Theory and Applications of Categories, Vol. 16, No. 28, 2006, pp. 771–784.

PULLBACK AND FINITE COPRODUCT PRESERVING FUNCTORS BETWEEN CATEGORIES OF PERMUTATION REPRESENTATIONS

ELANGO PANCHADCHARAM AND ROSS STREET

ABSTRACT. Motivated by applications to Mackey functors, Serge Bouc [Bo] characterized pullback and finite coproduct preserving functors between categories of permutation representations of finite groups. Initially surprising to a category theorist, this result does have a categorical explanation which we provide.

1. Introduction

For a finite group G, we write $G\text{-}\mathbf{set}_{\mathit{fin}}$ for the category of finite (left) G-sets (that is, of permutation representations of G) and equivariant functions. We write $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ for the category whose morphisms are isomorphism classes of spans between finite G-sets. Coproducts in $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ are those of $G\text{-}\mathbf{set}_{\mathit{fin}}$ and composition in $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ involves pullbacks in $G\text{-}\mathbf{set}_{\mathit{fin}}$.

According to Harald Lindner [Li], a Mackey functor M on a finite group H is a coproduct preserving functor $M : \mathbf{Spn}(H\text{-}\mathbf{set}_{\mathit{fin}}) \longrightarrow \mathbf{Mod}_k$. A functor $F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$ which preserves pullbacks and finite coproducts will induce a functor

$$\mathbf{Spn}(F) : \mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}}) \longrightarrow \mathbf{Spn}(H\text{-}\mathbf{set}_{\mathit{fin}})$$

preserving finite coproducts. By composition with $\mathbf{Spn}(F)$, each Mackey functor M on H will produce a Mackey functor $M \circ \mathbf{Spn}(F)$ on G.

This observation led Bouc [Bo] to a systematic study of pullback and finite coproduct preserving functors $F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$. He characterized them in terms of $G^{\mathrm{op}} \times H$-sets A (where G^{op} is G with opposite multiplication). This perplexed us initially, as the category $(G^{\mathrm{op}} \times H)\text{-}\mathbf{set}$ of such A is equivalent to the category of finite colimit preserving functors $L : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$; these L generally do not preserve pullbacks, while the F generally do not preserve coequalizers. Of course, Bouc's construction of L from a left H-, right G-set A is quite different from the standard module theory construction of F from A. We shall explain the two constructions.

The authors are grateful for the support of the Australian Research Council Discovery Grant DP0346047, and the first author for the support of an Australian International Postgraduate Research Scholarship, and an International Macquarie University Research Scholarship.

Received by the editors 2006-09-15 and, in revised form, 2006-11-13.

Transmitted by RJ Wood. Published on 2006-11-15.

2000 Mathematics Subject Classification: 58A03, 18A30.

Key words and phrases: topos, permutation representations, limits and colimits, adjunction, Mackey functors.

We put $(g,h)a = hag$ for $g \in G, h \in H$ and a in the $(G^{\mathrm{op}} \times H)$-set A, so that A becomes a left H-set and a right G-set. For each left G-set X, define the left H-set $A \otimes_G X$ to be the quotient of the set $A \times X = \{(a,x) \mid a \in A, x \in X\}$ by the equivalence relation generated by

$$(ag, x) \sim (a, gx), \quad a \in A, x \in X, g \in G.$$

Write $[a,x]$ for the equivalence class of (a,x) and define $h[a,x] = [ha,x]$. For A finite, this defines our functor $L = A \otimes_G - : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$ on objects; it is defined on morphisms $f : X \longrightarrow X'$ by $L(f)[a,x] = [a, f(x)]$. Certainly L preserves all colimits that exist in $G\text{-}\mathbf{set}_{\mathit{fin}}$ since it has a right adjoint $R : H\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow G\text{-}\mathbf{set}_{\mathit{fin}}$ defined on the left H-set Y by $R(Y) = H\text{-}\mathbf{set}_{\mathit{fin}}(A, Y)$ with action $(g, \theta) \longmapsto g\theta$ where $(g\theta)(a) = \theta(ag)$. All this is classical "module" theory.

Now we turn to Bouc's construction. Again let A be a $(G^{\mathrm{op}} \times H)$-set. Rather than a mere G-set X, we define a functor on all $(K^{\mathrm{op}} \times G)$-sets B where K, G, H are all finite groups. Put

$$A \wedge_G B = \{(a,b) \in A \times B \mid g \in G, ag = a \Rightarrow \text{there exists } k \in K \text{ with } gb = bk\}.$$

This becomes a $(K^{\mathrm{op}} \times G \times H)$-set via the action

$$(k,g,h)(a,b) = (hag^{-1}, gbk).$$

Then Bouc defines the $(K^{\mathrm{op}} \times H)$-set

$$A \circ_G B = (A \wedge_G B)/G,$$

to be the set of orbits $orb(a,b) = [a,b]$ of elements (a,b) of $A \wedge_G B$ under the action of G. In particular, when $K = \mathbf{1}$ and $B = X \in G\text{-}\mathbf{set}_{\mathit{fin}}$, we obtain $F(X) = A \circ_G X \in H\text{-}\mathbf{set}_{\mathit{fin}}$. This defines the functor

$$F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}.$$

1.1. THEOREM. *[Bo] Suppose K, G and H are finite groups.*

(i) If A is a finite $(G^{\mathrm{op}} \times H)$-set then the functor

$$A \circ_G - : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$$

preserves finite coproducts and pullbacks.

