Tangle with categories

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$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \arrow[ddr, bend left] & \\
\bullet \arrow[r] \arrow[drr, bend right] & \bullet \arrow[dr, dotted] & \\
& & \bullet
\end{tikzcd}
$$

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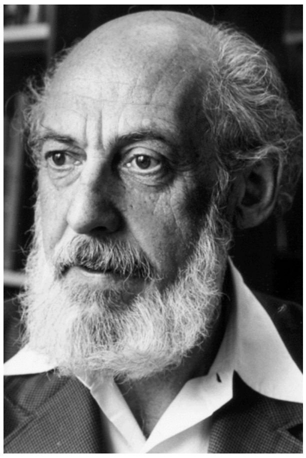
 

Figure 1: Saunders Mac Lane and Samuel Eilenberg

Part I

Introduction to category theory and links with knot theory

1 Introduction

1.1 Aim and History

> *Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial.*
>
> Peter Freyd

Category theory is a recent field in algebra ; set up by Saunders Mac Lane and Samuel Eilenberg (cf picture) in the 40's, it was aiming at formalising algebraic topology, but quickly became an interesting field on its own.

Saunders Mac Lane seems to be the first to consider category as more than a useful tool as he was the first to use a usual lower-case symbol to designate a category [1].

Plan : In this first part (mainly based on [8] and [3]) we are trying to show how category theory may build relationships between what appears to be very distinct fields : Knot theory and linear algebra.

The second part is a compendium of exercises from *Categories for the working mathematician* [5] dealing with a wide array of different kind of mathematics with the categorical point of view.

1.2 What is a Category

Definition 1 (*Category*)

A *Category* may be seen as a graph (with object set O and arrow set A with functions giving the source (*domain*) and target (*codomain*) of each arrow) equipped with extra structure allowing to compose pairs of arrows which follows one another.

More formally, it is a graph $A \rightrightarrows O$ together with :

- For all object a, an arrow $1_a : a \to a$.
- For all triple of objects (a, b, c), a map $\circ : \mathrm{Hom}(a, b) \times \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$ where $\mathrm{Hom}(x, y) = \{f \in A \mid \mathrm{domain}\, f = x \text{ and } \mathrm{codomain}\, f = y\}$. (We will denote $\circ(f, g)$ by $g \circ f$ or merely gf)

such that :

- for $f : a \to b$, one has $1_b \circ f = f = f \circ 1_a$.
- for all f, g and h arrows such that $\mathrm{codomain}\, f = \mathrm{domain}\, g$ and $\mathrm{codomain}\, g = \mathrm{domain}\, h$, one has $(h \circ g) \circ f = h \circ (g \circ f)$.

The arrows are also called *morphisms* or *maps*. The use of "morphisms" homogenise the nomenclature since we use the terms *isomorphism*, *endomorphism* and *automorphism*.

Remark

A one-object category is a (associative unitary) monoid where multiplication is the composition, hence a categories may be seen as a generalisation of monoids.

Examples

- The category **Rng** of rings has objects : rings (within a big set U) and arrows : ring homomorphisms between those. The composition is the usual composition, since the composite of two ring homomorphisms is an homomorphism.
- The category of sets has object the sets (within a big set U) and arrows the functions, the composition is the usual composition.

2 Functors and Natural Transformations

2.1 Functors an product of categories

Definition 2 (*Functor*)

A *functor* F from a category A to a category B is the data of

- A function (also called F) $O_A \to O_B$ on objects
- for each pair of objects (a, a') in A, a function (also called F) $\mathrm{Hom}_A(a, a') \to \mathrm{Hom}_B(F(a), F(a'))$

such that for all pair (f, g) of composable arrows in A, it satisfy :

$$T(g \circ f) = T(g) \circ T(f)$$

Definition 3 (*Product Category*)

Given two categories A and B, one may construct the *product category* denoted $A \times B$ whose objects are pairs in $O_A \times O_B$ and arrows are pairs of arrows : namely $\mathrm{Hom}_{A\times B}((a, b), (a', b')) = \mathrm{Hom}_A(a, a') \times \mathrm{Hom}_B(b, b')$

Remark

This product is in fact a categorical product in **Cat** since it has the universal property $\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C)$, naturally in the category C. See 7.1.3.

2.2 The very natural idea of natural transformations

2.2.1 Definition

Definition 4 (*Natural transformation*)

Given two functors $T, S : A \to B$, a *natural transformation* σ from T to S (denoted $\sigma : T \dot{\longrightarrow} S$) is a family of arrows $\sigma_a : T(a) \to S(a)$ for a object of A such that if $f : a \to a'$ then $\sigma_{a'} T(f) = S(f)\sigma_a$.

Which can be expressed by the following commutative diagram :

$$\begin{tikzcd} T(a) \arrow[r, "\sigma_a"] \arrow[d, "T(f)"] & S(a) \arrow[d, "S(f)"] \\ T(a') \arrow[r, "\sigma_{a'}"] & S(a') \end{tikzcd}$$

2.2.2 Natural isomorphism

Definition 5 (*Natural isomorphism*)

A natural transformation $\sigma : T \dot{\longrightarrow} S$ is a *natural isomorphism* if each σ_a is an isomorphism in B.

Remark

> There is a law of (vertical) composition of natural transformations, for which natural isomorphism are the invertible isomorphism. (where the identity natural transformation on a functor $T : a \to b$ is the family of identity of $T(a)$ in B.) [5] chapter 2 p. 42

Natural isomorphism is the strongest relation that can link two kind of objects (having a functorial construction) because the axioms of a category merely do not allow to derive equalities of functors.

Examples

> The main example could be the natural isomorphism $(A \times B) \times C \cong A \times (B \times C)$ (in the categories of sets, topological spaces, monoids, groups, rings, vector spaces ...)
>
> It states that one can associate a unique pair $((a, b), c)$ to each pair $(a, (b, c)) \in A \times (B \times C)$ (that's for the "iso" part) and that this association does not depend on A, B and C, it would have been defined the same way with other objects : It is "natural".
>
> Another useful example for the understanding of natural isomorphism is the bijection in **Set** between $C^{A\times B}$ and $(C^B)^A$:
>
> $$(a, b) \mapsto f(a, b) \quad \leftrightsquigarrow \quad b \mapsto (a \mapsto f(a, b))$$
>
> which definition also do not depend on A, B and C.

3 Monoidal Category

3.1 Axioms

3.1.1 Definition

Definition 6 (*Monoidal Category*)

> A *monoidal category* is a category M equipped with :
>
> - a functor $\otimes : M \times M \to M$
> - an object I of M.
> - a collection of isomorphisms $\alpha_{A,B,C} : A \otimes (B \otimes C) \tilde{\to} (A \otimes B) \otimes C$, for A, B, C objects of M, which is natural in A, B and C.
> - a collection of isomorphisms $\lambda_A : I \otimes A \tilde{\to} A$ which is natural in A object of M.
> - a collection of isomorphisms $\rho_A : A \otimes I \tilde{\to} A$ which is natural in A object of M.
>
> such that
>
> - $\alpha_{A,B,C\otimes D} \circ \alpha_{A\otimes B,C,D} = (1_A \otimes \alpha_{B,C,D}) \circ \alpha_{A,B\otimes C,D} \circ (\alpha_{A,B,C} \otimes 1_D)$.
> - $(1_A \otimes \lambda_B) \circ \alpha_{A,I,B} = \rho_A \otimes 1_B$.

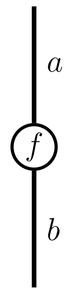
Examples

- $(\mathbf{Vect}_K, \otimes, K)$.
- $(\mathbf{Ab}, \otimes, \mathbb{Z})$.
- More generally $(R\text{-}\mathbf{Mod}, \otimes, R)$ for R commutative ring.
- $\mathbb{Z} = \{\cdots, -1, 0, 1, 2, \cdots\}$ as an ordered set with $n \otimes m = n + m$ and so $(a \leq b) \otimes (c \leq d) = a + c \leq b + d$ for arrows.

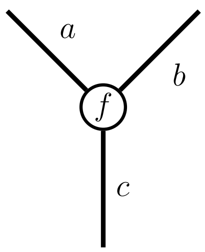
In the following, we may only consider "strict" monoidal categories : monoidal categories in which α, ρ and λ are identities.

3.2 String notations

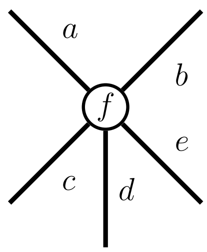
Usually morphisms are written as arrows, but Roger Penrose introduced the *string notation* :



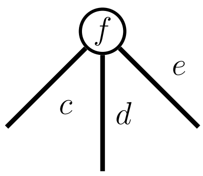
stands for $a \xrightarrow{f} b$



stands for $a \otimes b \xrightarrow{f} c$



stands for $a \otimes b \xrightarrow{f} c \otimes d \otimes e$



stands for $I \xrightarrow{f} c \otimes d \otimes e$

3.3 Duality between objects

Definition 7 (*duality in a monoidal category*)

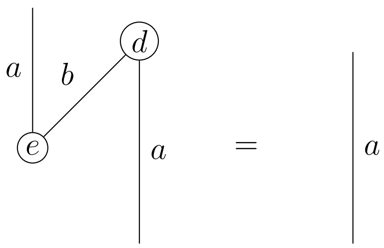
In a monoidal category $(M, \otimes, I, \alpha, \rho, \lambda)$, a *duality* between two objects a and b is a pair of morphisms :

$$e : a \otimes b \to I \quad \text{and} \quad d : I \to b \otimes a$$

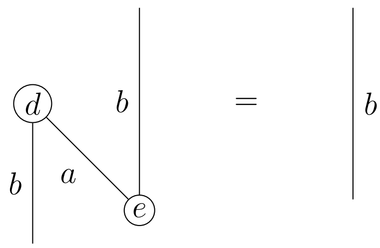
satisfying (ignoring α) the commutation of the followings diagrams :

$$\begin{tikzcd}
a \otimes I \arrow[r, "1 \otimes d"] \arrow[dr, "\rho_a"'] & a \otimes b \otimes a \arrow[r, "e \otimes 1"] & I \otimes a \arrow[dl, "\lambda_a"] \\
& a &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes b \arrow[r, "d \otimes 1"] \arrow[dr, "\lambda_b"'] & b \otimes a \otimes b \arrow[r, "1 \otimes e"] & b \otimes I \arrow[dl, "\rho_b"] \\
& b &
\end{tikzcd}$$

Using string notation, the two axioms can be translated this way :



and



Examples

- In the category $\mathbf{FinVect}_{\mathbb{K}}$ of finite dimensional vector spaces over K, with usual tensor product and neutral element $\mathbb{K}$, the usual duality $V \mapsto V^* = \mathcal{L}(V, \mathbb{K})$ is a duality in the categorical sense with

$$e : V \otimes V^* \to \mathbb{K} \qquad x \otimes f \mapsto f(x)$$

and

$$d : \mathbb{K} \to V^* \otimes V \qquad 1 \mapsto \sum_i v_i^* \otimes v_i$$

where $(v_i)_i$ is a basis of V and $(v_i^*)_i$ is the usual dual basis.

- The category **Hlb** of Hilbert spaces with morphisms continuous linear maps with dual $H \mapsto \{f : H \to \mathbb{C} \mid f \text{ continuous and linear}\}$ is a duality.

4 Braids and Tangles

4.1 The Braid Category

4.1.1 The group

Definition 8 (*Braid Group*)

The Braid group of n-strands braids can be defined by generators and relations as follows :

$$B_n = \langle \beta_1, \cdots, \beta_{n-1} | (1) \text{ and } (2) \rangle$$

Where the properties are :

(1) for all $i < n$, one has $\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}$.

(2) for all $i, j \leq n$ such that $i - j > 1$, one has $\beta_i \beta_j = \beta_j \beta_i$.

Remark

With extra property (3) : $\forall i, \beta_i^2 = 1$, then the group becomes the permutation group $\mathfrak{S}_n$. Therefore B_n can be seen as a group of permutations for which one remembers every transposition being used : If we consider a game of three-card trick, the permutation group only focus on the final state compared to the initial one, while the braid group remembers every transposition the gamer has made.

4.1.2 Category

Definition 9 (*Braid Category* $\mathbb{B}$)

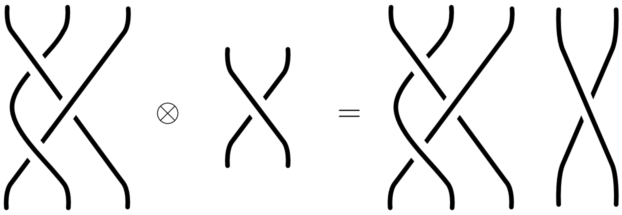
This category is the one where objects are natural numbers $0, 1, \cdots$ and Hom-sets are $\mathbb{B}(n, n) = B_n$ and $\mathbb{B}(n, m) = \emptyset$ for $n \neq m$.

4.1.3 Structure of Monoidal Category

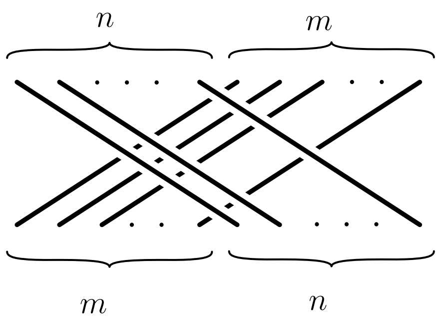
We can construct a tensor product $\otimes$ on $\mathbb{B}$:

$$\begin{array}{ccc} \mathbb{B}\times\mathbb{B} & \longrightarrow & \mathbb{B} \\ (n,m) & \mapsto & n+m \end{array}$$

With the following definition on arrows (braids) :



In this construction α, λ, ρ are identities. Moreover the braiding $\gamma_{n,m}$ is given by :

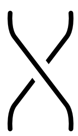


4.1.4 Freedom

Proposition 1

> The braid category $\mathbb{B}$ is the *free braided monoidal category* generated by a single object.

Let's call this object 1, the braiding $\gamma_{1,1}$ gives a braid :



which generates all braids via tensor product and composition : $\beta_i \in B_n$ can be written $1^{\otimes(i-1)} \otimes \gamma_{1,1} \otimes 1^{\otimes(n-i-1)}$ and so by definition it generates B_n.

