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★**Formal category theory: adjointness for 2-categories.**
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This volume contains seven sections of the first part of a proposed four-part work on categorical descriptions of aspects of the structure of category theory, i.e., on "formal category theory". This means that the author studies the category Cat of small categories "from outside" by considering it as an object of some category whose objects are instances of some enriched notion of category (usually 2-categories; but sometimes bicategories, double categories, etc.). Of course, these categories of enriched categories can in turn be studied from inside or outside. The author attempts to climb no higher up the infinite tree than this; already at this level, bicategories, 3-categories, and triple categories must be considered.

Properties of Cat are reviewed in Section 1, for two reasons. The first is to indicate the properties that we are trying to "understand" from outside. The second is to provide a starting point for generalizations of these properties to the appropriate enriched categories. Cartesian closedness, discrete categories, limits and colimits, the first four ordinals, Yoneda's lemma, adjointness, fibrations, the adjoint functor theorem, and Kan extensions are all discussed inside Cat.

Section 2 begins with a discussion of 2-categories as categories with homs enriched in Cat. Besides the Cat-natural transformations between 2-functors there are the more general "quasinatural transformations", which have been called 2-natural transformations elsewhere by the author [*Category theory, homology theory and their applications, III* (Battelle Inst. Conf., Seattle, Wash., 1968), pp. 242–312, Lecture Notes in Math., Vol. 99, Springer, Berlin, 1969; MR0249483], by the present name by M. Bunge [Reports of the Midwest Category Seminar, V (Zürich, 1970), pp. 70–122; Lecture Notes in Math., Vol. 195, Springer, Berlin, 1971; MR0292904], "catadeses" by D. Bourn [Cahiers Topologie Géom. Différentielle **14** (1973), 371–415; MR0354808], and "lax natural transformations" by G. M. Kelly and the reviewer [Category Seminar (Proc. Sem., Sydney, 1972/73), pp. 75–103; Lecture Notes in Math., Vol. 420, Springer, Berlin, 1974; MR0357542]. Given 2-categories A and B, a subcategory ${B_0}'$ of the underlying category B_0 of B, and a sub-2-category A' of A containing all the arrows of A, the author denotes by $\mathrm{Fun}(B,{B_0}';A,A')$ the 2-category whose objects are 2-functors from B to A, whose arrows are those quasi-natural transformations σ such that σ_f is a 2-cell in A' for all arrows f in ${B_0}'$, and whose 2-cells are modifications between such. When ${B_0}'$ is empty, this 2-category is denoted by $\mathrm{Fun}(B,A)$; when ${B_0}'=B_0$ and A' has only the identity 2-cells, this 2-category becomes the internal hom A^B for the Cartesian closed structure on the category 2-Cat_0 of small 2-categories. Another important case is the one in which A' consists of the 2-cells in A that are isomorphisms. Given 2-functors $F_1\colon A_1\to B$, $F_2\colon A_2\to B$, the 2-comma category $[F_1,F_2]$ is described; it can also be described as the 2-category for which there is a universal quasinatural transformation

$$\begin{array}{ccc} [F_1,\,F_2] & \xrightarrow{P_2} & A_2 \\ {\scriptstyle P_1}\downarrow & \longrightarrow & \downarrow{\scriptstyle F_2} \\ A_1 & \xrightarrow[F_1]{} & B \end{array}\quad .$$

The latter description is given in section 5, where the properties of the construction are discussed at length. At the end of Section 2, double categories, triple categories, and 3-categories are briefly dealt with.

Bicategories in the sense of J. Benabou [Reports Midwest Category Seminar, pp. 1–

77, Lecture Notes in Math., Vol. 47, Springer, Berlin, 1967; MR0220789] are recalled in Section 3, and some examples are given. The construction of the bicategory of all bimodules from the monoidal category (=bicategory with one object) of abelian groups is generalized to a construction of a bicategory Bim X from any bicategory X that has coequalizers in its hom-categories that are preserved by the composition in X. Categorical fibrations are described in terms of Bim(Spans(Cat)).

The 2-categories $\mathrm{Fun}(A, C)$ are shown in Section 4 to provide the internal-hom for a biclosed-monoidal-category structure (denoted by 2-$\mathrm{Cat}_\otimes$) on 2-Cat_0 that is different from the cartesian-closed structure. The relevant tensor product $A \otimes B$ of 2-categories is described in two different ways; it is unsymmetric, but the internal-hom on the other side is just

$$\mathrm{Fun}(A^{\mathrm{co}}, C^{\mathrm{co}})^{\mathrm{co}} (\cong \mathrm{Fun}(A^{\mathrm{op}}, C^{\mathrm{op}})^{\mathrm{op}}),$$

where A^{co} is the 2-category (also called ${}^{\mathrm{op}}A$) obtained from A by reversing 2-cells. "Quasi-functors of two variables" $A,\ B \to C$ are described as a means of handling 2-functors $A \otimes B \to C$ without actually having to examine the rather complicated 2-category $A \otimes B$.

The author drops down a level in Section 6 to consider just one 2-category (or bicategory) A and to discuss adjunction and Kan extension for arrows within A. This leads to definitions of co-completeness (relative to a given sub-2-category of A) for an object of A, and of finality for an arrow in A.

In the final section, the author returns to the higher level to consider notions of adjunction for 2-functors. Many possibilities are discussed although the important notion seems to be "quasi-adjunction" or some strengthening thereof. The definition of a quasi-adjunction is just the same as the "counit-unit definition" of an adjunction in a 2-category, where instead of objects, arrows and 2-cells we have 2-categories, 2-functors and quasinatural transformations (however, these do not form a 2-category). A quasiadjunction for 2-functors $F\colon B \to A$ and $U\colon A \to B$ leads to functors $S\colon [F, B] \to [A, U]$ and $T\colon [A, U] \to [F, B]$ and natural transformations $\phi\colon ST \to 1$, $\psi\colon 1 \to TS$, which all project down to identities under the projections from the 2-comma categories into $A \times B$. The quasi-adjunction is said to be strict when ϕ and ψ are counit and unit for an adjunction for S and T (this condition can be translated in terms of the original data). Universal properties of quasiadjunctions are discussed; similar results have been obtained by M. Bunge [Trans. Amer. Math. Soc. **197** (1974), 355–390; MR0344305]. With the notion of quasiadjunction established, corresponding notions of quasilimit (=quasiadjoint to the diagonal 2-functor $A \to \mathrm{Fun}(B, {B_0}'; A, A'))$ and quasi-Kan extension are derived for 2-functors. A multitude of examples is given, especially in relation to the existence of these constructions in Cat. The categorical comprehension scheme of the author [op. cit.] is an example of a strict quasiadjunction involving 2-categories $[\mathrm{Cat}^{\mathrm{co}}, X]$ and $\mathrm{Fun}(X, \mathrm{Cat})^{\mathrm{co}}$. The work concludes with a discussion of some 2-$\mathrm{Cat}_\otimes$-categories; in particular, the description of a 2-$\mathrm{Cat}_\otimes$-functor from 2-$\mathrm{Cat}_\otimes$ to Spans(2-Cat) is seen as a "global quasi-Yoneda lemma".

Some of the results of this volume have been announced elsewhere by the author [Bull. Amer. Math. Soc. **80** (1974), 142–147; MR0340369]. There is a view of $A \otimes B$ at a higher level given by Bourn [C. R. Acad. Sci. Paris Sér. A-B **277** (1973), A1025-A1028].

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