(ii) Every functor $F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow H\text{-}\mathbf{set}_{\mathit{fin}}$ which preserves finite coproducts and pullbacks is isomorphic to one of the form $A \circ_G -$.

(iii) The functor F in (ii) preserves terminal objects if and only if A is transitive (connected) as a right $G\text{-}\mathbf{set}_{\mathit{fin}}$.

(iv) If A is as in (i) and B is a finite $(K^{\mathrm{op}} \times G)$-set then the composite functor

$$K\text{-}\mathbf{set}_{\mathit{fin}} \xrightarrow{B \circ_K -} G\text{-}\mathbf{set}_{\mathit{fin}} \xrightarrow{A \circ_G -} H\text{-}\mathbf{set}_{\mathit{fin}}$$

is isomorphic to $(A \circ_G B) \circ_K -$.

Our intention in the present paper is to provide a categorical explanation for this Theorem.

In Section 2, before turning to the problem of preserving pullbacks, we examine finite limit preserving functors from categories like G-$\mathbf{set}_{\mathit{fin}}$ to $\mathbf{set}_{\mathit{fin}}$. We adapt the appropriate classical adjoint functor theorem to this "finite" situation. To make use of this for the purpose in hand, in Section 3, we need to adapt the result to include preservation of finite coproducts and reduce the further preservation of pullbacks to the finite limit case.

Section 4 interprets the work in the finite G-set case. In Section 5 we express the conclusions bicategorically. Implications for our original motivating work on Mackey functors are explained in the final Section 6.

2. Special representability theorem

In this section we provide a direct proof of the well-known representability theorem (see Chapter 5 [Ma]) for the case where "small" means "finite".

Recall that an object Q of a category $\mathscr{A}$ is called a *cogenerator* when, for all $f, g : A \longrightarrow B$ in $\mathscr{A}$, if $uf = ug$ for all $u : B \longrightarrow Q$, then $f = g$.

A *subobject* of an object A of $\mathscr{A}$ is an isomorphism class of monomorphisms $m : S \longrightarrow A$; two such monomorphisms $m : S \longrightarrow A$ and $m' : S' \longrightarrow A$ are isomorphic when there is an invertible morphism $h : S \longrightarrow S'$ with $m' \circ h = m$. We call $\mathscr{A}$ *finitely well powered* when each object A has only finitely many subobjects. Write $\mathit{Sub}(A)$ for the set of subobjects $[m : S \longrightarrow A]$ of A.

For each set X and object A of $\mathscr{A}$, we write A^X for the object of $\mathscr{A}$ for which there is a natural isomorphism

$$\mathscr{A}(B, A^X) \cong \mathscr{A}(B, A)^X$$

where Y^X is the set of functions from X to Y. Such an object may not exist; if $\mathscr{A}$ has products indexed by X then A^X is the product of X copies of A.

We write $\mathbf{set}_{\mathit{fin}}$ for the category of finite sets and functions. A functor $T : \mathscr{A} \longrightarrow \mathbf{set}_{\mathit{fin}}$ is *representable* when there is an object $K \in \mathscr{A}$ and a natural isomorphism $T \cong \mathscr{A}(K, -)$.

2.1. THEOREM. *(Special representability theorem) Suppose $\mathscr{A}$ is a category with the following properties:*

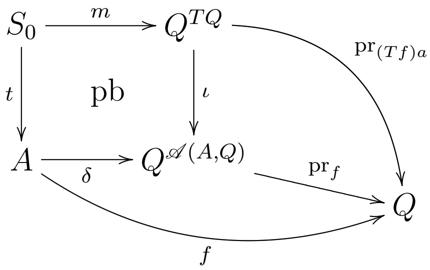
(i) each homset $\mathscr{A}(A, B)$ is finite;
(ii) finite limits exist;
(iii) there is a cogenerator Q;
(iv) $\mathscr{A}$ is finitely well powered.

Then every finite limit preserving functor $T : \mathscr{A} \longrightarrow \mathbf{set}_{\mathit{fin}}$ is representable.

PROOF. Using (ii) and (iv), we have the object

$$P = \prod_{[S] \in \mathit{Sub}(Q^{TQ})} S^{TS}.$$

We shall prove that, for each $A \in \mathscr{A}$ and $a \in TA$, there exists $p \in TP$ and $w : P \longrightarrow A$ such that $(Tw)p = a$. The following diagram defines δ, ι and S_0.

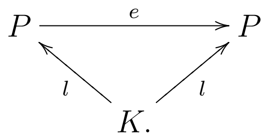


Now $\delta u = \delta v$ implies $fu = fv$ for all $f : A \longrightarrow Q$, so $u = v$ by (iii). So δ is a monomorphism. So $[S_0] \in Sub(Q^{TQ})$. Since T preserves pullbacks, there is a unique $s \in TS_0$ such that $(Tt)s = a$ and $(Tm)s$ transports to 1_{TQ} under $T(Q^{TQ}) \cong (TQ)^{TQ}$. Let p transport to $(1_{TS})_{[S]}$ under $TP \cong \prod_{[S]}(TS)^{TS}$. Then we can define w to be the composite

$$P \xrightarrow{\mathrm{pr}_{[S_0],s}} S_0 \xrightarrow{\ t\ } A$$

with $(Tw)p = a$.

