SEPARABLE ENRICHED CATEGORIES AND ABSOLUTE ENDS

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ABSTRACT. Our goal is to examine two notions of separability and the relationship between them. Both notions generalize to enriched categories the classical notion for algebras over a field. We look at modules over separable enriched categories and prove that tensor products and homs of such modules are obtained using absolute colimits and limits.

1. INTRODUCTION

More will appear here soon. Highlight [1].

2. SEPARABILITY TRANSFORMATIONS

Classically [2], an algebra A over a field is *separable* when it is equipped with a *separability element*; that is, an element $e = \sum_{i=1}^{n} a_i \otimes b_i \in A \otimes A$ such that

$$\sum_{i=1}^{n} a_i b_i = 1 \qquad \text{and} \tag{2.1}$$

$$\sum_{i=1}^{n} (ra_i) \otimes b_i = \sum_{i=1}^{n} a_i \otimes (b_i r) \tag{2.2}$$

for all $a \in A$. It follows that A is finitely generated and projective as a left A-, right A-bimodule.

An obvious generalization of this notion is as follows. Let $\mathscr{V}$ denote a monoidal category. A monoid A in $\mathscr{V}$ is called *separable* when it is equipped with a morphism

$$\rho : I \longrightarrow A \otimes A \tag{2.3}$$

such that

$$\mu\rho = \eta \qquad \text{and} \tag{2.4}$$

$$(\mu \otimes 1)(1 \otimes \rho) = (1 \otimes \mu)(\rho \otimes 1)\,. \tag{2.5}$$

Here I is the tensor unit object of $\mathscr{V}$ while $\eta : I \to A$ and $\mu : A \otimes A \to A$ are the unit and multiplication for the monoid A. Using the string notation of [3], we denote η, μ and ρ, respectively, as in Figure 1.

The axioms for separability then become as in Figure 2.

We wish to generalize this even further. A monoid A in $\mathscr{V}$ is a one object category enriched in $\mathscr{V}$. We wish to generalize separability to enriched categories with several objects. A monoidal category $\mathscr{V}$ is a one object bicategory. We shall allow our base for enrichment to be a bicategory $\mathscr{W}$ and

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FIGURE 1.

FIGURE 2.

not merely a monoidal category $\mathscr{V}$. We shall recall the definition of $\mathscr{W}$-category; that is, category enriched in $\mathscr{W}$. Suitable references are [4] and [5] .

Recall that a *chaotic category* is one whose homsets are singletons. There is a unique chaotic category S_{ch} whose objects are the elements of a given set S. A *category enriched in a bicategory* $\mathscr{W}$ is a lax functor (= morphism of bicategories) $|-| : (\mathrm{ob}\mathscr{A})_{\mathrm{ch}} \to \mathscr{W}$ where $\mathrm{ob}\mathscr{A}$ is a set. The elements of $\mathrm{ob}\mathscr{A}$ are called *objects* of $\mathscr{A}$. For each object A of $\mathscr{A}$, the object $|A|$ of $\mathscr{W}$ is called the *extent* of A. The value of $|-|$ at the single morphism $A \to B$ in $(\mathrm{ob}\mathscr{A})_{\mathrm{ch}}$ is denoted $\mathscr{A}(A,B) : |A| \to |B|$ in $\mathscr{W}$ and called a *hom morphism* of $\mathscr{A}$. The composition constraints for the lax functor $\mathscr{A}$ are 2-cells

$$\mu = \mu^B_{AC} : \mathscr{A}(B,C)\mathscr{A}(A,B) \Longrightarrow \mathscr{A}(A,C) \tag{2.6}$$

in $\mathscr{W}$ called *composition* for $\mathscr{A}$. The identity constraints for the lax functor $\mathscr{A}$ are 2-cells

$$\eta = \eta^A : 1_{|A|} \Longrightarrow \mathscr{A}(A,A) \tag{2.7}$$

in $\mathscr{W}$ called *identities* for $\mathscr{A}$.

Each object U of $\mathscr{W}$ determines a $\mathscr{W}$-category $\mathscr{I}_U$ with one object that we may as well denote by U, and where $|U| = U$ and $\mathscr{I}_U(U,U) = 1_U : U \to U$.

Suppose $\mathscr{A}$ and $\mathscr{X}$ are $\mathscr{W}$-categories. A *$\mathscr{W}$-functor* $F : \mathscr{A} \to \mathscr{X}$ consists of a function $F : \mathrm{ob}\mathscr{A} \to \mathrm{ob}\mathscr{X}$ which preserves extent, and 2-cells $\chi_{F;A,B} : \mathscr{A}(A,B) \Rightarrow \mathscr{X}(FA,FB)$ which preserve composition and identities.