It yields that given a braided monoidal category $(M, \otimes, I)$ with an object a, there is a unique tensor preserving functor $T : \mathbb{B} \to M$ such that $T(1) = a$.

In fact, there is a stronger result that may be stated this way :

Proposition 2

> For all braided monoidal category M, there is an equivalence of categories :
>
> $$\mathbf{B.StMon}(\mathbb{B}, M) \cong M$$
>
> Where **B.StMon** is the category of braided monoidal categories with arrows the functors preserving their structure ("St" stands for "strong"), hence $\mathbf{B.StMon}(\mathbb{B}, M)$ is the category of functors $\mathbb{B} \to M$ preserving the braiding and the tensor product.

Proof: First the statement is equivalent to the previous one with a good definition of freeness : in our case we can say that an object $f \in \mathcal{C}$ is free if one has a bijection $\mathcal{C}(f, a) \xrightarrow{\sim} \mathbf{Set}(\star, Ua)(\cong Ua)$ natural in $a \in \mathcal{C}$. Moreover this bijection can sometimes be made into an isomorphism in $\mathcal{C}$ when $\mathcal{C}(f, a)$ is an object of $\mathcal{C}$.

It is the case in **Grp** (with free object $\mathbb{Z}$), **Mon** (with $\mathbb{N}$) or $\mathbf{Vect}_{\mathbb{K}}$ (with $\mathbb{K}$) ; and also **BStMon**.

About the equivalence of category, it is given on objects by $F \mapsto F(1)$ and $\alpha \mapsto \alpha_1$ on arrows. Conversely given a $a \in M$, let's define $F : \mathbb{B} \to M$ by $F(n) = F(1^{\otimes n}) = a^{\otimes n}$.

And in order to define F on braids it suffice to describe F on β_i for all i in $\mathbb{N}$ as done before.

Those two construction are exactly inverses of each other so we have a stronger result M and $\mathbf{BStMon}(\mathbb{B}, M)$ are *isomorphic* as categories and hence equivalent. ■

4.2 Yang-Baxter operators

Definition 10 (*Yang-Baxter operator*)

A *Yang-Baxter operator* on an object a of monoidal category M is an isomorphism $y : a \otimes a \to a \otimes a$, such that the following diagram commutes (ignoring α) :

$$\begin{tikzcd} a^{\otimes 3} \arrow[r, "y \otimes 1"] \arrow[d, "1 \otimes y"] & a^{\otimes 3} \arrow[r, "1 \otimes y"] & a^{\otimes 3} \arrow[d, "y \otimes 1"] \\ a^{\otimes 3} \arrow[r, "y \otimes 1"] & a^{\otimes 3} \arrow[r, "1 \otimes y"] & a^{\otimes 3} \end{tikzcd}$$

Any braiding γ in a monoidal category gives a YB operator for each object a : $y = \gamma_{a,a}$

Definition 11 (*Category of Yang-Baxter operators [4]*)

Given a monoidal category $(M, \otimes, I)$, the category YB(M) has objects : couples $(V, y : V \otimes V \to V \otimes V)$ where V is an object of M and y a Yang-Baxter operator on V.

A morphism of Yang-Baxter operators $(V, y) \to (W, z)$ is a map $f : V \to W$ such that $(f \otimes f)y = z(f \otimes f)$

Proposition 3

$\mathbb{B}$ is the free monoidal category generated by an object bearing a non-trivial YB-operator. Which means that for any monoidal category M :

$$\mathbf{StMon}(\mathbb{B}, M) \cong \mathrm{YB}M$$

Where $\mathbf{StMon}(\mathbb{B}, M)$ is the category of tensor preserving functors from $\mathbb{B}$ to M.

Proof: The idea is the same that the previous proposition : The first part is obvious since the objects are of the form $a^{\otimes n}$ for $n \in \mathbb{N}$ and then the YB-operator $u : a \otimes a \to a \otimes a$ define a braid with the inductive definition (modulo α):

$$\begin{cases} \gamma_{a^{\otimes n}, a^{\otimes m}} & = & (1_a \otimes \gamma_{a^{\otimes n}, a^{\otimes (m-1)}})(\gamma_{a^{\otimes n}, a} \otimes 1_a^{\otimes (m-1)}) \\ \gamma_{a,a} & = & u \end{cases}$$

Now given any monoidal category M and object $a \in M$ equipped with an YB : $y : a \otimes a \to a \otimes a$, there is a unique tensor-preserving functor $Y : \mathbb{B} \to M$ such that $Y(1) = a$ and $Y(u) = y$. It maps the object $1^{\otimes n}$ to $a^{\otimes n}$ and a braid $b = \prod \bigotimes u^{\otimes i}$ to $\prod \bigotimes y^{\otimes i}$. ■

4.2.1 Representation

Application to $M = \mathbf{FinVect}_{\mathbb{C}}$. Given V a $\mathbb{C}$-vector space of finite dimension, and e_i a basis of V, then $(e_i \otimes e_j)$ is a basis of $V \otimes V$.

Let $q \in \mathbb{C}^*$, we define the following map $V \otimes V \to V \otimes V$:

$$R_q : e_i \otimes e_j \mapsto \begin{cases} q\ e_i \otimes e_j + (1-q)\ e_j \otimes e_i & \text{if } i > j \\ q\ e_i \otimes e_j & \text{if } i = j \\ q\ e_i \otimes e_j + (1-q^{-1})\ e_j \otimes e_i & \text{if } i < j \end{cases}$$

We can show that R_q is a YB-operator on V (its inverse is $R_{q^{-1}}$) and then the equivalence of categories above gives a tensor-preserving functor $\pi_q : \mathbb{B} \to \mathbf{FinVect}_{\mathbb{C}}$ sending the Hom-set B_n to a submonoid of $\mathcal{M}_n(\mathbb{C})$ which is then a group and so a subgroup of $\mathcal{G}\ell_n(\mathbb{C})$.

Therefore those YB-operators gives matrix representations of B_n.

4.3 Trace

Definition 12 (*Trace*)

Let $f : A \to A$ be an endomorphism in a braided monoidal category $(M, \otimes, I)$. We define $\mathrm{Tr}(f) : I \to I$ the *trace* of f by $\mathrm{Tr}(f) = e\gamma(1 \otimes f)d$:

$$I \xrightarrow[d]{} A^* \otimes A \xrightarrow[1 \otimes f]{} A^* \otimes A \xrightarrow[\gamma]{} A \otimes A^* \xrightarrow[e]{} I$$

Let consider, given a monoidal category $(M, \otimes, I)$ the subcategory $\mathcal{C}$ with a unique object I and $\mathrm{Hom}_{\mathcal{C}}(I, I) = \mathrm{Hom}_M(I, I)$. This category also is a monoidal category with the "induced"(modulo isomophism) tensor product from M with : $I \otimes I = I$ and $f \otimes g : I \to I$.

4.4 Tangles

The construction of the tangle category uses [3].

Definition 13 (*Geometric Tangle*)

Let P be the euclidian plane, and let consider $X = P \times [0, 1]$ seen as a subspace of $\mathbb{R}^3$. A geometric tangle is a compact oriented manifold of dimension 1 in X with its borders in $P \times \{0, 1\}$.

Example

In the figure 2, the border of the tangle (dots) can be labelled $+$ or $-$ with the induced orientation from the tangle in $P \times \{1\}$ and the opposite from the induced orientation in $P \times \{0\}$. (cf. Figure 3)

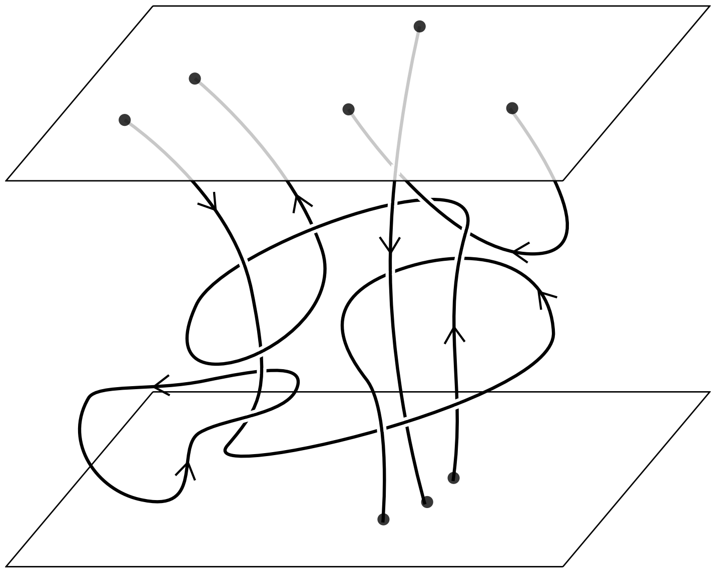


Figure 2: An example of geometric tangle

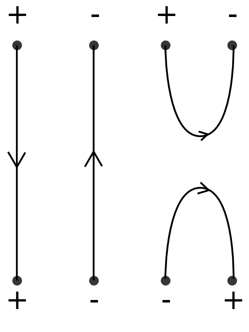


Figure 3: Example of orientation-labelling of the point

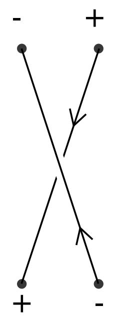
Definition 14 (*Tangle*)

A *tangle* is an equivalence classe of geometric tangles up to homotopy equivalences with fixed extremities and where the extremities of each side are aligned and equidistant (resp. in $P \times \{0\}$ and $P \times \{1\}$).

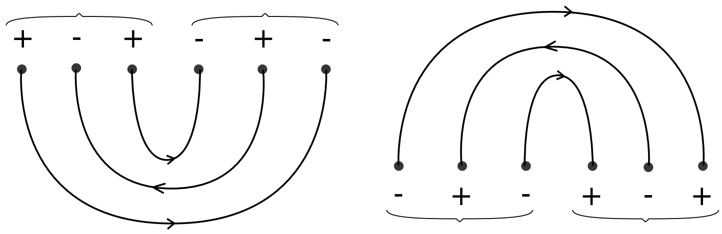
Definition 15 (*Autonomous braided tensor category of tangles*)

Let $\mathcal{T}$ be the category with objects words on the alphabet $\{+, -\}$ and arrows tangles between those.

Tensor product and composition are as for braids. The braiding is given by :



And the *autonomous structure* (existence and unicity up to isomorphisms of adjoints (simultaneously left and right adjoints)) is given by e and d as follows :



where the dual of $(+-+)$ is $(-+-)$.

Remark

- Tangles exists only between objects with the same parity of length and if satisfying some conditions stating there is always a source and a target for each path.
- $\mathcal{T}(\emptyset, \emptyset)$ is the set of oriented links (i.e. knots using several strings)
- The braid category can be seen as a subcategory of $\mathcal{T}$ with $n \mapsto +^{\otimes n}$.
- The trace of a tangle (in particular of a bread) is the Markov closure (cf. Figure 4) which is an oriented link.

Now given a tensor preserving functor $F : \mathcal{T} \to \mathbf{FVect}_{\mathbb{C}}$, it maps oriented links $\mathrm{Hom}_{\mathcal{T}}(\emptyset, \emptyset)$ to complex numbers (in fact elements of $\mathrm{Hom}_{FVect_{\mathbb{C}}}(\mathbb{C}, \mathbb{C})$), thus giving invariant of links.

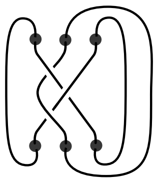


Figure 4: Markov closure of a braid (using its embedding in $\mathcal{T}$)

5 Conclusion

The Penrose notation applied to monoidal categories makes it totally explainable with braids. Similarly, Tangles must have a freeness property among differently structured categories than braided monoidal ones. The question is what should this structure be (stronger or weaker than braided monoidal) because the free vector space on a set is "smaller" than the free group which is "smaller" than the free monoid, but the free algebra is more complex because it already involves the free group.

The understanding of those free elements (and mainly the free element over a one-object set) is very fruitful for picturing the structure of those categories. However, it demands that the freeness property can be expressed in a much stronger way that the usual adjunction in a category $\mathcal{C}$ with a forgetful functor $U : \mathcal{C} \to \mathbf{Set}$:

$$\mathrm{Hom}_{\mathcal{C}}(f, a) \cong \mathrm{Hom}_{\mathbf{Set}}(\star, Ua)$$

Part II

Exercises from *Categories for the Working Mathematician* [5]

6 Categories, Functors and Natural Transformations

6.1 Exercises on sections 1, 2 and 3 p.15

6.1.1 Usual constructions seen as functors

Exercise

Show how each of the following constructions can be regarded as a functor:

- The field of quotient of an Integral domain
- The Lie-Algebra of a Lie-Group

1) Let **C** be the Category of integral domains but **with injective mappings only**. And consider the map : Frac:$A \mapsto Frac(A) = A \times A^*/\sim$. Where $(a, b) \sim (c, d)$ if and only if $ad = bc$. We denote the class of (a, b) by a/b.

If B is in **C** and $f : A \hookrightarrow B$ then let $x = a/b$ be in $Frac(A)$ we define $Frac(f)(x) = f(a)/f(b)$. As $b \neq 0_A$ then $f(b) \neq 0_B$ and $Frac(f)$ is well-defined.

In fact as f is one-to-one into, and as $B \hookrightarrow Frac(B)$ then there is an injection $A \hookrightarrow Frac(B)$, but by the universal property of the field of fractions, there is also a one-to-one into map $Frac(A) \to Frac(B)$ such that the unwritten diagram commutes.

Anyway the result is false for the whole category of integral domains : Let $A = \mathbb{Z}$ and $B = \mathbb{Z}/7\mathbb{Z}$, then there exists a morphism $f : A \to B$ but there is no morphism from $\mathbb{Q} = Frac(A)$ to $B = Frac(B)$ because ring-morphisms on fields need to be injective.

2) Let **LieG** be the category of lie-groups, with arrow the differentiable group homomorphisms.

Given a Lie group G of dimension n, one can consider the space $I(G)$ of vector fields on G. It is a lie-algebra with bracket the commutator of vector fields.

Given a morphism of lie-groups $f : G \to H$, it yields an application $F : I(H) \to I(G)$ given by $F(X) = X \circ f$. This is therefore a morphism of lie algebras.