Now let K be the equalizer of all the endomorphisms (including 1_P) of P (we are using (i)):



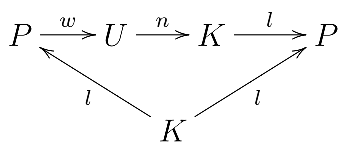
Since T preserves limits, there is a unique $k \in TK$ with $(Tl)k = p$. Define

$$\theta_A : \mathscr{A}(K, A) \longrightarrow TA$$

by $\theta_A(r) = (Tr)k$; this is natural in A. Moreover, θ_A is surjective since (K, k) clearly has the same property that we proved for (P, p).

It remains to prove θ_A injective. Suppose r and $r' : K \longrightarrow A$ are such that $(Tr)k = (Tr')k$.

Let $n : U \longrightarrow K$ be the equalizer of r and r', and let $u \in TU$ be unique with $(Tn)u = k$. By the property of (P, p), there exists $w : P \longrightarrow U$ with $(Tw)p = u$.



From the definition of K, we have $lnwl = l$. Yet l is a monomorphism (since it is an equalizer), so $nwl = 1$ and $r = rnwl = r'nwl = r'$, as required. ∎

For categories $\mathscr{A}$ and $\mathscr{X}$ admitting finite limits, write $\mathbf{Lex}(\mathscr{A}, \mathscr{X})$ for the full subcategory of the functor category $[\mathscr{A}, \mathscr{X}]$ consisting of the finite limit preserving functors.

2.2. Corollary. *For a category $\mathscr{A}$ satisfying the conditions of Theorem 2.1, the Yoneda embedding defines an equivalence of categories*

$$\mathscr{A}^{\mathrm{op}} \simeq \mathbf{Lex}(\mathscr{A}, \mathbf{set}_{\mathit{fin}}) \;, \quad A \longmapsto \mathscr{A}(A, -).$$

3. Finite coproducts

Suppose the category $\mathscr{A}$ has finite coproducts. An object C of $\mathscr{A}$ is called *connected* when the functor $\mathscr{A}(C, -) : \mathscr{A} \longrightarrow \mathbf{Set}$ preserves finite coproducts. Write $\mathbf{Conn}(\mathscr{A})$ for the full subcategory of $\mathscr{A}$ consisting of the connected objects.

Write $\mathbf{Cop}(\mathscr{A}, \mathscr{X})$ for the full subcategory of $[\mathscr{A}, \mathscr{X}]$ consisting of the finite coproduct preserving functors. Also $\mathbf{CopLex}(\mathscr{A}, \mathscr{X})$ consists of the finite coproduct and finite limit preserving functors. As an immediate consequence of Corollary 2.2 we have

3.1. Corollary. *For a category $\mathscr{A}$ with finite coproducts and the properties of Theorem 2.1, the Yoneda embedding defines an equivalence of categories*

$$\mathbf{Conn}(\mathscr{A})^{\mathrm{op}} \simeq \mathbf{CopLex}(\mathscr{A}, \mathbf{set}_{\mathit{fin}}).$$

Suppose $\mathscr{A}$ is a finitely complete category and $T : \mathscr{A} \longrightarrow \mathbf{set}_{\mathit{fin}}$ is a functor. For each $t \in T1$, define a functor $T_t : \mathscr{A} \longrightarrow \mathbf{set}_{\mathit{fin}}$ using the universal property of the pullback

$$\begin{array}{ccc} T_t A & \xrightarrow{\iota_t A} & TA \\ \big\downarrow & \text{pb} & \big\downarrow {\scriptstyle T\,!} \\ 1 & \xrightarrow[t]{} & T1. \end{array}$$

Clearly $T \cong \sum_{t \in T1} T_t$. Taking $A = 1$ in the above pullback, we see that each T_t preserves terminal objects. The following observations are obvious.

3.2. Proposition. *Suppose $\mathscr{A}$ is a finitely complete category, $T : \mathscr{A} \longrightarrow \mathbf{set}_{\mathit{fin}}$ is a functor, and the functors $T_t : \mathscr{A} \longrightarrow \mathbf{set}_{\mathit{fin}}$, $t \in T1$, are defined as above.*

(i) If T preserves pullbacks then each T_t preserves finite limits.

(ii) Each T_t preserves whatever coproducts that are preserved by T.

For any small category $\mathscr{C}$, we write $\mathbf{Fam}(\mathscr{C}^{\mathrm{op}})$ for the free finite coproduct completion of $\mathscr{C}^{\mathrm{op}}$. The objects are families $(C_i)_{i \in I}$ of objects C_i of $\mathscr{C}$ with indexing set I finite. A morphism $(\xi, f) : (C_i)_{i \in I} \longrightarrow (D_j)_{j \in J}$ consists of a function $\xi : I \longrightarrow J$ and a family $f = (f_i)_{i \in I}$ of morphisms $f_i : C_{\xi(i)} \longrightarrow D_i$ in $\mathscr{C}$.

There is a functor

$$\mathscr{Z}_{\mathscr{C}} : \mathbf{Fam}(\mathscr{C}^{\mathrm{op}}) \longrightarrow [\mathscr{C}, \mathbf{Set}]$$

defined by

$$\mathscr{Z}_{\mathscr{C}}(C_i)_{i\in I} = \sum_{i\in I} \mathscr{C}(C_i, -)$$

which is fully faithful. So $\mathbf{Fam}(\mathscr{C}^{\mathrm{op}})$ is equivalent to the closure under finite coproducts of the representables in $[\mathscr{C}, \mathbf{Set}]$.