We identify the object A of the $\mathscr{W}$-category $\mathscr{A}$ with the $\mathscr{W}$-functor $\mathscr{I}_{|A|} \to \mathscr{A}$ whose value at the object $|A|$ is A.

Definition 2.1. Let $\mathscr{A}$ be a $\mathscr{W}$-category. A *separability transformation at* $B \in \mathscr{A}$ is a family of 2-cells

$$\rho = \rho_A : 1_{\mathscr{A}A} \Longrightarrow \mathscr{A}(B,A)\mathscr{A}(A,B), \qquad A \in \mathscr{A}, \tag{2.8}$$

in $\mathscr{W}$ such that

$$\mu^B_{AA}\rho_A = \eta^A \qquad \text{and} \tag{2.9}$$

$$\begin{array}{ccc} \mathscr{A}(A,C) & \xrightarrow{\mathscr{A}(A,C)\rho} & \mathscr{A}(A,C)\mathscr{A}(B,A)\mathscr{A}(A,B) \\ {\scriptstyle \rho\mathscr{A}(A,C)}\downarrow & = & \downarrow{\scriptstyle \mu\mathscr{A}(A,B)} \\ \mathscr{A}(B,C)\mathscr{A}(C,B)\mathscr{A}(A,C) & \xrightarrow[\mathscr{A}(B,C)\mu]{} & \mathscr{A}(B,C)\mathscr{A}(A,B) \end{array} . \tag{2.10}$$

Notice that condition (2.10) expresses the dinaturality of ρ in the variable A. A relationship between separability transformations and structure considered by the third author [6] is apparent from the following result.

Proposition 2.2. *For each separability transformation ρ at B, the diagonal of (2.10) defines a family of 2-cells*

$$\delta = \delta_{AC} : \mathscr{A}(A,C) \Longrightarrow \mathscr{A}(B,C)\mathscr{A}(A,B), \qquad A, C \in \mathscr{A}, \tag{2.11}$$

in $\mathscr{W}$ such that

$$\mu^B_{AC}\delta_{AC} = 1_{\mathscr{A}(A,C)} \qquad \text{and} \tag{2.12}$$

$$\left(\mu^C_{BD}\mathscr{A}(A,B)\right)\left(\mathscr{A}(C,D)\delta_{AC}\right) = \delta_{AD}\mu^C_{AD} = \left(\mathscr{A}(B,D)\mu^C_{AB}\right)\left(\delta_{CD}\mathscr{A}(A,C)\right). \tag{2.13}$$

The equation

$$\rho_A = \delta_{AA}\eta^A \tag{2.14}$$

establishes a bijection between separability transformations and "cocomposition" families (2.11 satisfying (2.12 and (2.13).

Proof. The bijective correspondence between families (2.8) and families (2.11) according to (2.14) follows by the Yoneda Lemma. Also by Yoneda, to check (2.12) it suffices to check it at the identity, which amounts to (2.9). The equality of the left and right terms of (2.13) amounts to naturality of δ in both A and C, and this is well known [7] to correspond to dinaturality (2.10 in A under Yoneda. Equality of the first and middle terms of (2.13) is an easy consequence of the other conditions, which provides a good exercise in the string notation about to be introduced. □

We can adapt the string notation for working in monoidal categories to the current situation. We respectively depict the morphisms (2.7), (2.6), (2.11) and (2.8) as in Figure 3.

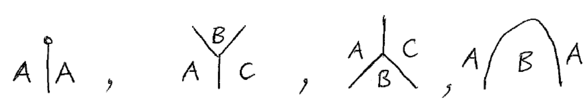

FIGURE 3.

Lemma 2.3. *For a separability transformation ρ at B, the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{A}(B,A)\mathscr{A}(A,B) & \xrightarrow{\mathscr{A}(B,A)\rho\mathscr{A}(A,B)} & \mathscr{A}(B,A)\mathscr{A}(C,B)\mathscr{A}(B,C)\mathscr{A}(A,B) \\ {\scriptstyle\rho}\uparrow & = & \downarrow{\scriptstyle\mu\mu} \\ 1_{\mathscr{A}A} & \xrightarrow[\rho]{} & \mathscr{A}(C,A)\mathscr{A}(A,C) \end{array} \tag{2.15}$$

Proof. A string proof is provided by Figure 4. □

3. MODULES

For $\mathscr{W}$-category $\mathscr{A}$ and $\mathscr{X}$, a *module* $M : \mathscr{X} \to \mathscr{A}$ consists of a family of morphisms

$$M(A,X) : |A| \longrightarrow |X|, \qquad A \in \mathscr{A}, X \in \mathscr{X}, \tag{3.1}$$

in $\mathscr{W}$ equipped with a family of 2-cells

$$\mu = \mu^{BX}_{AY} : \mathscr{X}(X,Y)M(B,X)\mathscr{A}(A,B) \Longrightarrow M(A,Y) \tag{3.2}$$

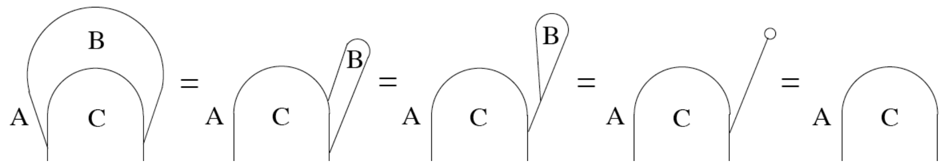

Figure 4.