Given an element $g \in G$, the left translation by g, $l_g : G \to G$ gives an automorphism L_g of $I(G)$. The set of fixed points under all these automorphisms is the lie algebra of G, denoted $L(G)$.

Consequently again if $f : G \to H$ is a morphism of lie groups, $F : I(H) \to I(G)$ is a morphism of lie algebras and moreover it commutes with the translations in the following sense : $L_g F = F L_{f(g)}$. Then it induce a morphism of lie algebras $L(F) : L(H) \to L(G)$.

As composition is respected, L is a (contravariant) functor from the category of lie groups the one of lie algebras.

6.1.2 Functors from the first ordinals

Exercise

> Show that functors $\mathbf{1} \to \mathbf{C}$, $\mathbf{2} \to \mathbf{C}$ and $\mathbf{3} \to \mathbf{C}$ correspond respectively to objects, arrows and composable pairs of arrows in $\mathbf{C}$.

1) Let consider $\varphi : \mathrm{Hom}(\mathbf{1}, \mathbf{C}) \longrightarrow \mathbf{C}$ defined by $\varphi(F) = F(\mathbf{0})$ where $\mathbf{0}$ is the unique object of $\mathbf{1}$. This is obviously a bijection, as a functor $F \in \mathrm{Hom}(\mathbf{1}, \mathbf{C})$ is totally defined by the datum of $F(\mathbf{0})$, and that for all objects $x \in \mathbf{C}$ we can consider a functor $F : \mathbf{1} \to \mathbf{C}$ with $F(\mathbf{0}) = x$

2) Let denote A the set of arrows of C, and $\mathbf{2} : \mathbf{0} \to \mathbf{1}$, where the arrow is called f. And let consider $\varphi : \mathrm{Hom}(\mathbf{2}, \mathbf{C}) \longrightarrow A$ defined by $\varphi(F) = F(f)$.

We can prove the bijection by constructing the converse function $\psi : A \longrightarrow \mathrm{Hom}(\mathbf{2}, \mathbf{C})$ defined by for all $g \in A$, we create the functor F on $\mathbf{2}$ with $F(\mathbf{0}) = \mathrm{domain}(g)$, $F(\mathbf{1}) = \mathrm{codomain}(g)$ and $F(f) = g$.

3) With the same notations $\mathbf{3} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ with morphisms $f : \mathbf{0} \to \mathbf{1}$ and $g : \mathbf{1} \to \mathbf{2}$ and $B \subset A \times A$ the set of composable pairs of arrows.

Let $\varphi : \mathrm{Hom}(\mathbf{3}, \mathbf{C}) \longrightarrow B$ defined by $\varphi(F) = (F(g), F(f))$.

The converse function is given by $(u, v) \mapsto F$ where $F(\mathbf{0}) = \mathrm{domain}(v)$, $F(\mathbf{1}) = \mathrm{codomain}(v) = \mathrm{domain}(u)$ as $u \circ v$ exists and $F(\mathbf{2}) = \mathrm{codomain}(u)$ with $F(f) = v$ and $F(g) = u$.

6.1.3 Functor interpretation

Exercise

> Interpret "functor" in different contexts :

a) Lets $(I, \leq_I)$ and $(J, \leq_J)$ be two preordered sets seen as categories ($a \leq b$ **ssi** $\exists f : a \to b$).

Let T be a functor between I and J, then $a \leq_I b$ involve the existence of $T(f) : T(a) \to T(b)$ i.e. $T(a) \leq T(b)$. Therefore a functor on those type of categories is always a monotonic function.

b) Let G, H be two groups seen as categories of 1 element (namely $\star_G$ and $\star_H$).

And T a functor between $\mathbf{G}$ and $\mathbf{H}$, its associated map on morphisms is a function $T : G \to H$ satisfying $T(g \circ g') = T(g) \circ T(g')$ and $T(e_G) = e_H$. It is then a morphism of monoids on groups thus a morphism of groups.

Conversely, a morphism of groups from G to H permit to build a functor between $\mathbf{G}$ and $\mathbf{H}$ by just adding the obvious object function.

c) A functor $T : G \to \mathbf{Set}$ gives a permutation representation of G while a functor $G \to \mathbf{Matr}_K$ gives a matrix representation.

In the first case, let's call K the set image of the element "$\star_G$" by T. Then $\mathrm{End}_{\mathbf{Set}}(K)$ has a structure of monoid and (as show in 6.1.3) T induce a morphism of monoids from G to $\mathrm{End}_{\mathbf{Set}}(K)$, as G is a group, the image is a group then a subset of $\mathfrak{S}(K)$. And so we've got a morphism of groups $G \to \mathfrak{S}(K)$ i.e. an action of G on K.

In the matrix case, let fix $n = T(G)$, then the same idea work : T is a morphism of monoids between G and $\mathrm{End}_{\mathbf{Matr}}(n) = \mathcal{M}_n(K)$, as we've got groups it could be restricted to a morphism $G \to GL_n(K)$ of groups.

6.1.4 Functor center on groups does not exists

Exercise

Prove that there is no functor **Grp** $\to$ **Ab** sending each group to its center.

Let suppose that T satisfy this condition.

$\mathfrak{S}_3$ acts on $\mathfrak{S}_3\,/H$ by left translation where $H =< (1,2,3) >$. Then it gives a morphisms $\mathfrak{S}_3 \to \mathfrak{S}_2$. Moreover the induced action of $\mathfrak{S}_2$ (through the inclusion morphisms) is the identity.

Therefore we've got the following diagram :

$$\begin{tikzcd}
\mathfrak{S}_2 \arrow[r] \arrow[d, mapsto, "T"] & \mathfrak{S}_3 \arrow[r] \arrow[d, mapsto, "T"] & \mathfrak{S}_2 \arrow[d, mapsto, "T"] \\
\mathfrak{S}_2 \arrow[r] & \{e\} \arrow[r] & \mathfrak{S}_2
\end{tikzcd}$$

And we've got the upper diagram giving the identity arrow $\mathfrak{S}_2 \to \mathfrak{S}_2$, but the functor T maps this arrow to the nil arrow $\theta : \mathfrak{S}_2 \to \mathfrak{S}_2$, which is absurd.

Therefore T does not exists.

6.1.5 Different functors on Grp

Exercise

Find two different functors T : **Grp** $\longrightarrow$ **Grp** with object function the identity : $T(G) = G$ for every group G.

The identity functor obviously suits.

Then let's fix for each group G an element r_G (axiom of choice on categories). Then let's consider the function on arrows on **Grp** given by $T : (f : G \to H) \mapsto (int_{r_H} \circ f \circ int_{r_G^{-1}})$. Where $int_a : g \mapsto aga^{-1}$.

Then we can see that $T(\mathrm{id}_G) = int_{r_G} \circ int_{r_G^{-1}} = \mathrm{id}_G$. And moreover $T(f \circ g) = int_{r_K} \circ (f \circ g) \circ int_{r_G^{-1}} = int_{r_K} \circ f \circ int_{r_H^{-1}} \circ int_{r_H} \circ g \circ int_{r_G^{-1}} = T(f) \circ T(g)$.

The same result can be proved if one chooses, instead of $(r_G)_G$, a family f_G for all groups G of automorphisms of G.

6.2 Exercises on section 4 : Natural transformations p.18

6.2.1 Natural transformation on the functor "functions from S to _"

Exercise

Let S be a fixed set. Show that $X \mapsto X^S$ is the object function of a functor **Set** $\to$ **Set**, and that evaluation $e_X : X^S \times S \to X$ defined by $e_X(h, s) = h(s)$, the value of the function h at s, is a natural transformation.

I assume that the second part of the question is in fact : let $s \in S$ fixed. Show that $e_X : h \mapsto h(s)$ is a natural transformation between the functor $X \mapsto X^S$ and the identity on **Set**.

Let $f : X \to Y$, then we define $S(f) : X^S \to Y^S$ by $(\varphi : S \to X) \mapsto (f \circ \varphi)$.
It thus give a functor structure to the function $X \mapsto X^S$.

Let fix $s \in S$ (if S is the empty set, the above functor is the final functor $X \mapsto 0$ and then it coincides with the natural transformation... I guess)
Then consider the transformation $e_X : X^S \to X,\ h \mapsto h(s)$. It is natural as if $f : X \to Y$ then $(e_Y \circ S(f))(h) = (f \circ h)(s) = f(e_X(h)) = (f \circ e_X)(h)$.

6.2.2 Functor $H \times \_$ on groups

Exercise

> If H is a fixed group, show that $G \mapsto H \times G$ defines a functor $(H\times\_) : \mathbf{Grp} \to \mathbf{Grp}$ and that any morphism $f : H \to K$ gives a natural transformation $(H\times\_) \to (K\times\_)$.

Let $u \in \mathrm{Hom}_{\mathbf{Grp}}(G, G')$, we construct the morphism $(\mathrm{Id}_H \times u) : H \times G \to H \times G'$ as usual : $(h, g) \mapsto (h, u(g))$. This give the arrow function of the functor $(H\times\_)$.

Let consider $f : H \to K$. For G we define $f_G : (H \times G) \to (K \times G)$ by $f_G(h, g) = (f(h), g)$ i.e. $f_G = f \times \mathrm{Id}_G$.
Then the family $(f_G)_G$ is a natural transformation : For $u \in \mathrm{Hom}_{\mathbf{Grp}}(G, G')$, let's call $v = (H\times\_)(u)$; then $(v \circ f_G)(h, g) = v(f(h), g) = (f(h), u(g)) = f_{G'}(h, u(g)) = (f_{G'} \circ v)(h, g)$.

6.2.3 Functor on one-object category groups

Exercise

> If B and C are groups (regarded as categories with one object each) and $S, T : \mathbf{B} \to \mathbf{C}$ are functors (then morphisms, see 6.1.3), show that there is a natural transformation $S \to T$ iff S and T are conjugate in C.

There is a natural transformation from S to T iff there is a $\alpha : \star_C \to \star_C$ such that for all $f \in B$, one has $\alpha \circ S(f) = T(f) \circ \alpha$ which means, if we consider α as an element of C, then $\forall f \in B, \alpha S(f)\alpha^{-1} = T(f)$. It show thus the equivalence between being conjugated and the existence of a natural transformation.

6.2.4 Inequalities between functors on preorders

Exercise

> For functors $S, T : \mathbf{C} \to P$ where P is a preorder, show that there is a natural transformation $S \to T$ (which is then unique ?) iff $Sc \leqq Tc$ for each $c \in \mathbf{C}$.

The unicity results of the fact that for any two element a, b of P there exist at most one morphism from a to b. Therefore is there is a natural transformation two such transformation would have the same arrows.

Suppose S and T are functors and $\exists c \in \mathbf{C}$ such that $S(c) \nleq T(c)$ then there is no morphism $\alpha_c : S(c) \to T(c)$ and so there cannot exist a natural transformation between S and T.

Conversely, in a preorder, any diagram immediately commutes (because of the unicity of the arrows), so as soon as the arrows exists, there is a natural transformation.

6.2.5 Natural transformations and functions on arrows

Exercise

> Let $S, T : \mathbf{C} \to \mathbf{B}$. Show that every natural transformation $\tau : S \to T$ defines a function (also called τ) which sends each arrow $f : c \to c'$ of $\mathbf{C}$ to an arrow $\tau f : S(c) \to T(c')$ of $\mathbf{B}$ such that $Tg \circ \tau f = \tau(gf) = \tau g \circ Sf$ for each composable pair (g, f).
>
> Conversely show that every such function on arrows τ comes from a unique natural transformation with $\tau_c = \tau(id_c)$

The first statement does derive from definition, as if $f : c \to c'$ then the arrow $\tau_{c'} \circ S(f) = T(f) \circ \tau_c : S(c) \to T(c')$ can be called τf as it only depends on f.

And then, for a pair of composable arrows $(g, f) : c \to c' \to c''$, one has $\tau(gf) = T(gf)\tau_c = T(g)T(f)\tau_c = Tg\tau f$ and the same way, one can show $\tau g Sf = \tau(gf)$.

Conversely, if τ is a function on arrows, such that $Tg \circ \tau f = \tau(gf) = \tau g \circ Sf$ then let call $\tau_c = \tau(\mathrm{id}_c)$

If $f : c \to c'$ then $Tf\tau_c = \tau(f\mathrm{id}_c) = \tau(\mathrm{id}_{c'}f) = \tau_{c'}Sf$. Therefore τ is a natural transformation $T \stackrel{\cdot}{\longrightarrow} S$.

6.2.6 Skeleton of the category of finite-dimensional vector spaces over F

Exercise

> Let F be a field (fixed). Show that the category of all finite-dimensional vector spaces over F (with linear transformations as morphisms) written $\mathbf{C}$ is equivalent to the category $\mathbf{Matr}_F$.

First of all, let fix for all $E \in \mathbf{C}$ a basis B_E (axiom of choice over categories !)

Let consider the functor $T : \mathbf{C} \to \mathbf{Matr}_F$ defined by $T(E) = \dim(E)$ and for each linear transformation $f : E \to F$ let associate the matrix of f in the basis B_F and B_E.

Then we consider the "inverse" functor $S : \mathbf{Matr}_F \to \mathbf{C}$ with $S(n) = F^n$ and $S(M) = F^n \ni X \mapsto MX \in F^m$ for any matrix M "from" n "to" m.

Therefore the composition $S \circ T$ associate any vector space of dim n with the space F^n, and any linear function to the linear function on the coords in the respective basis.

Then the isomorphism given for a fixed E of dim n between $(S \circ T)(E) = F^n$ and E is a natural isomorphism between $S \circ T$ and Id.

While $T \circ S$ is the identity functor, as soon as we choose the basis B_E to be the canonical one if $E = F^n$.

This is therefore an equivalence of categories.

6.3 Exercises on section 5 : Monics, Epis and Zeros p.21

6.3.1 Example of non equivalence monic and epi with bijectivity

Exercise

Find an arrow which is both epi and monic but not invertible.

Let consider $f : \mathbb{Q} \hookrightarrow \mathbb{R}$ in Top. This is a monic as it is injective in Top and that in this category monic and injective morphisms coincide.

Moreover let suppose $u \circ f = v \circ f$ then u and v are continuous functions from $\mathbb{R}$ that coincide on a dense subset namely $\mathbb{Q}$. Therefore they are equals, so f is an epi.