We write $\mathbf{Pb}(\mathscr{A}, \mathscr{X})$ for the full subcategory of $[\mathscr{A}, \mathscr{X}]$ consisting of pullback preserving functors. Also $\mathbf{CopPb}(\mathscr{A}, \mathscr{X})$ has objects restricted to those preserving finite coproducts and pullbacks.

3.3. PROPOSITION. *Suppose the category $\mathscr{A}$ is as in Corollary 3.1. The functor $\mathscr{Z}_{\mathscr{A}}$ induces an equivalence of categories*

$$\mathbf{Fam}(\mathbf{Conn}(\mathscr{A})^{\mathrm{op}}) \simeq \mathbf{CopPb}(\mathscr{A}, \mathbf{set}_{\mathit{fin}}).$$

PROOF. Clearly $\mathbf{Fam}(\mathbf{Conn}(\mathscr{A})^{\mathrm{op}})$ is a full subcategory of $\mathbf{Fam}(\mathscr{A}^{\mathrm{op}})$ and $\mathscr{Z}_A$ restricts to a fully faithful functor

$$\mathbf{Fam}(\mathbf{Conn}(\mathscr{A})^{\mathrm{op}}) \longrightarrow [\mathscr{A}, \mathbf{set}_{\mathit{fin}}].$$

It remains to identify the essential image of this functor as those $T : \mathscr{A} \longrightarrow \mathbf{set}_{\mathit{fin}}$ which preserve finite coproducts and pullbacks. However, we have seen in Proposition 3.2 that such a T has the form $T \cong \sum_{t\in T1} T_t$ where each T_t preserves finite coproducts and is left exact. By Corollary 3.1, we have

$$T_t \cong \mathscr{A}(C_t, -)$$

where each C_t is connected. ■

4. Application to permutation representations

A *permutation representation* of a finite group G, also called a *finite left G-set*, is a finite set X together with a function $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$, called the *action* such that

$$1x = x \quad \text{and} \quad g_1(g_2x) = (g_1g_2)x.$$

If X and Y are such G-sets, a *(left) G-morphism* $f : X \longrightarrow Y$ is a function satisfying $f(gx) = gf(x)$. We write G-$\mathbf{set}_{\mathit{fin}}$ for the category of finite left G-sets and left G-morphisms.

The terminal object of G-$\mathbf{set}_{\mathit{fin}}$ is the set $\mathbf{1}$ with only one element with its unique action. The pullback in G-$\mathbf{set}_{\mathit{fin}}$ of two morphisms $f : X \longrightarrow Z$ and $k : Y \longrightarrow Z$ is given by $\{(x, y) \in X \times Y | f(x) = k(y)\}$ with componentwise action $g(x, y) = (gx, gy)$. So G-$\mathbf{set}_{\mathit{fin}}$ has finite limits.

Since the set G-$\mathbf{set}_{\mathit{fin}}(X, Y)$ is a subset of the set $\mathbf{set}_{\mathit{fin}}(X, Y) = Y^X$, it is finite.

The set $\mathscr{P}G$ of subsets of G becomes a G-set by defining the action as

$$gS = \{h \in G \mid hg \in S\}$$

for $S \subseteq G$ and $g \in G$. For each $x \in X \in G\text{-}\mathbf{set}_{\mathit{fin}}$, we can define a G-morphism $\chi_x : X \longrightarrow \mathscr{P}G$ by

$$\chi_x(z) = \{h \in G \mid hz = x\}.$$

If $x, y \in X \in G\text{-}\mathbf{set}_{\mathit{fin}}$ then $\chi_x(x) = \chi_x(y)$ implies $1 \in \chi_x(y)$, so $y = x$. It follows that $\mathscr{P}G$ is a cogenerator for $G\text{-}\mathbf{set}_{\mathit{fin}}$.

Subobjects of $X \in G\text{-}\mathbf{set}_{\mathit{fin}}$ are in bijection with sub-G-sets of X. So $G\text{-}\mathbf{set}_{\mathit{fin}}$ is finitely well powered.

From Section 2, we therefore have:

4.1. Corollary. *Every limit preserving functor* $T : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow \mathbf{set}_{\mathit{fin}}$ *is representable. The Yoneda embedding induces an equivalence of categories*

$$G\text{-}\mathbf{set}_{\mathit{fin}}^{\mathrm{op}} \simeq \mathbf{Lex}(G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}}).$$

Recall that a G-set X is called *transitive* when it is non-empty and, for all $x, y \in X$, there exists $g \in G$ with $gx = y$.

For any G-set X and any $x \in X$, we put

$$\mathit{stab}(x) = \{g \in G \mid gx = x\}$$

which is a subgroup of G called the *stabilizer* of x. We also put

$$\mathit{orb}(x) = \{gx \mid g \in G\}$$

which is a transitive sub-G-set X called the *orbit* of x. We write X/G for the set of orbits which can be regarded as a $G\text{-}\mathbf{set}_{\mathit{fin}}$ with trivial action so that $\mathit{orb} : X \longrightarrow X/G$ is a surjective G-morphism. If $u \in X/G$, we also write X_u for the orbit u as a sub-G-set of X. So every G-set is the disjoint union of its orbits.