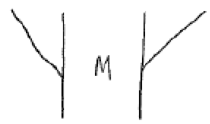

Figure 5.

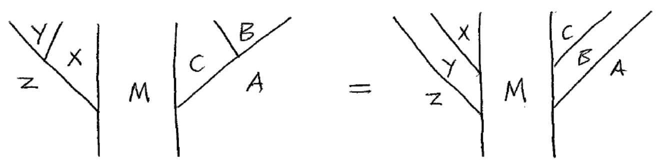

Figure 6.

satisfying the action conditions. In terms of strings, this action is depicted as in Figure 5 so that the action conditions appear as in Figure 6. We also put

$$\mu^{X}_{AY} = \mu^{AX}_{AY}(11\eta^{A}) : \mathscr{X}(X,Y)M(A,X) \Longrightarrow M(A,Y) \tag{3.3}$$

for the left action and

$$\mu^{B}_{AX} = \mu^{BX}_{AX}(\eta^{X}11) : M(B,X)\mathscr{A}(A,B) \Longrightarrow M\,(A,X) \tag{3.4}$$

for the right action.

Suppose $M : \mathscr{X} \to \mathscr{A}$ and $M' : \mathscr{X} \to \mathscr{A}$ are modules. A *module morphism* $\phi : M \Rightarrow M'$is a family of 2-cells

$$\phi_{AX} : M(A,X) \Longrightarrow M'(A,X), \qquad A \in \mathscr{A}, X \in \mathscr{X}, \tag{3.5}$$

in $\mathscr{W}$ commuting with the actions (3.2); equivalently, it commutes with the left (3.3 and right (3.4 actions. We obtain a category $\mathscr{W}\text{-}Mod(\mathscr{X},\mathscr{A})$ of modules from $\mathscr{X}$ to $\mathscr{A}$.

Suppose we have modules $M : \mathscr{X} \to \mathscr{A}$, $N : \mathscr{A} \to \mathscr{K}$ and $L : \mathscr{X} \to \mathscr{K}$. A *module bimorphism* $\psi : (M,N) \Rightarrow L$ is a family of 2-cells

$$\psi^{A}_{KX} : M(A,X)N(K,A) \Longrightarrow L(K,X), \qquad K \in \mathscr{K}, A \in \mathscr{A}, X \in \mathscr{X}, \tag{3.6}$$

such that the following three conditions hold:

$$\mu^{X}_{KY}(1\psi^{A}_{KX}) = \psi^{A}_{KY}(\mu^{X}_{AY}1), \tag{3.7}$$

$$\psi^{A}_{KX}(\mu^{B}_{AX}1) = \psi^{B}_{KX}(1\mu^{A}_{KB}), \quad \text{and} \tag{3.8}$$

$$\mu^{K}_{HX}(\psi^{A}_{KX}1) = \psi^{A}_{HX}(1\mu^{K}_{HA}). \tag{3.9}$$

The fifth author has also used the term "form" to cover the cases of module morphisms, module bimorphisms, and higher. Write $\langle M, N; L\rangle$ for the set of module bimorphisms $(M, N) \Rightarrow L$. This gives the assignment on objects of an obvious functor

$$\langle -, -; -\rangle : \mathscr{W}\text{-}Mod(\mathscr{X}, \mathscr{A})^{\mathrm{op}} \times \mathscr{W}\text{-}Mod(\mathscr{A}, \mathscr{K})^{\mathrm{op}} \times \mathscr{W}\text{-}Mod(\mathscr{X}, \mathscr{K}) \longrightarrow \mathrm{Set}. \tag{3.10}$$

Definition 3.1. A *composite* $N \circ M : \mathscr{X} \to \mathscr{K}$ of modules $M : \mathscr{X} \to \mathscr{A}$, $N : \mathscr{A} \to \mathscr{K}$ is a representing object for the functor $\langle M, N; -\rangle$. A *left hom* ${}^{N}L : \mathscr{X} \to \mathscr{A}$ of modules $N : \mathscr{A} \to \mathscr{K}$, $L : \mathscr{X} \to \mathscr{K}$ is a representing object for the functor $\langle -, N; L\rangle$. A *right hom* $L^{M} : \mathscr{A} \to \mathscr{K}$ of modules $M : \mathscr{X} \to \mathscr{A}$, $L : \mathscr{X} \to \mathscr{K}$ is a representing object for the functor $\langle M, -; L\rangle$.

