal{A}} : \mathcal{A}^{op} \otimes \mathcal{A} \longrightarrow \mathcal{V}$$

defined by

$$W_{\mathcal{A}}(C, D) = \int_A \mathcal{A}(A, D) \otimes \mathcal{A}(C, A).$$

By the general theory of cocompletion, it follows from Corollary 18, if $\mathcal{A}$ is BV-separable, the end $\int_A$ is the weighted colimit

$$\mathrm{colim}(W_{\mathcal{A}}, -)$$

and has right adjoint

$$\begin{aligned} \mathcal{V} &\longrightarrow [\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{V}] \\ X &\longmapsto \left((C, D) \longmapsto \int_A [X, \mathcal{A}(A, D) \otimes \mathcal{A}(C, A)] \right). \end{aligned}$$

Recall that a module $M : \mathcal{I} \nrightarrow \mathcal{B}$ has a right adjoint in the bicategory $\mathcal{V}$-Mod if and only if $M \in [\mathcal{B}^{op}, \mathcal{V}]$ is small projective. Similarly, a module $N : \mathcal{C} \nrightarrow \mathcal{I}$ has a left adjoint if and only if $N \in [\mathcal{C}, \mathcal{V}]$ is small projective (since it is the same as $N^{op} : \mathcal{I} \nrightarrow \mathcal{C}^{op}$ having a right adjoint).

Recall also that we have modules

$$E_{\mathcal{A}} : \mathcal{A}^{op} \otimes \mathcal{A} \nrightarrow \mathcal{I} \quad \text{and} \quad N_{\mathcal{A}} : \mathcal{I} \nrightarrow \mathcal{A} \otimes \mathcal{A}^{op},$$

for any $\mathcal{V}$-category $\mathcal{A}$, defined by

$$E_{\mathcal{A}}(*, A, B) = \mathcal{A}(A, B) = N_{\mathcal{A}}(B, A, *).$$

(In fact these are counit and unit for a biduality $\mathcal{A}^{op} \dashv_b \mathcal{A}$ in the monoidal bicategory $\mathcal{V}$-Mod.)

<u>Proposition 19</u> <u>The following conditions on</u> <u>a</u> <u>$\mathcal{V}$-category</u> $\mathcal{A}$ <u>are equivalent</u>:

(i) $\mathcal{A}$ <u>is BV-separable</u>;

(ii) $E_{\mathcal{A}} : \mathcal{A}^{op} \otimes \mathcal{A} \nrightarrow \mathcal{I}$ <u>has a left adjoint module</u>; <u>and</u>

(iii) $N_{\mathcal{A}} : \mathcal{I} \nrightarrow \mathcal{A} \otimes \mathcal{A}^{op}$ <u>has a right adjoint module</u>.

Proof Under the equivalences

$$[\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{V}] \simeq \mathcal{V}\text{-Mod}(\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{I}) \simeq \mathcal{V}\text{-Mod}(\mathcal{I}, \mathcal{A} \otimes \mathcal{A}^{op}),$$

the hom $\mathcal{A}(-,-)$ is taken to $E_{\mathcal{A}}$ which is then taken to $N_{\mathcal{A}}$. If one is small projective, all are. □

Proposition 20 If $\mathcal{A}$ is BV-separable then end

$$\int_A : [\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{X}] \longrightarrow \mathcal{X}$$

and coend

$$\int^A : [\mathcal{A}^{op} \otimes \mathcal{A}, \mathcal{X}] \longrightarrow \mathcal{X}$$

are absolute (weighted) limits and colimits, respectively. They exist for any Cauchy complete $\mathcal{V}$-category $\mathcal{X}$.

Proof We have

$$\lim(E_A, S) = \int_{A,B} [\mathcal{A}(A,B), S(A,B)]$$

$$\cong \int_A S(A,A) \quad \text{and}$$

$$\mathrm{colim}(N_{\mathcal{A}}, S) = \int^{A,B} \mathcal{A}(B,A) \otimes S(A,B)$$
$$\cong \int^{A} S(A,A),$$

showing that end is limit weighted by $E_{\mathcal{A}}$ while coend is colimit weighted by $N_{\mathcal{A}}$. By [St: Aciec], $\lim(E_{\mathcal{A}}, -)$ is absolute since $E_{\mathcal{A}}$ has a left adjoint and $\mathrm{colim}(N_{\mathcal{A}}, -)$ is absolute since $N_{\mathcal{A}}$ has a right adjoint (i.e. is a Cauchy module). $\square$

Given modules as in the triangle

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "M" slash] \arrow[dr, "L"' slash] & & \mathcal{B} \arrow[dl, "N" slash] \\ & \mathcal{C} & \end{tikzcd},$$

we have

$$\underset{\sim}{H}_{\mathcal{A}}(M, L) = \int_A [M(-,A), L(\sim,A)]$$

$$N \circ M = M \otimes_{\mathcal{B}} N = \int^{B} M(B,-) \otimes N(\sim,B)$$

$$\underset{\sim}{H}^{\mathcal{C}}(N, L) = \int_C [N(C,-), L(C,\sim)]$$

involving absolute ends and coends when $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are $B\mathcal{V}$-separable, respectively.