Finally it is obvious that f is no an iso as it would in particular be a bijection between $\mathbb{Q}$ and $\mathbb{R}$ which is impossible.

6.3.2 Properties on arrows

Exercise

Prove that the composite of monics is monic and likewise for epis.

Let (f, g) be a pair of monic composable arrows. Let h_1, h_2 be two arrows from A to $\text{domain}(g) = \text{domain}(f \circ g)$ such that $(f \circ g) \circ h_1 = (f \circ g) \circ h_2$ then the associative condition gives $f \circ (g \circ h_1) = f \circ (g \circ h_2)$ therefore $g \circ h_1 = g \circ h_2$ and then $h_1 = h_2$.

The second result is given by applying this one in the opposite category (we just have to make sure that this result is not used in the construction of the dual category).

6.3.3 Similarity with usual notions

Exercise

If a composite $g \circ f$ is monic, so is f. What about g ?

Let h_1, h_2 two arrows from the same object to $\text{domain}(f)$. Then suppose $f \circ h_1 = f \circ h_2$, then $(g \circ f) \circ h_1 = g \circ (f \circ h_1) = g \circ (f \circ h_2) = (g \circ f) \circ h_2$. So as $(g \circ f)$ is monic, $h_1 = h_2$.

g is not necessarily monic : example in **Set** where monic is equivalent to one-to-one into, $f : \mathbb{N} \to \mathbb{Z}$ the usual inclusion, and $g : \mathbb{Z} \to \mathbb{N},\ x \mapsto |x|$; then g is not one-to-one, but $g \circ f$ is injective as it is the identity.

6.3.4 Epi non surjective

Exercise

Show that the inclusion $i : \mathbb{Z} \to \mathbb{Q}$ is epi in **Rng**.

Suppose h_1, h_2 are two ring morphisms from $\mathbb{Q}$ to a ring A such that $h_1 \circ i = h_2 \circ i$.

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \rightrightarrows A$$

Let distinguish two cases :

- $A = \{0\}$ then $h_1 = h_2$
- $A \neq \{0\}$, then h_1 and h_2 are injective maps as morphisms of rings from field. Then there is an injection $\mathbb{Z} \hookrightarrow A$ and so A has nil characteristic.

  Let $x = p/q \in \mathbb{Q}^*$, therefore $qh_1(x) = h_1(p) = h_2(p) = qh_2(x)$ and by simplifying by q one has shown $h_1 = h_2$.

However, let consider:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook, "\delta"] & \mathbb{Z} \times \mathbb{Z} \arrow[r, bend left, "P"] \arrow[r, bend right, "Q"'] & \mathbb{Z} \end{tikzcd}$$

Here is an example with an injective map δ from $\mathbb{Z}$, the the initial property of $\mathbb{Z}$ in the category of ring compels $P\delta = Q\delta$ but $P \neq Q$.

6.3.5 Epi are surjective on groups

Exercise

In **Grp** prove that every epi is surjective.

Let consider $f : G \to H$ epi and M be the image of G under f, a subgroup of H

Let $X = M \backslash H = \{Mh \mid h \in H\}$ the set of right-classes of H under M-equivalence. There is an action of H on this set, by left translation, this gives a morphism $a : H \to \mathfrak{S}(X)$. One can easily see that $af = G \to \mathbf{1} \to \mathfrak{S}(X)$ is the nil morphism from G to $\mathfrak{S}(X)$.

Then if $b : H \to \mathbf{1} \to \mathfrak{S}(X)$, one has $bf = af$, thus $a = b$ as f is epi.

So the left-translation action of H on X is the identity. Then $\forall x \in H, x(Me) = (Me)$ in particular $x = xe \in Me$ so $H \subset M$ i.e. $H = M$ which means f is surjective.

6.3.6 Idempotent in Set

Exercise

In **Set** show that every idempotent split.

Let $f : A \to A$ be an arrow in **Set** such that $f \circ f = f$.

Let C denote the set $\{f(x) \mid x \in A\}$, then f is the identity on C : if $y \in C$ then $y = f(x)$ and so $f(y) = f(f(x)) = f(x) = y$.

Let consider $g : A \to C$ to be equal to f pointwise and $h : C \to A$ to be the inclusion injection. Then $h \circ g = f$ by construction and $g \circ h = \mathrm{id}_C$ as proved above.

6.3.7 Regularity in Set

Exercise

In a category C, an arrow $f : a \to b$ is said *regular* if there exists an arrow $g : b \to a$ such that $fgf = f$.

Show that if f as either a right or a left inverse then it is regular.

Show that every arrow in **Set** is regular.

if $fg = \mathrm{id}_b$ then $fgf = f$, and similarly if $gf = \mathrm{id}_a$ then $fgf = f$.

Let $f : a \to b$ be an arrow in **Set** then using the axiom of Choice, let chose for each y in the image of f an antecedent $g(y) \in a$. This g is an arrow from $\mathrm{Im}\ f$ to a, it can be extended to an arrow form b to a this way : Let $\alpha \in a$, then define

$$\tilde{g} : y \mapsto \begin{cases} g(y) & \text{if } y \in \mathrm{Im} f \\ a & \text{otherwise} \end{cases}$$

We can then easily prove that $(f, \tilde{g})$ and $(\tilde{g}, f)$ are composable pairs of arrows and that $f\tilde{g}f = f$.

6.3.8 Initial object

Exercise

> Consider the category $\mathbf{C}$ with objects (X, e, t), where X is a set, $e \in X$, and $t : X \to X$, and with arrows $f : (X, e, t) \to (X', e', t')$ the functions $f : X \to X'$ such that $fe = e'$ and $ft = t'f$.
>
> Prove this category has an initial object in which X is the set of natural numbers, $e = 0$ and t is the successor function.

$O = (\mathbb{N}, 0, s)$ is an object of $\mathbf{C}$.

Let $\chi = (X, e, t) \in \mathbf{C}$, one wants to show that there is an unique morphism $f : O \to \chi$.

If $f : O \to \chi$, then $f(0) = e$ and thus by induction $f(n) = fs^n(0) = t^n f(0) = t^n(e)$. Conversely this define a arrow of $\mathbf{C}$ which is then unique.

6.3.9 Usual property II

Exercise

> If a functor $T : C \to B$ is faithful and Tf is monic, prove f monic.

Consider $f : c \to c'$, and let $g, h : a \to c$ such that $fg = fh$, then $TfTg = TfTh$, as Tf is monic : $Tg = Th$ and as T is faithful such equality can only occurs if $g = h$. Thus f is monic.

6.4 Exercises on section 6 : Foundations p.24

6.4.1 Memorandum

A universe U is a set satisfying the following properties :

(i) $x \in u \in U$ implies $x \in U$.

(ii) $u, v \in U$ imply

  (a) $\{u, v\} \in U$

  (b) $(u, v) \in U$

  (c) $u \times v \in U$

(iii) $x \in U$ implies

 (a) $\mathcal{P}(x) \in U$

 (b) $\cup x \in U$

(iv) $\mathbb{N} = \omega \in U$

(v) if $f : a \to b$ is a surjective function with $a \in U$ and $b \subset U$ then $b \in U$.

6.4.2 Cartesian product

Exercise

Given a universe U and a function $f : I \to b$ with domain $I \in U$ and with $\forall i \in I, f_i \in U$, prove that the usual cartesian product $\prod_i f_i$ is an element of U.

1. As $\forall i \in I f_i \in U$; $x = \{f_i \mid i \in I\} \subset U$.
2. Prop. (iii)-(b) and 1. give $\{w \mid \exists i \in I, w \in f_i\} = \cup x \in U$.
3. Prop. (ii)-(c), 2. and $I \in U$ give $I \times \cup x \in U$.
4. Prop. (iii)-(a) and 3. give $\mathcal{P}(I \times \cup x) \in U$.
5. As $I \in U$ then $\cup x \in U$ by applying twice prop. (ii)-(a) with $u = v$ one gets $\{I\}, \{\cup x\} \in U$.
6. Prop. (ii)-(c) used twice and 4., 5. give $A = \{(I, \Gamma, \cup x) \mid \Gamma \subset I \times \cup x\} = \{I\} \times \mathcal{P}(I \times \cup x) \times \{\cup x\} \in U$.
7. Prop. (iii)-a and 6. give $\mathcal{P}(A) \in U$.
8. Then let $y \in \prod f_i$, by definition $y = (I, Y, \cup x)$ with $Y \subset I \times \cup x$ (and such that... –various properties–), thus $y \in A$, therefore $\prod f_i \subset A$ i.e. $\prod f_i \in \mathcal{P}(A)$.
9. Prop (i), 7. and 8. give $\prod f_i \in U$.

6.4.3 Equivalence of definitions of universe

Exercise

(α) Given a universe U and a function $f : I \to b$ with domain $I \in U$, show that the usual union $\cup_i f_i$ is a set of U.

(β) Show that this property (α) may replace (v) and (iii)-(b) in the definition of a universe.

(α) : The result as stated is false in general... maybe one needs to assume $b \subset U$.

Example : Let $b = \{\mathcal{P}(U)\}$ and $I = \{0\} \in \omega \in U$, then $f : 0 \mapsto \mathcal{P}(U) = f_0$ verify $\cup_i f_i = f_0 = \mathcal{P}(U) \not\subseteq U$.

Proof with hypothesis $b \subset U$ and with conclusion $\cup_i f_i \in U$ (being "a set of U" is understood as "an element of U"). The statement (α) is then :

If $f : I \to b$ with $b \subset U$ and $I \in U$, then $\cup_i f_i \in U$.

1. let consider the function $g : I \to \{f_i \mid i \in I\} = x$ defined by $g(i) = f(i)$ for all $i \in I$. This function is by definition surjective.
2. then as $x \subset b \subset U$, prop. (v) gives $x \in U$.
3. Prop. (iii)-(b) and 2. thus give $\cup_i f_i = \cup x \in U$.

(β) : The proof above only uses props (v) and (iii)-(b) of the definition of an universe. Then those two implies prop. (α).

Conversely, if $f : I \to b \subset U$ is surjective and $I \in U$, then (α) gives that $\cup_i f_i \in U$ and yet $b = \{f_i \mid i \in I\}$ so $\cup b \in U$. And then $b \in \mathcal{P}(\mathcal{P}(\cup b)) \in U$ using property (iii)-(a). So $b \in U$.

And if $x \in U$ one wants to prove that $\cup x = \{z \mid \exists y \in x, z \in y\}$ is an element of U but by considering the application $f : x \to U$ given by $y \mapsto y \in x \in U$, it appears that $\cup_y y \in U$ i.e. $\cup x \in U$.

7 Constructions on Categories

7.1 Exercises on section 1,2 and 3 p.39

7.1.1 Special cases of products of categories

Exercise

> Show that the product of categories coincides with the usual product of Monoids, Groups (categories with one element) and sets (discrete categories).

Monoids Let M, N be monoids, the product category has a unique object $\star = (\star_M, \star_N)$, and $\hom(\star, \star) = \hom(\star_M, \star_M) \times \hom(\star_N, \star_N) = M \times N$.

It is thus also a category with one element, and the hom-set has a neutral element $e = (e_M, e_N)$ and an internal composition law $\circ : (M \times N) \times (M \times N) \to M \times N$ given by $(f, g) \circ (u, v) = (fu, gv)$. The hom-set is therefore isomorphic to the product monoid.

Groups The construction is the same as above, except that one has a function i on the hom-set, which satisfy $if \circ f = f \circ if = e$ given by $i(f, g) = (f^{-1}, g^{-1})$.

The structure of the hom-set is therefore the same as the product group.

Sets The result could be stated as following :*The product of two discrete categories is a discrete category*; as it is obvious that then the underlying set of the product is the set-product of the previous categories object sets.

And the proof of the result is given by : let f be an arrow in the product category, then $f = (f_a, f_b)$ but as f_a and f_b are arrows of discrete categories, then $f_a = \mathrm{id}_a$ and $f_b = \mathrm{id}_b$ and thus $(f_a, f_b) = \mathrm{id}_{(a,b)}$.

7.1.2 Product of preorders

Exercise

> Show that a product of two preorders is a preorder.

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two preorders. A category is a preorder if and only if for each couple of objects (a, b) there exists at most one arrow $f : a \to b$.

Then let consider the product category, suppose there is two arrows $f = (f_P, f_Q), g = (g_P, g_Q) : (p, q) \to (p', q')$ then f_P and g_P are two arrows in $\mathbf{P}$ between p and p' i.e. $f_P = p \leq_P p' = g_P$. By the same way, one can show $f_Q = g_Q$ and therefore $f = g$.

7.1.3 Product of a family of categories

Exercise

> If $(\mathbf{C}_i)_{i\in I}$ is a family of categories indexed by a set I, describe the product $\mathbf{C} = \prod_i \mathbf{C}_i$, its projections $P_i : \mathbf{C} \to \mathbf{C}_i$ and establish the universal property of these projections.

The object set of $\mathbf{C}$ is the usual cartesian product of the object sets of the $\mathbf{C}_i$, and an arrow $f : (c_i)_i \to (c'_i)_i$ is a family of arrow $(f_i)_i$ which for each $i \in I$; $f_i : c_i \to c'_i$.

One may summarise this as $\mathrm{Obj}(\mathbf{C}) = \prod_i \mathrm{Obj}(\mathbf{C}_i)$ and $\forall (c_i)_i, (c'_i)_i \in \mathbf{C}$ the hom-set is given by $\hom(c, c') = \prod_i \hom(c_i, c'_i)$.

Then $P_i : \mathbf{C} \to \mathbf{C}_i$ is a functor given by $P_i((c_j)_j) = c_i$ on objects and $P_i((f_j)_j) = f_i$ on arrows.

The family of functors $(P_i)_i$ thus satisfy the following property :

Given any category D with a family of functors $(Q_i)_{i\in I}$ where $Q_i : D \to \mathbf{C}_i$, there exists a unique functor $R : D \to \mathbf{C}$ such that $\forall i \in I,\ P_i R = Q_i$.

R is given by $d \mapsto (Q_i d)_i$ on objects and $f \mapsto (Q_i f)_i$ on arrows. And it is unique as one can show by contrapositive : if $Q : D \to \mathbf{C}$ is not R, then there exists an object or an arrow a in D such that $Qa = q \neq r = Ra$, i.e. there exists $i \in I$ such that $q_i \neq r_i$ therefore $P_i Qa \neq P_i Ra = Q_i a$ and so Q does not satisfy the property.