Each $\mathscr{W}$-functor $F : \mathscr{X} \to \mathscr{A}$ determines $\mathscr{W}$-modules $F_* : \mathscr{X} \to \mathscr{A}$ and $F^* : \mathscr{A} \to \mathscr{X}$ defined by

$$F_*(A, X) = \mathscr{A}(A, FX) \qquad \text{and} \qquad F^*(X, A) = \mathscr{A}(FX, A), \tag{3.11}$$

with actions

$$\mathscr{X}(X, Y)\mathscr{A}(B, FX)\mathscr{A}(A, B) \overset{\chi_F 11}{\Longrightarrow} \mathscr{A}(FX, FY)\mathscr{A}(B, FX)\mathscr{A}(A, B) \overset{\mu_3}{\Longrightarrow} \mathscr{A}(A, FY), \tag{3.12}$$

$$\mathscr{A}(A, B)\mathscr{A}(FY, A)\mathscr{X}(X, Y) \overset{11\chi_F}{\Longrightarrow} \mathscr{A}(A, B)\mathscr{A}(FY, A)\mathscr{A}(FX, FY) \overset{\mu_3}{\Longrightarrow} \mathscr{A}(FX, B). \tag{3.13}$$

The identity functor $1_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ leads to an "identity module" $1_{\mathscr{A}*} : \mathscr{A} \to \mathscr{A}$ which is denoted by $\mathscr{A}$. There is a canonical module bimorphism $\varepsilon_F : (F^*, F_*) \Rightarrow \mathscr{A}$ defined to be composition:

$$\mu : \mathscr{A}(FX, B)\mathscr{A}(A, FX) \to \mathscr{A}(A, B). \tag{3.14}$$

Yoneda's lemma takes the following form.

Proposition 3.2. *For a $\mathscr{W}$-functor $F : \mathscr{X} \to \mathscr{A}$ and modules $M : \mathscr{K} \to \mathscr{A}$, $N : \mathscr{A} \to \mathscr{K}$,*

1. *the composite $N \circ F_* : \mathscr{X} \to \mathscr{K}$ exists and has $(N \circ F_*)(K, X) = N(K, FX)$,*
2. *the composite $F^* \circ M : \mathscr{K} \to \mathscr{X}$ exists and has $(F^* \circ M)(X, K) = M(FX, K)$,*
3. *the left hom ${}^{F^*}N : \mathscr{X} \to \mathscr{K}$ exists and has $({}^{F^*}N)(K, X) = N(K, FX)$, and*
4. *the right hom $M^{F_*} : \mathscr{K} \to \mathscr{X}$ exists and has $(M^{F_*})(X, K) = M(FX, K)$.*

There are other cases where composition and homs of modules are formed easily.

Proposition 3.3. *For modules $M : \mathscr{X} \to \mathscr{A}$, $N : \mathscr{A} \to \mathscr{K}$ and $L : \mathscr{X} \to \mathscr{K}$,*

1. *if $\mathscr{A} = \mathscr{I}_U$ then $N \circ M : \mathscr{X} \to \mathscr{K}$ exists and has $(N \circ M)(K, X) = N(K, U)M(U, X)$,*
2. *if $\mathscr{K} = \mathscr{I}_U$ and $\mathscr{W}$ admits right liftings then ${}^{N}L : \mathscr{X} \to \mathscr{A}$ exists and ${}^{N}L(A, X) = {}^{N(U,A)}L(U, X)$, and*
3. *if $\mathscr{X} = \mathscr{I}_U$ and $\mathscr{W}$ admits right extensions then $L^{M} : \mathscr{A} \to \mathscr{K}$ exists and $L^{M}(K, A) = L(K, U)^{M(A,U)}$.*

Suppose we have modules $M : \mathscr{X} \to \mathscr{A}$ and $N : \mathscr{A} \to \mathscr{K}$ where $\mathscr{A}$ is separable. Define the family of 2-cells

$$\varepsilon^{AB}_{XK} : M(A, X)N(K, A) \Longrightarrow M(B, X)N(K, B) \tag{3.15}$$

to be the composite

(3.16)

$$M(A, X)N(K, A) \overset{M(A,X)\rho N(K,A)}{\Longrightarrow} M(A, X)\ \mathscr{A}(B, A)\ \mathscr{A}(A, B)\ N(K, A) \overset{\mu\mu}{\Longrightarrow} M(B, X)N(K, B),$$

which is depicted as in Figure 7.

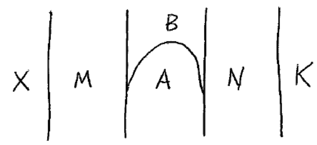

FIGURE 7.

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