The empty coproduct is the empty set $\mathbf{0}$ with its unique action. The coproduct of two G-sets X and Y is their disjoint union $X + Y$ with action such that the coprojections $X \longrightarrow X + Y$ and $Y \longrightarrow X + Y$ are G-morphisms. So every G-set X is a coproduct

$$X \cong \sum_{u \in X/G} X_u$$

of transitive G-sets (the orbits X_u).

Each subgroup H of G determines a transitive G-set

$$G/H = \{xH \mid x \in G\}$$

where $xH = \{xh \mid h \in H\}$ is the left coset of H containing x, and where the action is

$$g(xH) = (gx)H.$$

Every transitive G-set X is isomorphic to one of the form G/H; we can take $H = stab(x)$ for any $x \in X$.

The G-morphisms $f : G/H \longrightarrow X$ are in bijection with those $x \in X$ such that $H \leq stab(x)$. The G-sets G/H and G/K are isomorphic if and only if the subgroups H and K are conjugate (that is, there exists $x \in G$ with $Hx = xK$).

We provide a proof of the following well-known fact.

4.2. Proposition. *A finite G-set X is transitive if and only if X is a connected object of G-$\mathbf{set}_{\mathit{fin}}$.*

Proof. A G-set X is non-empty if and only if G-$\mathbf{set}_{\mathit{fin}}(X, \mathbf{0})$ is empty; that is, if and only if G-$\mathbf{set}_{\mathit{fin}}(X, -)$ preserves empty coproducts.

A morphism $G/H \longrightarrow Y + Z$ is determined by an element of $Y + Z$ stable under H; such an element must either be an element of Y or an element of Z stable under H. So transitive G-sets are connected.

Assume X is connected. We have already seen that X is non-empty so choose $x \in X$. Then

$$X = orb(x) + U$$

for some sub-G-set U of X. We therefore have the canonical function

$$G\text{-}\mathbf{set}_{\mathit{fin}}(X, orb(x)) + G\text{-}\mathbf{set}_{\mathit{fin}}(X, U) \longrightarrow G\text{-}\mathbf{set}_{\mathit{fin}}(X, X)$$

which is invertible since X is connected. So the identity function $X \longrightarrow X$ is in the image of the canonical function and so factors through $orb(x) \subseteq X$ or $U \subseteq X$. Since $x \notin U$, we must have $orb(x) = X$. So X is connected. ∎

We have thus identified $\mathbf{Conn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ as consisting of the transitive G-sets. This category has a finite skeleton $\mathscr{C}_G$ since there are only finitely many G-sets of the form G/H. Corollary 3.1 yields:

4.3. Corollary. *The Yoneda embedding induces an equivalence of categories*

$$\mathscr{C}_G^{\mathrm{op}} \simeq \mathbf{CopLex}(G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}}).$$

Let $N : \mathscr{C}_G \longrightarrow G$-$\mathbf{set}_{\mathit{fin}}$ denote the inclusion functor and define the functor

$$\tilde{N} : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow [\mathscr{C}_G^{\mathrm{op}}, \mathbf{set}_{\mathit{fin}}]$$

by $\tilde{N}X = G\text{-}\mathbf{set}_{\mathit{fin}}(N-, X)$.

4.4. Proposition. *The functor $\tilde{N}$ induces an equivalence of categories*

$$G\text{-}\mathbf{set}_{\mathit{fin}} \simeq \mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}}).$$

Proof. We first prove that N is dense; that is, that $\tilde{N}$ is fully faithful. Let $\theta : \tilde{N}X \longrightarrow \tilde{N}Y$ be a natural transformation. For each $u : C \longrightarrow D$ in $\mathscr{C}_G$ we have a commutative square

$$\begin{array}{ccc} G\text{-}\mathbf{set}_{\mathit{fin}}(D,X) & \xrightarrow{\theta_D} & G\text{-}\mathbf{set}_{\mathit{fin}}(D,Y) \\ {\scriptstyle -\circ u}\downarrow & & \downarrow{\scriptstyle -\circ u} \\ G\text{-}\mathbf{set}_{\mathit{fin}}(C,X) & \xrightarrow[\theta_C]{} & G\text{-}\mathbf{set}_{\mathit{fin}}(C,Y). \end{array}$$

Since the single-object full subcategory of G-$\mathbf{set}_{\mathit{fin}}$ consisting of G is dense ($G^{\mathrm{op}} \longrightarrow G$-$\mathbf{set}_{\mathit{fin}}$ is a Yoneda embedding), by restricting C and D to be equal to $G \in \mathscr{C}_G$, we obtain a G-morphism $f : X \longrightarrow Y$ defined uniquely by $f(x) = \theta_G(\hat{x})(1)$ where $\hat{x} : G \longrightarrow X$ is given by $\hat{x}(g) = gx$. Then, for all $w : D \longrightarrow X$ and $d \in D$, the above commutative square, with $C = G$, yields

$$\theta_D(w)(d) = (\theta_D(w) \circ \hat{d})(1) = \theta_G(w \circ \hat{d})(1) = \theta_G(\widehat{w \circ d})(1) = (f \circ w)(d).$$

So $\theta_D = G$-$\mathbf{set}_{\mathit{fin}}(D, f)$ for a unique G-morphism f.