7.1.4 An example of opposite category

Exercise

> Describe the opposite of the category $\mathbf{Matr}_K$.

In this category, objects are integers as in $\mathbf{Matr}_K$. But a morphism $n \to m$ is a matrix $A : K^m \to K^n$ and the composition of two morphisms is given by $A \circ_{op} B = BA$ where the second is the usual matrix product.

But it can be seen as the category in which $\mathrm{Hom}(n, m) = \{A^T \mid A \text{ matrix from } n \text{ to } m\}$ and $A^T \circ_{op} B^T = (BA)^T$

7.1.5 Functor of the set of continuous real-valuated functions

Exercise

> Show that the ring of continuous real-valuated function on a topological space is the object function of a contravariant functor on **Top** to **Rng**.

For $X \in$ **Top** let denote the ring $\mathcal{C}(X, \mathbb{R})$. $\mathcal{C}(\cdot, \mathbb{R})$ is going to be the object function of the desired functor.

Then for $f : X \to Y$ a continuous map, one can define a ring-morphism $\tilde{f} : \mathcal{C}(Y, \mathbb{R}) \to \mathcal{C}(X, \mathbb{R})$ given by $\tilde{f}(\varphi) = \varphi \circ f$.

This function of $\tilde{f}$ is a ring-morphism as for $\varphi, \psi \in \mathcal{C}(Y, \mathbb{R})$, $\tilde{f}(\varphi \cdot \psi) = (\varphi \cdot \psi) \circ f = (\varphi \circ f) \cdot (\psi \circ f) = \tilde{f}(\varphi) \cdot \tilde{f}(\psi)$. And moreover $f \mapsto \tilde{f}$ does conserve identities.

Then it's the arrow function of the functor $\mathcal{C}(\cdot, \mathbb{R})$.

7.2 Exercises on section 4 : Functor Categories p.42

7.2.1 The category of R-modules

Exercise

> For R a ring, describe R-**Mod** as a full subcategory of the functor category $\mathbf{Ab}^{\mathbf{R}}$.

The category **R**,is the datum of 1 object $\star$, and a hom-set : $\mathrm{Hom}(\star, \star) = R$, where multiplication is composition. Then a functor is (as in the group case) a morphism for the multiplication, but not necessarily a morphism for the addition.

Let M be a R-module, there is a ring-morphism $\varphi : R \to End(M)$. One can associate this morphism with the functor $F : \mathbf{R} \to \mathbf{Ab}$ where $F(\star) = M$ and $\forall f \in R$, $Ff = \varphi(f)$. This is a functor because $\varphi(1_R) = id_M$ as compelled by unitary ring-morphisms and $F(fg) = \varphi(fg) = \varphi(f) \circ \varphi(g)$.

This association is then an injection as if two R-module are giving the same functor, they have the same abelian group as basis and moreover the R-action is given by the same ring-morphism.

Now let consider a morphism of R-modules : $f : m \to n$ such that $\forall r \in R,\ \varphi_n(r) \circ f = f \circ \varphi_m(r)$ (ie $r.f(x) = f(r.x)$)

Then if we consider S, T the respectively associated functors, one can see that f gives a natural transformation $S \dot{\longrightarrow} T$ as :

$$\begin{tikzcd}
m \arrow[r, "f"] \arrow[d, "S(r)"] & n \arrow[d, "T(r)"] \\
m \arrow[r, "f"] & n
\end{tikzcd}$$

is commutative.

Moreover, for any natural transformation $S \dot{\longrightarrow} T$, it is a morphism of abelian groups from $m = S(\star)$ to $n = T(\star)$ satisfying $S(r) \circ f = f \circ T(r)$ for all $r \in R$. Then the functor defined above $\mathbf{R}$-Mod $\to \mathbf{Ab}^{\mathbf{R}}$ is fully faithful.

7.2.2 Category of functors on a discrete category

Exercise

> Describe B^X for X a finite set (seen as a finite discrete category).

A functors $F : X \to B$ is totally determined by a function $F : X \to B$ i.e. a family $(b_x)_x$ of elements of B.

Let call n the cardinal of X, and φ be a bijection $\varphi : X \to n$. Then instead of writing $(b_x)_x$ one may write $(b_k)_{1\leq k\leq n}$.

A natural transformation between two such functors S, T is a family $(f_k)_{1\leq k\leq n}$ of arrows where $f_k : Sx_k \to Tx_k$ of B verifying no other condition.

Then functor category has object-set $\mathrm{Obj}(B)^n$ and Hom-sets $\mathrm{Hom}((b_k)_k, (b'_k)_k) = \prod_{1\leq k\leq n} \mathrm{Hom}(b_k, b'_k)$.

7.2.3 Category of graded abelian groups

Exercise

> Let $\mathbb{N}$ be the discrete category of natural numbers. Describe the functor category $\mathbf{Ab}^{\mathbb{N}}$ (commonly known as the category of graded abelian groups).

This is pretty much the same thing as the exercise above. This category is the category of sequences of abelian groups with morphisms the sequences of morphisms $(\varphi_n)_n$ where φ_n is an arrow between the n-th elements.

7.2.4 Category of functors on preorders

Exercise

> if P and Q are preorders, describe the functor category Q^P and show it a preorder.

As shown in reffunctorsonpreorders : if S, T are functors on preorders then any natural transformation between the is unique so it can be called $S \leq T$.

And it happens iff $\forall p, S(p) \leq_q T(p)$.

Anyway the functor category Q^P is a preorder, with the order relation $\leq$ defined "point-wise".

7.2.5 Category of permutation representation of a group

Exercise

> If **Fin** is the category of finite sets, and G a finite group, describe $\mathbf{Fin}^{\mathbf{G}}$ (the category of permutation representation of G).

Let $F : \mathbf{G} \to \mathbf{Fin}$ be a functor, such that $F(\star_G) = X$. Then for all $g \in G$, the image $F(g)$ is a function $X \to X$, but as G is a group $F(g)F(g^{-1}) = \mathrm{id}_X$ so $F(g) \in \mathfrak{S}(X)$. Moreover as F induce a morphism of monoids on the Hom-sets and as G is a group, F gives a group morphism $G \to \mathfrak{S}(X)$ ie a finite action of G on X.

A natural transformation of two such functors S, T is an function $\alpha : X = S(\star) \to Y = T(\star)$ satisfying $\forall g \in G$, $\alpha S(g) = T(g)\alpha$. I.e. $\forall g \in G, \forall x \in X$, $\alpha(g.x) = g.\alpha(x)$.

7.2.6 Equivalence and similitude on matrices with a categorical point of view

Exercise

> Let $\mathbf{M}$ be the infinite cyclic monoid (elements are $1, m, m^2, \cdots$). In the functors categories $(\mathbf{Matr}_K)^2$ and $(\mathbf{Matr}_K)^{\mathbf{M}}$ show that objects are matrices and isomorphic objects are equivalent and respectively equivalent and similar matrices, in the usual sense of linear algebra.

Equivalence : A functor $F : \mathbf{2} \to \mathbf{Matr}_K$ is given by an arrow of $\mathbf{Matr}_K$ (namely the image of the unique arrow of $\mathbf{2}$) i.e. a matrix $A : K^{n_0} \to K^{n_1}$.

A morphism of matrices A, B (natural transformation between the associated functors) is a couple of matrices $Q = (Q_0, Q_1)$ such that $BQ_0 = Q_1A$ as shown in the following commutative diagram:

$$\begin{tikzcd} n_0 \arrow[r, "Q_0"] \arrow[d, "A"] & n'_0 \arrow[d, "B"] \\ n_1 \arrow[r, "Q_1"] & n'_1 \end{tikzcd}$$

Then A is isomorphic to B **ssi** there is a natural transformation $Q = (Q_0, Q_1)$ such that Q_0 and Q_1 are isomorphisms i.e. invertible. Then the commutation could be written as $Q_1^{-1}BQ_0 = A$ which means A and B are equivalent.

Similarity : A functor $F : \mathbf{M} \to \mathbf{Matr}_K$ is given by an arrow of $\mathbf{Matr}_K$ (namely the image of m) i.e. a matrix $A : K^n \to K^n$ where n is the image of the "point" $\star_M$.

A morphism of matrices A, B (natural transformation between the associated functors) is a unique matrix Γ such that $B\Gamma = \Gamma A$ as shown in the following commutative diagram:

$$\begin{tikzcd} n \arrow[r, "\Gamma"] \arrow[d, "A"] & n' \arrow[d, "B"] \\ n \arrow[r, "\Gamma"] & n' \end{tikzcd}$$

Then A is isomorphic to B **ssi** there is a natural transformation Γ which is an isomorphisms i.e. invertible. Then the commutation could be written as $\Gamma^{-1}B\Gamma = A$ which means A and B are similar.

7.2.7 Functor to the category of arrows

Exercise

> Given two categories B, C and the functor category $B^{\mathbf{2}}$, show that each functor $H : C \to B^{\mathbf{2}}$ determines two functors $S, T : C \to B$ and a natural transformation $\tau : S \dot{\longrightarrow} T$ and show that this assignment $H \mapsto (S, \tau, T)$ is a bijection.

$B^{\mathbf{2}}$ is (equivalent to) the category of arrows of B with morphisms between f and g, the pairs of arrows u, v such that $fu = vg$.

Then a functor $H \to B^{\mathbf{2}}$ gives a function which associate to each $h \in H$ an arrow f_h of B.

Therefore, one can construct functors $S, T : H \to B$ defined by $Sh = \text{domain}\, f_h$, and $Th = \text{codomain}\, f_h$ on objects and by $Hg = (Sg, Tg)$ on arrows.

It also gives a family $(\tau_h)_h$ of arrows of B defined by $\tau_h = f_h : Sh \to Th$. One can easily prove that this family is a natural transformation from S to T as if $g : h \to k$ then $Tg \circ \tau_h = Tg \circ f_h = f_K \circ Sg$. Because (Sg, Tg) is a morphism between f_h and f_k as shown in the following commutative diagram :

$$\begin{tikzcd} Sh \arrow[r, "f_h"] \arrow[d, "Sg"] & Th \arrow[d, "Tg"] \\ Sk \arrow[r, "f_k"] & Tk \end{tikzcd}$$

7.2.8 Functor to the category of arrows II

Exercise

> Relate the functor H of exercise 7 to $F : C \times \mathbf{2} \to B$, given by for $f : c \to c'$, its images are $F(f, \mathbf{0}) = S(f)$, $F(f, \mathbf{1}) = T(f)$, $F(f, \downarrow) = Tf \circ \tau_c = \tau_{c'} \circ Sf$

Let's call $U, V : \mathbf{Cat} \to \mathbf{Cat}$ the functors respectively $\_ \times \mathbf{2}$ and $(\_)^{\mathbf{2}}$.

As shown in the following exercise, U is the left ? adjoint of V, so there is a natural bijection between $\hom(C \times \mathbf{2}, B)$ and $\hom(C, B^{\mathbf{2}})$. This bijection maps F to H.

Indeed : Let S, T associated with H as above. Let $c \in \mathbf{C}$. $H(c)(0) = Sc = F(c, 0)$ and $H(c)(1) = Tc = F(c, 1)$ then $H(c)$ and $F(c, \_)$ coincide on objects for all c, and similarly with the definition above of S and T, one can show that for all f arrow in $\mathbf{C}$, $H(f)$ and $F(f, \_)$ coincide.

7.3 Exercises on section 5 : The Category of All Categories p.45

7.3.1 Higher level of adjunction

Exercise

> For small categories A,B and C establish a bijection $\mathbf{Cat}(A \times B, C) \cong \mathbf{Cat}(A, C^B)$.
>
> And show it natural in A, B and C. Hence show that $\_ \times B : \mathbf{Cat} \to \mathbf{Cat}$ has a right adjoint.

$\mathbf{Cat}(A \times B, C)$ can be seen in three ways :

- A category of functors $C^{A \times B}$
- A hom-set in $\mathbf{Cat}$: $\hom_{\mathbf{Cat}}(A \times B, C)$
- A functor $\mathbf{Cat}^{op} \times \mathbf{Cat}^{op} \times \mathbf{Cat} \to \mathbf{Cat}$

The exercise ask for a bijection, in the vision as Hom-sets, but when it is asked for a naturality of the bijection then it has to be a natural transformation between the functors.

Let consider the function $\Omega : \mathbf{Cat}(A \times B, C) \to \mathbf{Cat}(A, C^B)$ given by $\Omega(F) : a \mapsto F(a, \_)$ and for $f : a \to a'$; $\Omega(F)(f)$ is the natural transformation τ between $F(a, \_)$ and $F(a', \_)$ given by $\tau_b = F(f, 1_b)$. As F is a functor, the following diagram for all $\varphi : b \to b'$ commutes:

$$\begin{tikzcd} F(a,b) \arrow[r, "F(1_a,\varphi)"] \arrow[d, "\tau_b = F(f,1_b)"] & F(a,b') \arrow[d, "\tau_{b'}"] \\ F(a',b) \arrow[r, "F(1_{a'},\varphi)"] & F(a',b') \end{tikzcd}$$

One has just defined a function between $\mathbf{Cat}(A \times B, C)$ and $\mathbf{Cat}(A, C^B)$. This one is a bijection as it exists a converse function Γ defined as follows :

For $F : A \to C^B$ let set $\Gamma(F) \in \mathbf{Cat}(A \times B, C)$ as $(a, b) \mapsto F(a)(b)$ on objects and for σ natural transformation between two functors $S, T : A \to C^B$, one may define $\Gamma(\sigma)_{(a,b)} : \Gamma(S)(a, b) \to \Gamma(T)(a, b)$ by $(\sigma_a)_b$ –because σ_a is a natural transformation between functors $B \to C$–.

One can easily verify that $\Gamma\Omega = I_{\mathbf{Cat}(A \times B, C)}$ and $\Omega\Gamma = I_{\mathbf{Cat}(A, B^C)}$.

Naturality : Let A, B, C and A', B', C' be in $\mathbf{Cat}$ and $f : A' \to A$, $g : B' \to B$, $h : C \to C'$.

One has :

$$(f^{op}, g^{op}, h) : (A, B, C) \to (A', B', C')$$

Then :

$$\begin{tikzcd} (A \times B, C) \arrow[d, "(f^{op} \times g^{op}, h)"] & (A, C^B) \arrow[d, "(f^{op}, h \circ \_ \circ g)"] \\ (A' \times B', C') & (A', (C')^{B'}) \end{tikzcd}$$

And therefore by applying the Hom-functor, one gets

$$\begin{tikzcd} \mathbf{Cat}(A \times B, C) \arrow[r, "\Omega_{A,B,C}"] \arrow[d, "h \circ \_ \circ (f \times g)"] & \mathbf{Cat}(A, C^B) \arrow[d, "h \circ (\_ \circ f)() \circ g"] \\ \mathbf{Cat}(A' \times B', C') \arrow[r, "\Omega_{A',B',C'}"] & \mathbf{Cat}(A', (C')^{B'}) \end{tikzcd}$$

This one is commutative. Indeed : given a $u : A \times B \to C$, one has $(h \circ [\Omega_{ABC}(u) \circ f]() \circ g)(a')(b') = h[\Omega_{ABC}(u)(f(a'))(g(b'))] = h(u(f(a'), g(b')))$ while $\Omega_{A'B'C'}[h \circ u \circ (f \times g)](a')(b') = h[u(f(a'), g(b'))]$.

So the transformation is natural in (A, B, C).

Adjoint of functor $\_ \times B$: Let denote F the functor $\_ \times B$ and G the functor $(\_)^B$. Then the natural bijection $\mathrm{Hom}(F(A), C) \cong \mathrm{Hom}(A, G(C))$ means G is the left adjoint of F.

7.3.2 Natural isomorphisms

Exercise

> For categories A, B and C establish natural isomorphisms:
>
> - $(A \times B)^C \cong A^C \times B^C$
> - $C^{A\times B} \cong (C^B)^A$

1. Let consider the following function $\tau : (A \times B)^C \to A^C \times B^C$ defined by $\tau f = (Pf, Qf)$ where P, Q are the respective usual projections $A \times B \to A, B$. In other words, $\tau = (P \circ \_, Q \circ \_)$.

It is a bijection as for all $f : C \to A$ and for all $g : C \to B$, the unique map $C \to A \times B$ such that $\tau(u) = (f, g)$ is $u = (f, g)$.

This bijection is moreover natural in A, B, C as if one renames τ as τ_{ABC} then : If $(f, g, h^{op}) : (A, B, C) \to (A', B', C')$, then $(f \times g, h^{op}) : (A \times B, C) \to (A' \times B', C')$.

Therefore, one has the following diagram :

$$\begin{tikzcd}
(A \times B)^C \arrow[r, "\tau_{ABC}"] \arrow[d, "(f \times g) \circ _ \circ h"] & A^C \times B^C \arrow[d, "(f \circ _ \circ h) \times (g \circ _ \circ h)"] \\
(A' \times B')^{C'} \arrow[r, "\tau_{A'B'C'}"] & A'^{C'} \times B'^{C'}
\end{tikzcd}$$

This one is commutative because given $\psi : C \to A \times B$. Then $\tau_{ABC}\psi = (P\psi, Q\psi)$ so $(f^h \times g^h) \circ \tau_{ABC}\psi = \langle fP\psi h, gQ\psi h\rangle$.

On the other side $\tau_{A'B'C'} \circ ((f \times g)^h)\psi = \tau_{A'B'C'}(fP\psi h, gQ\psi h)) = (fP\psi h, gQ\psi h)$.

Moreover it also commutes on arrows, but it starts to be too awful to be written.

2. Magical proof by Ross :

Let D be a category, and let call π the natural isomorphism constructed in exercise 7.3.1.

$$\begin{array}{rcll}
\mathbf{Cat}(D, C^{A\times B}) & \cong & \mathbf{Cat}(D \times (A \times B), C) & (\pi^{-1}) \\
& \cong & \mathbf{Cat}((D \times A) \times B, C) & (\text{"associativity"}) \\
& \cong & \mathbf{Cat}(D \times A, C^B) & (\pi) \\
& \cong & \mathbf{Cat}(D, (C^B)^A) & (\pi)
\end{array}$$

Now let D be $C^{A\times B}$, one has therefore $\omega : \mathbf{Cat}(C^{A\times B}, C^{A\times B}) \cong \mathbf{Cat}(C^{A\times B}, (C^A)^B)$, thus the image of the identity functor is a bijective functor*, and its naturality is a consequence of the naturality of the above isomorphism ω in (A, B, C).

* : It is bijective as it is equal to π on objects, and as shown in exercise 7.3.1, π is a bijection. Moreover it maps an arrow $(\alpha_{ab})_{a,b}$ in $C^{A\times B}$ to the sequence of sequences of C-arrows $(\alpha_b)_a$ for $a \in A$ and $b \in B$, which obviously makes this functor fully-faithful. And so an isomorphism of categories.

7.3.3 Horizontal composition is a functor

Exercise

> Use theorem 1. to show that horizontal composition is a functor : $\circ : A^B \times B^C \to A^C$.

The function $\circ$ is a function $A^B \times B^C \to A^C$ on objects (functors) given by $(F, G) \mapsto F \circ G$.

Now let $\sigma = (\sigma_1, \sigma_2)$ be a natural transformation $(F, G) \dot{\longrightarrow} (S, T)$:

$$\begin{tikzcd} C \arrow[r, bend left, "G"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\downarrow\sigma_1"] & B \arrow[r, bend left, "F"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\downarrow\sigma_2"] & A \end{tikzcd}$$

then one can define $\circ(\sigma) = \sigma_2 \circ \sigma_1 : F \circ G \dot{\longrightarrow} S \circ T$.

Now let's take two natural transformations σ, τ as follows :

$$\begin{tikzcd} C \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=3ex, "\downarrow\sigma_1"] \arrow[r, phantom, shift right=3ex, "\downarrow\tau_1"] & B \arrow[r, bend left=50, "R"] \arrow[r, "S" description] \arrow[r, bend right=50, "T"'] \arrow[r, phantom, shift left=3ex, "\downarrow\sigma_2"] \arrow[r, phantom, shift right=3ex, "\downarrow\tau_2"] & A \end{tikzcd}$$

Then the interchange law gives $(\sigma_2 \circ \sigma_1) \cdot (\tau_2 \circ \tau_1) = (\sigma_2 \cdot \tau_2) \circ (\sigma_1 \cdot \tau_1)$, which can be written $\circ(\sigma_1 \cdot \tau_1, \sigma_2 \cdot \tau_2) = \circ(\sigma \cdot \tau)$.

One has just shown $\circ(\sigma) \cdot \circ(\tau) = \circ(\sigma \cdot \tau)$, so $\circ$ is a functor.

7.3.4 Interchangeable laws on a topological group

Exercise

> Let G be a topological group with identity e. One considers the set of continuous path from e to e equipped with the pointwise product $(\cdot)$ and the concatenation of paths $(\circ)$. Prove that the interchange law holds.

The pointwise product of two continuous path u, v is the path $u \cdot v : [0, 1] \to G$ defined by $(u \cdot v)(t) = u(t) \cdot v(t)$. One can easily show that this is also a continuous path from e to e.

Let denote Eu for $\Omega \subset [0, 1]$ and $u : \Omega \to G$, the function $[0, 1] \to G$ which extend u by e outside Ω. Moreover one may denote $\tau : t \mapsto t + \frac{1}{2}$ and $\mu : t \mapsto \frac{1}{2}t$.

The composition (concatenation) of two path u, v is thus the function : $E(u \circ \mu) \cdot E(v \circ \tau \circ \mu)$. For security reasons, one may avoid using the usual composition $\circ$ and keep that notation for the concatenation of paths. Therefore $u \circ v = Eu\mu \cdot Ev\tau\mu$.

Now let consider $(u \circ v) \cdot (u' \circ v') = Eu\mu \cdot Ev\tau\mu \cdot Eu'\mu \cdot Ev'\tau\mu = (Eu \cdot Ev\tau \cdot Eu' \cdot Ev'\tau)\mu$.

Now one may notice that those functions of the kind Eu and $Ev\tau$ commutes for $(\cdot)$ as if for a fixed t, one is not equal to e then the other is.

Then $(u \circ v) \cdot (u' \circ v') = (Eu \cdot Ev\tau \cdot Eu' \cdot Ev'\tau)\mu = (Ev\tau \cdot Ev'\tau \cdot Eu \cdot Eu')\mu$. And as u and u' as the same domain, then on can write $(Ev\tau \cdot Ev'\tau \cdot Eu \cdot Eu') = (E(v \cdot v')\tau \cdot E(u \cdot u')\tau) = (E(u \cdot u') \cdot E(v \cdot v')\tau)$.

Finally $(u \circ v) \cdot (u' \circ v') = (E(u \cdot u')\mu \cdot E(v \cdot v')\tau\mu) = (u \cdot u') \circ (v \cdot v')$, i.e. the interchange law holds.

7.3.5 Hilton-Heckmann result on unitary interchangeable laws

Exercise

> Let S be a set with two (everywhere defined) binary operations $\cdot, \circ : S \times S \to S$ which both have the same (two-sided) unit element e and which satisfy the interchange identity. Prove that $\cdot = \circ$ and that each is commutative.

For fixed $x, y \in S$, let consider $x \circ y = (x \cdot e) \circ (e \cdot y) = (x \circ e) \cdot (e \circ y) = x \cdot y$. Moreover $x \circ y = (e \cdot x) \circ (y \cdot e) = (e \circ y) \cdot (x \circ e) = y \cdot x$. Thus the two laws are equals and "anti-equals" so commutative.

7.3.6 π_1 of a topological group

Exercise

> Combine exercises 4 and 5 to prove that the fundamental group of a topological group is abelian.

As shown in exercise 4, the set S of continuous path on G (topological group fixed from now on) is given with two laws $\cdot$ and $\circ$ which satisfy the hypotheses of Hilton-Heckmann theorem (exercise 7.3.5) with identity the constant function to e.

Then if one calls S this set of paths, S is a commutative monoid with law $\circ = \cdot$.

As usual, one may define the homotopic equivalence of paths on $(S, \circ)$ and prove that relation respect composition. Then the quotient, $S/\sim = \pi_1(G, e)$ is the quotient of a commutative monoid, then it is abelian (and it is a group as usual).

Maybe one can construct π_1 by seeing S as a groupoid with arrows homotopic transformations.

7.3.7 Natural transformation between Hom-functors

Exercise

> If $T : A \to D$ is a functor, show that its arrow functions $T_{a,b} : A(a,b) \to D(Ta, Tb)$ define a natural transformation between functors $A^{op} \times A \to \mathbf{Set}$.

One has two functors $A^{op} \times A \rightrightarrows \mathbf{Set}$ namely $\mathrm{Hom}_A(\_,\_)$ and $\mathrm{Hom}_D(T\_, T\_)$. Then because T is a functor, for all $(f, g) : (a, b) \to (a', b')$ in $A^{op} \times A$, the following diagram is commutative :

$$(a,b) \xmapsto{T^{op} \times T} (Ta, Tb) \qquad (a',b') \xmapsto[T^{op} \times T]{} (Ta', Tb')$$

$$\begin{tikzcd} A(a,b) \arrow[r, "T_{a,b}"] \arrow[d, "A(f,g)"'] & D(Ta,Tb) \arrow[d, "D(Tf,Tg)"] \\ A(a',b') \arrow[r, "T_{a',b'}"'] & D(Ta',Tb') \end{tikzcd}$$

Then $(T_{a,b})_{a,b}$ is a natural transformation.

7.3.8 Natural transformations of the identity functor

Exercise

> For the identity functor $I_{\mathbf{C}}$ of any category, the natural transformations $\alpha : I_{\mathbf{C}} \dot{\longrightarrow} I_{\mathbf{C}}$ form a commutative monoid. Find this monoid in the cases $\mathbf{C} = \mathbf{Grp}, \mathbf{Ab}$ and $\mathbf{Set}$.

General results : The considered set of natural transformations between $I_{\mathbf{C}}$ functor is in fact the Hom-set $\mathbf{C}^{\mathbf{C}}(I_{\mathbf{C}}, I_{\mathbf{C}})$. It is as usual a monoid for the horizontal composition law $\circ$. Moreover it is also a monoid with the same (two-sided) identity for the vertical composition. Therefore using result of exercise 7.3.5, one has that this monoid is commutative.

Such a natural transformation is family $\sigma = (\sigma_c)_{c \in \mathbf{C}}$ and it also "maps" an arrow $f : c \to c'$ to an arrow of $\mathbf{C^2}$ between σ_c and $\sigma_{c'}$ given by the couple (f, f).

So σ is a functor $\mathbf{C} \to \mathbf{C^2}$. Now one has the following diagram :

$$\begin{tikzcd}
\mathbf{C} \arrow[r, bend left, "\sigma"] & \mathbf{C^2} \arrow[l, bend left, "\text{domain}"] \arrow[r, bend right, "\text{codomain}"'] & \mathbf{C} \arrow[l, bend right, "\sigma"']
\end{tikzcd}$$

with the property $D\sigma = C\sigma = I_{\mathbf{C}}$.

Groups : Given a such transformation σ, lets call $s = \sigma_{\mathbb{Z}}(1)$. Then lets take a group G and an element $g \in G$.

Then one has the following commutative diagram :

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\sigma_{\mathbb{Z}}"] \arrow[d, "\hat{g}"] & \mathbb{Z} \arrow[d, "\hat{g}"] \\
G \arrow[r, "\sigma_G"] & G
\end{tikzcd}$$

Where $\hat{g} : \mathbb{Z} \to G$ is the group homomorphism that maps 1 to g (it is unique as defined on the generator of the free group $\mathbb{Z}$).

The commutativity thus gives $\sigma_G(g) = \hat{g}(s) = g^s$. So any such natural transformation is determined by a unique $s \in \mathbb{Z}$ and gives for all group G a map $g \mapsto g^s$.

Conversely as for all $g \in G \to_f H$, and for all s, $f(g^s) = f(g)^s$, all such family of arrows are natural transformation between the identity functor.

Abelian groups : The above results holds as arrows of abelian groups are exactly the group arrows.