The proof of the equivalence of categories will be completed by characterizing the essential image of $\tilde{N}$ as finite coproducts of representables in $[\mathscr{C}_G^{\mathrm{op}}, \mathbf{set}_{\mathit{fin}}]$. If $F \in [\mathscr{C}_G^{\mathrm{op}}, \mathbf{set}_{\mathit{fin}}]$ is a finite coproduct of representables then we have a finite family $(C_i)_{i \in I}$ of objects of $\mathscr{C}_G$ and an isomorphism $F \cong \sum_i \mathscr{C}(-, C_i)$. We have the calculation:

$$\begin{aligned} \sum_i \mathscr{C}(-, C_i) &\cong \sum_i G\text{-}\mathbf{set}_{\mathit{fin}}(N-, C_i) \\ &\cong G\text{-}\mathbf{set}_{\mathit{fin}}(N-, \sum_i C_i) \\ &\cong \tilde{N}(\sum_i C_i). \end{aligned}$$

So F is in the essential image of $\tilde{N}$. Conversely, every $X \in G$-$\mathbf{set}_{\mathit{fin}}$ is a coproduct $X \cong \sum_i C_i$ of connected G-sets. So the same calculation, read from bottom to top, shows that $\tilde{N}(X)$ is a finite coproduct of representables. ∎

4.5. Corollary. *There is an equivalence of categories*

$$G\text{-}\mathbf{set}_{\mathit{fin}} \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}})$$

taking the left G-set C to the functor

$$\sum_{w \in C/G} G\text{-}\mathbf{set}_{\mathit{fin}}(C_w, -)$$

where C_w is the orbit w as a sub-G-set of C.

There is an isomorphism of categories

$$\ell_G : G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow G\text{-}\mathbf{set}_{\mathit{fin}}$$

which preserves the underlying sets. If A is a right G-set then $\ell_G A = A$ as a set with left action ga in $\ell_G A$ equal to ag^{-1} in A. As a special case of the construction in the Introduction, for a right G-set A and a left G-set X, we have

$$\begin{aligned} A \wedge_G X &= \{(a, x) \in \ell_G A \times X \mid \mathit{stab}(a) \leq \mathit{stab}(x)\} \text{ and} \\ A \circ_G X &= (A \wedge_G X)/G. \end{aligned}$$

Each $(a, x) \in A \wedge_G X$ defines a G-morphism

$$\theta_X(a, x) : \ell_G A_u \longrightarrow X$$

where $u = \mathit{orb}(a)$ and $\theta_X(a, x)(ag^{-1}) = gx$ (which is well defined since $ag_1^{-1} = ag_2^{-1} \Rightarrow g_2^{-1}g_1 \in \mathit{stab}(a) \Rightarrow g_2^{-1}g_1 \in \mathit{stab}(x) \Rightarrow g_1x = g_2x$). This defines a function

$$\theta_X : A \wedge_G X \longrightarrow \sum_{u \in G\backslash A} G\text{-}\mathbf{set}_{\mathit{fin}}(\ell_G A_u, X)$$

naturally in $X \in G\text{-}\mathbf{set}_{\mathit{fin}}$ (where $G\backslash A$ is the set of orbits of the right action). Clearly

$$\theta_X(a, x) = \theta_X(b, y) \quad \text{if and only if} \quad \mathit{orb}(a, x) = \mathit{orb}(b, y).$$

This proves:

4.6. Proposition. *For all $A \in G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}}$, the natural transformation θ induces a natural isomorphism*

$$\bar{\theta} : A \circ_G - \cong \sum_{u \in G\backslash A} G\text{-}\mathbf{set}_{\mathit{fin}}(\ell_G A_u, -)$$

between functors from $G\text{-}\mathbf{set}_{\mathit{fin}}$ to $\mathbf{set}_{\mathit{fin}}$.

4.7. Corollary. *There is an equivalence of categories*

$$G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}} \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}}) \ , \quad A \longmapsto A \circ_G -.$$

5. A bicategory of finite groups

The goal of this section is to consolidate our results in terms of a homomorphism of bicategories which is an equivalence on homcategories. We construct a bicategory whose objects are finite groups and whose morphisms are permutation representations between them. This bicategory is the domain of the homomorphism. The codomain is the 2-category of categories of the form G-$\mathbf{set}_{\mathit{fin}}$ and pullback-and-finite-coproduct-preserving functors between them.

Suppose G and H are finite groups. There is a monad $H \times -$ on G^{op}-$\mathbf{set}_{\mathit{fin}}$ whose Eilenberg-Moore algebras are the $(G^{\mathrm{op}} \times H)$-sets. For the endofunctor

$$H \times - : G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}},$$

we simply regard H as a trivial G^{op}-$\mathbf{set}_{\mathit{fin}}$ $(hg = h)$ and take A to $H \times A$ in G^{op}-$\mathbf{set}_{\mathit{fin}}$. The unit $A \longrightarrow H \times A$ and multiplication $H \times H \times A \longrightarrow H \times A$ for the monad are defined by $a \longmapsto (1, a)$ and $(h_1, h_2, a) \longmapsto (h_1 h_2, a)$.