Again $\mathrm{Hom}_{[\mathbf{Ab},\mathbf{Ab}]}(I, I) = \{(\sigma_G)_G \mid \exists s \in \mathbb{Z}, \forall G \in \mathbf{Ab}, \sigma_G : g \mapsto g^s\}$.

Set : In **Set**, the identity functor is isomorphic to $\mathrm{Hom}(\mathbf{1}, \_)$, then the set of natural transformations $I \dot{\longrightarrow} I$ is isomorphic to the set $\mathrm{Nat}(\mathrm{Hom}(\mathbf{1}, \_), \mathrm{Hom}(\mathbf{1}, \_))$.

But by Yoneda's lemma, this is isomorphic to $\mathrm{Hom}(\mathbf{1}, \mathbf{1})$, so there is a unique natural transformation between the identity functor in **Set**.

7.4 Exercises on section 6 : Comma Categories p.48

7.4.1 Category of commutative rings under a commutative field

Exercise

> If K is a commutative ring, show that the comma category $(K \downarrow \mathbf{CRng})$ is the usual category of all small commutative K-algebras.

A commutative ring under K, is a couple $(R, f : K \to R)$ where $R \in \mathbf{CRng}$.

Therefore f is a monomorphism and K can be seen as a sub-ring of R, then R is a K-algebra with the obvious K-linear multiplication $\lambda.r = f(\lambda) \times r$.

Moreover any commutative K-algebra, R comes with a ring-morphism $f : K \ni \lambda \mapsto \lambda.1_R$.

Now let's consider a morphism $h : R \to S$ of commutative rings $(R, f), (S, g)$ under K. Then $hf = g$. Which means, for each $\lambda \in K$, h maps $f(\lambda)$ to $g(\lambda)$.

So $f : R \to S$ is a ring-morphism that respects the embedding of K in R and S it is exactly what is called a morphism of K-algebras.

7.4.2 Category of objects over a terminal object

Exercise

> If t is a terminal object in $\mathbf{C}$, prove that $(\mathbf{C} \downarrow t)$ is isomorphic to $\mathbf{C}$.

As states the definition of a terminal object, for each $c \in \mathbf{C}$, there exists a unique $f : c \to t$. So there is an injection on objects $\mathbf{C} \hookrightarrow (\mathbf{C} \downarrow t)$. And this is a bijection as the converse function is given by $(c, f) \mapsto c$.

Now let $g : c \to c'$ be an arrow in $\mathbf{C}$, because of the uniqueness of the arrows $c \to t$, the following diagram is necessarily commutative :

$$\begin{tikzcd}
c \arrow[rr, "g"] \arrow[dr, "!f"'] & & c' \arrow[dl, "!f'"] \\
& t &
\end{tikzcd}$$

Then any arrow $g : c \to c'$ in $\mathbf{C}$ is also an arrow in $(\mathbf{C} \downarrow t)$ and conversely. So the above function is in fact a bijective functor on objects as well as on arrows.

These two categories are therefore isomorphic.

7.4.3 Complete the definition

Exercise

Express P, Q, R of the definition of a comma category, on arrows :

$$\begin{tikzcd}
 & (T \downarrow S) \arrow[dl, "P"'] \arrow[d, "R"] \arrow[dr, "Q"] & \\
E \arrow[d, "T"] & \mathbf{C^2} \arrow[dl, "\mathbf{C}^{d_0}"] \arrow[dr, "\mathbf{C}^{d_1}"'] & D \arrow[d, "S"] \\
\mathbf{C} & & \mathbf{C}
\end{tikzcd}$$

On objects : Given $x = (e, d, f : Te \to Sd)$ an object of $(T \downarrow S)$, the projections Px, Qx, Rx are defined as follows:

- $Px = e$
- $Qx = d$
- $Rx = f$

On arrows : Given two objects of the comma category $x = (e, d, f)$ and $x' = (e', d', f')$ and a morphism $y = (g, h) : x \to x'$. Then one can define the projections of y as follows :

- $Py = g$
- $Qy = h$
- $Ry = (Tg, Sh)$

The definition of Ry gives indeed an arrow of $\mathbf{C^2}$ as the diagram :

$$\begin{tikzcd}
T(e) \arrow[r, "f"] \arrow[d, "Tg"] & S(d) \arrow[d, "Sh"] \\
T(e') \arrow[r, "f'"] & S(d')
\end{tikzcd}$$

is supposed commutative, in the definition of an arrow of $(T \downarrow S)$.

7.4.4 Natural transformations seen as functor on comma category

Exercise

Given functors $S, T : \mathbf{D} \to \mathbf{C}$, show that a natural transformation $\tau : T \dot{\longrightarrow} S$ is the same thing as a functor $\tau : \mathbf{D} \to (T \downarrow S)$ such that $P\tau = Q\tau = \mathrm{id}_D$ (where P, Q are the projections $(T \downarrow S) \to \mathbf{D}$).

Let $\tau : T \dot{\longrightarrow} S$ be a natural transformation.
One can construct the functor $\tau : D \to (T \downarrow S)$ by :

- $d \mapsto (d, \tau_d, d)$ on objects.
- And for $f : d \to d'$, one can associate the pair of morphisms (f, f). It is a morphism in $(T \downarrow S)$ as the following diagram commutes because of the naturality of τ :

$$\begin{tikzcd} T(d) \arrow[r, "Tf"] \arrow[d, "\tau_d"] & T(d') \arrow[d, "\tau_{d'}"] \\ S(d) \arrow[r, "Sf"] & S(d') \end{tikzcd}$$

It follows the definition that on objects $d \in \mathbf{D}$, $P\tau(d) = Q\tau(d) = d$, and moreover, for $f : d \to d'$, $P\tau(f) = P(f, f) = f$, $Q\tau(f) = Q(f, f) = f$.

7.4.5 The comma category is a "pullback"

Exercise

Given any X such that the following diagram commutes :

$$\begin{tikzcd} & X \arrow[dl, "P'"'] \arrow[d, "R'"] \arrow[dr, "Q'"] & \\ E \arrow[d, "T"] & \mathbf{C^2} \arrow[dl] \arrow[dr] & D \arrow[d, "S"] \\ \mathbf{C} & & \mathbf{C} \end{tikzcd}$$

Prove that there is a unique functor $L : X \to (T \downarrow S)$ such that $P' = PL, Q' = QL, R' = RL$.

Let consider the functor $L = (P', Q', R') : X \to (T \downarrow S)$ as for $x \in X$, $R'x : TP'x \to SQ'x$. Then one can see that this functor factorise P', Q', R'.

Conversely let G be an other functor that factorise P', Q', R', then for $x \in X$, $Gx = (PGx, QGx, RGx) = (P'x, Q'x, R'x) = Lx$ and similarly with the arrows $Gf = (PGf, QGf, RGf) = (P'f, Q'f, R'f) = Lf$.

7.4.6 The functor "comma category"

Exercise

(a) For fixed small C, D, E, show that $(T, S) \mapsto (T \downarrow S)$ is the object function of a functor $(C^E)^{op} \times (C^D) \to \mathbf{Cat}$.

(b) Describe a similar functor for variable C, D, E.

(a) The transformation $(T,S) \to (T \downarrow S)$ is indeed a function on objects of the category $(C^E)^{op} \times C^D$ to the objects of **Cat**.

Now let $u = (\tau, \sigma)$ be a morphism $(C^E)^{op} \times C^D$ i.e. it is a pair of natural transformations $(T', S) \dot{\longrightarrow} (T, S')$.

One may define a functor $U : (T \downarrow S) \to (T' \downarrow S')$ given by :

- On objects : $U : (e,d,f) \mapsto (e,d,\sigma_d \circ f \circ \tau_e)$, as:

$$T'(e) \xrightarrow{\tau_e} T(e) \xrightarrow{f} S(d) \xrightarrow{\sigma_d} S'(d)$$

- On arrows : given two elements (e,d,f) and (e',d',f') and an arrow (h,k), let set $U : (h,k) \mapsto (h,k)$, as given by:

$$\begin{tikzcd}
T'(e) \arrow[r, "\tau_e"] \arrow[d, "T'h"] & T(e) \arrow[r, "f"] \arrow[d, "Th"] & S(d) \arrow[r, "\sigma_d"] \arrow[d, "Sk"] & S'(d) \arrow[d, "S'k"] \\
T'(e') \arrow[r, "\tau_{e'}"] & T(e') \arrow[r, "f'"] & S(d') \arrow[r, "\sigma_{d'}"] & S'(d')
\end{tikzcd}$$

  For which the external rectangle is commutative because the central square commutativity is given by the definition of arrows in the comma category and the two other ones are stating that σ and τ are natural transformations.

Then U is a functor as the definition on arrows obviously respects composition.

So far, one has shown the "arrow category"-function acts also on morphisms. Let now prove that $u \mapsto U$ respects composition to show it a functor.

Let $u = (\tau, \sigma)$ and $v = (\tau', \sigma')$ two naturals transformations as follows:

$$\alpha : T'' \xrightarrow{\tau'} T' \xrightarrow{\tau} T$$

$$\beta : S \xrightarrow{\sigma} S' \xrightarrow{\sigma'} S''$$

The functor UV is given by the identity on arrows and $(e,d,f) \mapsto (e,d,\sigma_d \sigma'_d f \tau'_e \tau_e)$ on objects.

On the other side, let consider the transformation $w = (\alpha, \beta) = (\tau\tau', \sigma'\sigma)$. Then the functor W is given by the identity on arrows and $(e,d,f) \mapsto (e,d,\sigma'_d \sigma_d f \tau_e \tau'_e)$.

Then $UV = W$, and the considered application is a covariant functor.

(b)

Naturality : Let call $\downarrow_{C,D,E}$ the above functor $(T,S) \mapsto (T \downarrow S)$.

Then one may consider the functor $\mathbf{Cat} \times \mathbf{Cat}^{op} \times \mathbf{Cat}^{op} \to \mathbf{Cat}^{\mathbf{Cat}}$ defined by : $(C,D,E) \mapsto \downarrow_{CDE}$ on objects. And maps a triplet of arrows $(f,g,h) : (C,D,E) \to (C',D',E')$ to the natural transformation $(f\_h \downarrow f\_g) : \downarrow_{C,D,E} \dot{\longrightarrow} \downarrow_{C',D',E'}$.

This proves that $\downarrow_{C,D,E}$ is natural in C, D and E.

Functor : Let consider the coma category $(\mathbf{Cat} \times \mathbf{Cat} \downarrow \Delta_{\mathbf{Cat}})$ whose objects are triples made of a pair of source categories (E, D), a category target C and a pair of functors $(T, S) : (E, D) \to \Delta_{\mathbf{Cat}} C = (C, C)$.

Then one can define the functor $\downarrow$: $(\mathbf{Cat} \times \mathbf{Cat} \downarrow \Delta_{\mathbf{Cat}}) \to \mathbf{Cat}$ given by $(T, S) \mapsto (T \downarrow S)$.

The definition of this functor on arrows (i.e. triples of functors (U, V, F) such that $T'U = FT$ and $S'V = FS$) is given by the universality of the comma category (c.f. 7.4.5) which gives a functor $(T \downarrow S) \to (T' \downarrow S')$.

Remark : The naturality property stated above may be translatable into the fact that $\downarrow$ is a 2-functor.

7.5 Exercises on section 7 : Graphs and Free Categories p.51

7.5.1 Opposite and product of Graph(s)

Exercise

Define "opposite graph" and "product of graphs" to agree with the corresponding definitions for categories (ie, so that the forgetful functor U will preserve opposite and product).

Opposite Let G be a O-graph with arrow set A equipped with two functions $d, c : A \rightrightarrows O$. then define G' the opposite graph, as the O-graph with arrow set A but with function domain $d' = c$ and codomain $c' = c$.

Thus it can easily be shown that for a category $\mathbf{C}$, $U(\mathbf{C}^{op}) = U(\mathbf{C})^{op}$, as if an arrow f lies in $\mathbf{C}$ then it is an arrow of the underlying graph of $\mathbf{C}$ but furthermore exchanging is domain and codomain is done in the same way in graphs and categories.

Product Given two graphs G, G' respectively $O, A \rightrightarrows O$ and $O', A' \rightrightarrows O'$. Let consider the graph H with object set $\Omega = O \times O'$ (as usual product of sets) and the arrow set $B = A \times A'$ equipped with two functions $\delta_0 \times \delta_0'$ and $\delta_1 \times \delta_1'$ form B to Ω.

One can notice that the object set thus defined is also the object set of a produce of two categories $\mathbf{C}$ and $\mathbf{C}'$ seen as graphs.

7.5.2 Free the ordinals

Exercise

Show that finite ordinal numbers are free categories.

Let consider for $n \in \mathbb{N}$ the following graph G_n : $\mathbf{0} \to \mathbf{1} \to \mathbf{2} \to \cdots \to \mathbf{n\text{-}1}$

Objects are $\{\mathbf{0}, \mathbf{1}, \cdots \mathbf{n\text{-}1}\}$ as like in the category $\mathbf{n}$. Arrows are mapping one object (ordinal) to its successor.

Then the free category on G_n is $\mathbf{n}$. Indeed : let F be this free category, and let suppose there is two arrows from u to v, then each arrow can be decomposed into a sequence of arrows of G_n. Necessarily the sequences has the same size, as v could not be the n-th and the m-th successor of u. And then they are equals as two sequences of same bound and same size in G_n could not differ. Thus F is a preorder.

Moreover F is totally ordered as if $u, v \in F$, one can suppose $u \le v$ as integers, then there is a sequence of arrows $u \to u+1 \to \cdots \to v$ in G_n and therefore an arrow $u \le v$ in F.

F is then a totally ordered set with elements $\mathbf{0} \le \mathbf{1} \le \cdots \le \mathbf{n\text{-}1}$, therefore it is $\mathbf{n}$.

7.5.3 Free groupoids

Exercise

> Show that each graph G generates a free groupoid F (which satisfy the same universality as the free category on G). Deduce as a corollary that every set X generates a free group.

"A groupoid is a small category in which every morphism is an isomorphism, and hence invertible. " Wikipedia

Construction For a fixed O-graph G with set of arrows A, let consider the O-graph G' with arrow set $A \cup \{f' : b \to a \mid f : (a \to b) \in A\}$. Then one can construct the free groupoid F on G' as follows :

Objects of F are those of G' and G namely O.