In the case $G = \mathbf{1}$, we have the monad $H \times -$ on $\mathbf{set}_{\mathit{fin}}$. This lifts pointwise to a monad $H \times -$ on $[G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}}]$. There is a canonical isomorphism

$$\begin{array}{ccc} G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}} & \longrightarrow & [G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}}] \\ {\scriptstyle H\times -}\downarrow & \cong & \downarrow{\scriptstyle H\times -} \\ G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}} & \longrightarrow & [G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}}], \end{array}$$

which is compatible with the monad structures, where the horizontal functors are both $A \longmapsto A \circ_G -$. The component

$$A \circ_G (H \times X) \cong H \times (A \circ_G X)$$

of the isomorphism at X is induced by the G-set isomorphism

$$\ell_G A \times (H \times X) \cong H \times (\ell_G A \times X)\ , \quad (a, (h, x)) \longmapsto (h, (a, x)).$$

Notice that, for $(h, x) \in H \times X$, $\mathit{stab}(h, x) = \mathit{stab}(x)$. It follows that $A \longmapsto A \circ_G -$ induces a functor

$$(G^{\mathrm{op}} \times H)\text{-}\mathbf{set} \longrightarrow [G\text{-}\mathbf{set}_{\mathit{fin}}, H\text{-}\mathbf{set}_{\mathit{fin}}].$$

If $F : G\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow \mathbf{set}_{\mathit{fin}}$ preserves pullbacks and finite coproducts, so does $H \times F-$. Also, the forgetful functor $H\text{-}\mathbf{set}_{\mathit{fin}} \longrightarrow \mathbf{set}_{\mathit{fin}}$ creates both finite coproducts and pullbacks. Therefore, Corollary 4.7 yields:

5.1. THEOREM. *There is an equivalence of categories*

$$(G^{\mathrm{op}} \times H)\text{-}\mathbf{set} \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{\mathit{fin}}, H\text{-}\mathbf{set}_{\mathit{fin}})\ , \quad A \longmapsto A \circ_G -.$$

Parts $(i), (ii)$ and (iii) of Theorem 1.1 follows from Theorem 5.1. It is also clear, in the setting of Theorem 1.1(iv), that there exists a $(K^{\mathrm{op}} \times H)$-set C such that

$$A \circ_G (B \circ_K -) \cong C \circ_K -$$

since the composite of pullback and finite coproduct preserving functors also preserves them. It remains to identify C as $A \circ_G B$. We do this directly.

5.2. Proposition. *If A, B and Z are respectively $(G^{\mathrm{op}} \times H)$-, $(K^{\mathrm{op}} \times G)$-, and K-sets then*

$$A \circ_G (B \circ_K Z) \cong (A \circ_G B) \circ_K Z \ , \quad [a,[b,z]] \mapsto [[a,b],z]$$

is an isomorphism of H-sets.

Proof. To say $(a,[b,z]) \in A \wedge_G (B \circ_K Z)$ is to say that $ag = a$ implies $g[b,z] = [b,z]$; that is, there exists $k \in K$ such that $gb = bk$ and $hz = z$. In particular, this means that $(a,b) \in A \wedge_G B$. We need to see that $([a,b],z) \in (A \circ_G B) \wedge_K Z$. So suppose $[a,b]k = [a,b]$. Then there exists $g \in G$ with $a = ag$ and $bk = gb$. The former implies there is $k_1 \in K$ such that $gb = bk_1$ and $k_1 z = z$. Then $bk = gb = bk_1$, so $bkk_1^{-1} = b$. Since $(b,z) \in B \wedge_K Z$, we have $kk_1^{-1}z = z$; so $kz = z$. One also sees that

$$\begin{aligned} [a,[b,z]] = [a',[b',z']] &\iff \exists g \in G, k \in K \ : \ a = a'g, \ gb = b'k, \ kz = z' \\ &\iff [[a,b],z] = [[a',b'],z']. \end{aligned}$$

This proves the bijection. Clearly the H-actions correspond. ∎

Let **CopPb** denote the 2-category whose objects are categories admitting pullbacks and finite coproducts, whose morphisms are functors which preserve these, and all natural transformations between such functors. This is a sub-2-category of the 2-category **Cat** of small categories. Let **CopLex** be the sub-2-category of **CopPb** consisting of the categories which also have terminal objects (and so are finitely complete), and the functors which preserve them.

We can now summarize the results in:

5.3. Theorem. *There is a bicategory* **Bouc** *whose objects are finite groups, whose hom-categories are*

$$\mathbf{Bouc}(G,H) = (G^{\mathrm{op}} \times H)\text{-}\mathbf{set},$$

and whose composition functors are

$$\mathbf{Bouc}(G,H) \times \mathbf{Bouc}(K,G) \longrightarrow \mathbf{Bouc}(K,H) \ , \quad (A,B) \mapsto A \circ_G B.$$

There is a homomorphism of bicategories

$$\Phi : \mathbf{Bouc} \longrightarrow \mathbf{CopPb}$$

defined on objects by $\Phi G = G\text{-}\mathbf{set}_{fin}$ and on homcategories by the equivalences of Theorem 5.1. The restriction of the morphisms $A : G \longrightarrow H$ of **Bouc** *to those $G^{op} \times H$-sets which are connected as right G-sets is a sub-bicategory* **Boucc** *of* **Bouc***, and Φ restricts to a homomorphism*

$$\Phi_C : \mathbf{Boucc} \longrightarrow \mathbf{CopLex}$$

which is also an equivalence on homcategories.

6. Application to Mackey functors

Given a finite group G, there is a bicategory of spans in the category G-$\mathbf{set}_{fin}$ [Bé]. We write $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$ for the category obtained by taking isomorphism classes of spans as morphisms. This category is compact closed and the coproduct in G-$\mathbf{set}_{fin}$ becomes direct sum (that is, it is also the product) in $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$. These matters are made explicit in [PS].

We fix a field k and write **vect** for the category of finite dimensional vector spaces over k with linear functions as morphisms. A *finite dimensional Mackey functor on G* is a functor $M : \mathbf{Spn}(G\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{vect}$ which preserves finite direct sums. We write $\mathbf{Mky}_{fin}(G)$ for the category of finite dimensional Mackey functors on G; the morphisms are natural transformations. It is an abelian k-linear category.

Each pullback preserving functor $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ induces a functor $\mathbf{Spn}(F) : \mathbf{Spn}(G\text{-}\mathbf{set}_{fin}) \longrightarrow \mathbf{Spn}(H\text{-}\mathbf{set}_{fin})$ (since composition of spans only involves pullbacks). If F also preserves finite coproducts then $\mathbf{Spn}(F)$ preserves direct sums. In that case, we obtain an exact functor

$$\bar{F} : \mathbf{Mky}_{fin}(H) \longrightarrow \mathbf{Mky}_{fin}(G)$$

defined by $\bar{F}(N) = N \circ \mathbf{Spn}(F)$ for all $N \in \mathbf{Mky}_{fin}(H)$. Moreover, $\bar{F}$ has a left adjoint

$$\mathbf{Mky}_{fin}(F) : \mathbf{Mky}_{fin}(G) \longrightarrow \mathbf{Mky}_{fin}(H)$$

defined by

$$\mathbf{Mky}_{fin}(F)(M) = \int^{C} \mathbf{Spn}(H\text{-}\mathbf{set}_{fin})(F(C), -) \otimes MC$$

where C runs over the connected G-sets as objects of $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$ and the tensor product is that of additive (commutative) monoids.

Let $\mathbf{AbCat}_k$ denote the 2-category of abelian k-linear categories, k-linear functors with right exact right adjoints, and natural transformations.

6.1. Corollary. *There is a homomorphism of bicategories*

$$\mathbf{Mky}_{fin} : \ \mathbf{Bouc} \longrightarrow \mathbf{AbCat}_k$$

$$(A : G \longrightarrow H) \longmapsto (\mathbf{Mky}_{fin}(A \circ_G -) : \mathbf{Mky}_{fin}(G) \longrightarrow \mathbf{Mky}_{fin}(H)).$$

Actually, $\mathbf{Mky}_{fin}(G)$ is much more than an abelian k-linear category. It is monoidal under the Day convolution tensor product [Da] using the monoidal structure on $\mathbf{Spn}(G\text{-}\mathbf{set}_{fin})$ coming from cartesian product in G-$\mathbf{set}_{fin}$. In fact, $\mathbf{Mky}_{fin}(G)$ is $*$-autonomous in the sense of [Ba]; details can be found in [PS].

If $F : G\text{-}\mathbf{set}_{fin} \longrightarrow H\text{-}\mathbf{set}_{fin}$ is left exact and finite coproduct preserving then $\mathbf{Spn}(F)$ is strong monoidal (=tensor product preserving). It follows (for example from [DS]) that $\mathbf{Mky}_{fin}(F)$ is strong monoidal. It also preserves the $*$-autonomous structure.

Let $*$-$\mathbf{AbCat}_k$ denote the 2-category of $*$-autonomous monoidal abelian k-linear categories, $*$-preserving strong-monoidal k-linear functors with right exact right adjoints, and natural transformations.

6.2. Corollary. *The homomorphism of bicategories in Corollary 6.1 induces a homomorphism*

$$\mathbf{Mky}_{\mathit{fin}} : \mathbf{Boucc} \longrightarrow *\text{-}\mathbf{AbCat}_k.$$

References

[Ba] M. Barr, ∗-Autonomous Categories, *Lecture Notes in Math.*, **752** (Springer-Verlag, Berlin, 1979).

[Bé] J. Bénabou, Introduction to bicategories, *Lecture Notes in Math.*, **47** (Reports of the Midwest Category Seminar) (Springer-Verlag, Berlin, 1967), 1–77.

[Bo] S. Bouc, Construction de foncteurs entre catégories de G-ensembles, *J. Algebra*, **183** (1996), 737–825.

[Da] B. J. Day, On closed categories of functors, *Lecture Notes in Math.*, **137** (Springer, Berlin, 1970), 1–38.

[DS] B. Day and R. Street, Kan extensions along promonoidal functors, *Theory and Applic. of Categories*, **1** (1995), 72–77.

[Li] H. Lindner, A remark on Mackey functors, *Manuscripta Math.*, **18** (1976), 273–278.

[Ma] S. Mac Lane, Categories for the Working Mathematician, *Lecture Notes in Math.*, **5** (Springer-Verlag, New York, 1971).

[PS] E. Panchadcharam and R. Street, Mackey functors on compact closed categories, *submitted.*

Centre of Australian Category Theory, Macquarie University,
New South Wales, 2109, Australia
Email: elango@maths.mq.edu.au
street@maths.mq.edu.au

This article may be accessed via WWW at http://www.tac.mta.ca/tac/ or by anonymous ftp at ftp://ftp.tac.mta.ca/pub/tac/html/volumes/16/28/16-28.{dvi,ps,pdf}