An arrow of F is a class of equivalence on finite sequence of arrows of G : Let consider the set of arrows A_{free} as in the case of the free category. Then one can define on that set a relation $\sim$ by $(a, f, b, f', a) \sim (a)$ and $(b, f', a, f, b) \sim (b)$ for all $f : a \to b$, and to extend that definition to any arrow by $u \sim v$, $x \sim y$ implies $ux \sim vy$ as soon as these arrow are composable. One can verify that this relation respects domains and codomains.

Then an arrow of F will be an element of $A_{\text{free}}/\sim$.

Thus, F can be seen as the quotient of the free category on G' by $\sim$.

This construction gives for every graph G a functor Q from the free category on G', namely C to the free groupoid on G : $F = C/\sim$. And then the following diagram can be completed :

$$\begin{tikzcd}
G \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] \arrow[dr, dotted, "\pi"'] & G' \arrow[r, "p"] \arrow[d, dotted, "UQp"] & UC \arrow[dl, "UQ"] \\
& UF &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C \arrow[d, "Q"] \\
F
\end{tikzcd}$$

Where $i : f \mapsto f$ and $j : f \mapsto f'$. One can define $\pi = UQ \circ p \circ i$.

Universality Given a O-graph G, let show that the free groupoid on G, denoted F, is universal from G to U the forgetful functor $\mathbf{Gpd} \to \mathbf{Grph}$.

Let A be a groupoid, and $g : G \to UA$ an arrow in the category of graphs. A will be considered as a category with a contravariant functor $Y : A \to A$, namely the inversion, which satisfy for all f arrow in A, $Yf \circ f = id_{\text{domain}\, f}$ and $f \circ Yf = id_{\text{codomain}\, f}$.

As A is a groupoid, g can be extended to an arrow $g' : G' \to UA$ in a unique way such that $UYg'j = g = g'i$. It is possible because G' has been constructed as the disjoint union of two sets and using i and j, one can define g on both those two.

Then by universality of C the free category on G', there is a functor $K : C \to A$ such that $UKp = g'$. So $U(YK)pi = UYg = UKpj$.

It yields that $KpiKpj = KpiYKpi = 1$, as the relation $\sim$ is defined by $pi \sim Ypj$ and extended by composition, K factors through Q (which is in fact the coequalizer of pi, Ypj).

Then there exists a unique $H : F \to A$ such that $HQ = K$. Therefore $UHUQpi = UH\pi = g$ and so g has the universal property of objects from G to U.

Corollary Given a set X, it can be seen as a graph G with object set $\{\star\}$ and arrows X.

Then the free groupoid on G is a groupoid with a unique element so it is a group F_X. With the property that for any group H of underlying set Y and function $f : X \to Y = UH$, there is an homomorphism of the groups $h : F_X \to H$ such that $Uhi = f$ as functions (where i is the injection $X \to UF_X$).

7.6 Exercises on section 8 : Quotient Categories p.52

7.6.1 A simple free category

Exercise

> Show that the category generated by the graph:
>
> $$\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "f"'] & \cdot \arrow[d, "f'"] \\ \cdot \arrow[r, "g'"] & \cdot \end{tikzcd}$$
>
> with the one relation $g'f = f'g$ has four identity arrows and exactly five non-identity arrows $f, g, f', g', g'f = f'g$.

The given graph has four objects so the free category ($\mathbf{C}$) on it has also four (distinct) objects and therefore exactly four identity arrows.

Moreover $f, g, g', f', g'f$ are distinct in $\mathbf{C}$, because any two of them don't have the same couple (domain,codomain). So there is at least 5 non-identity arrows.

Conversely in the graph, any sequence of composable arrows is at most of length 2, and a non-identity arrow is a sequence of length at least 1.

By hypothesis there is exactly four arrows in the graph i.e. four sequences of length 1, and moreover there is two sequences of length 2 but they are supposed equals by the relation so a unique other arrow is given.

7.6.2 Normal subgroups

Exercise

> If $\mathbf{C}$ is a group G seen as a category with one object. Show that to each congruence R on $\mathbf{C}$, there is a normal subgroup N of G with fRg iff $g^{-1}f \in N$.

A congruence R on $\mathbf{C}$ is a binary relation $R_{\star,\star} = R$ on G.

Moreover the quotient category $\mathbf{C}/R$ has a structure of monoid. And so the functor $Q : \mathbf{C} \to \mathbf{C}/R$ gives a monoid morphism $G \to G/R$.

Let $N \triangleleft G$ be the kernel of this morphism. Given $f, g \in G$, fRg iff $Qf = Qg$ i.e. $Q(f^{-1}g) = 1$ i.e. $f^{-1}g \in N$.

8 Universals and Limits

8.1 Exercises on section 1 : Universal arrows p.59

8.1.1 Universality of familiar constructions

Exercise

Show how each of the following familiar constructions can be interpreted as a universal arrow :

(a) The integral group ring of a group (better, a monoid).

(b) The tensor algebra of a vector space.

(c) The exterior algebra of a vector space.

(a) I only manage to do it with monoid, the group case seems more complex.

Let consider the functor $U : \mathbf{IRng} \to \mathbf{Mon}$ from integral domains to monoids. Given by $(R, +, \cdot) \mapsto (R^*, \cdot)$ on objects and by the restriction to R^* on morphisms.

Given a fixed monoid M, let consider the couple $(\mathbb{Z}[M], u : M \hookrightarrow \mathbb{Z}[M]^*)$. I claim it universal from M to U.

Indeed : Let $(A, v : M \to A^*)$ be a couple where A is an integral ring. Then define $w : \mathbb{Z}[M] \to A$ as follows : $w(m) = v(m)$ for $m \in M$, and let expand it on $\mathbb{Z}[M]$ such that it becomes a ring homomorphism.

Then it result from the definition that $(Uw)u = v$.

(b) Given a fixed field K, let call U the forgetful functor K-$\mathbf{Alg} \to K$-$\mathbf{Vect}$.

For a vector space V, one may construct the tensor algebra on V :

$$T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n} = K \oplus V \oplus (V \otimes V) \oplus \cdots$$

It is a K-vector space by construction, and this definitions gives immediately an linear map $i : V \to T(V)$ by $x \mapsto 0 \oplus x \oplus 0 \oplus \cdots$. Moreover, on may define a product the following way :

Given two integers $m, n \in \mathbb{N}$, one has the following isomorphism $V^{\otimes n} \otimes V^{\otimes m} \cong V^{\otimes(n+m)}$.

As the multiplication $\mu : T(V) \times T(V) \to T(V)$ is bilinear it is totally defined by the data of the family of bilinear functions :

$$\mu_{n,m} : V^{\otimes n} \times V^{\otimes m} \to T(V)$$

The previous isomorphism allows us to define $\mu_{n,m}$ as the mere injection $V^{\otimes n+m} \to T(V)$.

I claim $(T(V), i : V \to UT(V))$ is universal from V to the forgetful functor U.

Let's take an algebra A, and an arrow $f : V \to UA$. Given a $n \in \mathbb{N}$, one can associate with f an arrow $f^{\otimes n}$ this way :

$$V^n \xrightarrow[n \text{ times}]{(f, f, \cdots, f)} (UA)^n \xrightarrow[n-1 \text{ times}]{\mu_A \times \cdots \times \mu_A} UA$$

Where $\mu_A : UA \times UA \to UA$ is the multiplication in the algebra A.

Then one can associate to the above function an arrow $f^{\otimes n} : V^{\otimes n} \to UA$.

So as we have given an arrow on each "coordinate" of $T(V)$, then we have an arrow $T(f) : T(V) \to UA$ which satisfy $T(f)i = f$. Moreover this arrow maps a product in $T(V)$ to a product in A (and the neutral element to the neutral element), then it is an arrow of the category K-**Alg**.

The universality follows.

(c) For a fixed field K, and given a vector space V, the exterior algebra $\Lambda(V)$ of V is the quotient $T(V)/I$ where $I = \langle x \otimes x \mid x \in V \rangle$ is a two-sided ideal.

Let us call again U the forgetful functor $\mathbf{Alg}_K \to \mathbf{Vect}_K$. The property of the exterior algebra can be stated as follows : for every linear map $f : V \to UA$ such that $\mu(f, f) = 0$ (where μ is the multiplication in A) there exists a unique $\varphi : \Lambda V \to A$ which extend f.

This gives a bijection

$$\{f : V \to UA \mid \mu(f, f) = 0\} \cong \mathrm{Hom}_{Alg}(\Lambda V, A)$$

8.1.2 Universal element for the powerset functor

Exercise

Find a universal element for the contravariant functor $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Set}$.

Let consider the set $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$. Then the couple $(\mathbf{2}, \{\mathbf{1}\})$ is a universal element for $\mathcal{P}$.

Indeed : Let X be a set and $Y \in \mathcal{P}(X)$, then there exists a function $X \to \mathbf{2}$ which "define" Y in the sense that it maps an element of Y to $\mathbf{1}$ and an element which is not in Y to $\mathbf{0}$.

Then the powerset of this function $\mathcal{P}f : \{\emptyset, \{\mathbf{0}\}, \{\mathbf{1}\}, \mathbf{2}\} \to \mathcal{P}(X)$ is defined by $A \mapsto f^{-1}\langle A \rangle$.

So one has $\emptyset \mapsto \emptyset$, $\{\mathbf{0}\} \mapsto X \setminus Y$, $\{\mathbf{1}\} \mapsto Y$ and $\mathbf{2} \mapsto X$. Finally one has a function which powerset maps $\{\mathbf{1}\}$ to Y, and any powerset of a function f satisfying this property implies f is the characteristic function of the subset Y and therefore one has the unicity of the constructed function.

8.1.3 Universal arrows for forgetful functors

Exercise

Find (from any given object) universal arrows to the following forgetful functors:

(a) $\mathbf{Ab} \to \mathbf{Grp}$

(b) $\mathbf{Rng} \to \mathbf{Ab}$ (forget the second law)

(c) $\mathbf{Top} \to \mathbf{Set}$

(d) $\mathbf{Set}_* \to \mathbf{Set}$

(a) Let's call U the forgetful functor $\mathbf{Ab} \to \mathbf{Grp}$ and let fix a group G. One may denote $D(G)$ the normal subgroup of G generated by the commutators.

Then $(G/D(G), p : G \to G/D(G))$ is a universal arrow from G to U.

Indeed : given another abelian group A and a morphism $f : G \to UA$, one has necessarily $f(D(G)) = \{0_A\}$ so f does factorise through p : $\exists g : G/D(G) \to UA$ such that $gp = f$. Moreover as g is a group homomorphism, it is also a morphism for the category $\mathbf{Ab}$.

(b) Let's call $U : \mathbf{Rng} \to \mathbf{Ab}$ the forgetful functor. And let A be an abelian group. Since the category of abelian groups is equivalent (even isomorphic) to the category of $\mathbb{Z}$-modules (well known result), A will be considered as a $\mathbb{Z}$-module on its own.

One may remark that the construction of the tensor algebra (c.f. exercise 8.1.1) never uses the vector space structure, therefore it is still valid for modules. Let thus consider $T(A)$ the tensor algebra on A, it is a $\mathbb{Z}$-algebra.

But it is also known that $\mathbb{Z}$-algebras are exactly rings with unit the image of $1_{\mathbb{Z}}$, so $T(A)$ is a ring. Moreover the construction of $T(A)$ gives a morphism of groups $i : A \to UT(A)$ (as $T(A) = \cdots \oplus A \oplus \cdots$).

I claim that $(T(A), i : A \to U(T(A))$ is universal from A to U.

Indeed given a group homomorphism $f : A \to U(R)$, by reasoning as in exercise 8.1.1, one can construct an arrow $f_n : A^n \to U(R)$ which is n-$\mathbb{Z}$-linear, so it gives a morphism of group $A^{\otimes n} \to UR$. Gathering those arrows gives then a unique morphism of groups $g : UT(A) \to UR$ such that $gi = f$. The uniqueness result from the fact that each arrow from the n-th tensor power of A to UR is unique.

Moreover one can see that those morphisms respect the product in $T(A)$, therefore g can be seen as a ring homomorphism and finally it is unique among those morphism $T(A) \to R$ which satisfy $Ugi = f$.

The universality follows.

(c) Let's again call U the forgetful function $\mathbf{Top} \to \mathbf{Set}$. And let fix a set X. One may denote A the topological space X with the discrete topology (ie $\mathcal{P}(X)$).

Let B be a topological space with underlying set $Y = UB$ and topology τ. Given a function $v : X \to UB = Y$, it gives a continuous map $c : A \to B$ as the topology on A makes every function continuous.

And as its underlying function v factorise through the identity $X \to UA$, one has shown that $(A, id : X \to A)$ is universal from X to U.

(d) Let's one last time call U the forgetful functor $\mathbf{Set}_* \to \mathbf{Set}$. And let fix a set X. Then one may call A the set $X \sqcup \{X\}$ and (A, a) the pointed set in $a = \{X\}$.

Then given a pointed set (B, b) and an arrow $f : X \to B$, it can be extended to A by demanding $f(a) = b$. Therefore it gives a morphism $f' : (A, a) \to (B, b)$ which underlying function factorise through the injection $i : X \hookrightarrow A$.

The unicity of the arrow f' among those which functions are factorisable in this way, result from the pointed set condition $a \mapsto b$.

Then one has shown that $((A, a), i)$ is universal from X to U.

8.1.4 Results in group theory

Exercise

Use only universality of the projections to prove the following isomorphisms of groups :

(a) Given $M, N \lhd G$ with $M \subset N$, one has $(G/M)/(N/M) \cong G/N$;

(b) Given $N \lhd G$ and S subgroup of G, then $SN/N \cong S/(S \cap N)$.

Lemmas

Injection of subgroups : If H is a subgroup of G then the injection $i : H \to G$ is monic.

It results from the fact that set injections are monics, that the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ is faithful and on the result of exercise 6.3.9.

Projection to the quotient : If $N \lhd G$, then the projection $p : G \to G/N$ is epi.

Indeed : if $a, b : G/N \to H$ such that $ap = bp$, then the arrow ap satisfy $api = 0$ where $i : N \to G$ and $0 : N \to \{e\} \to H$.

So ap is an element of the set of arrows from G which kills N, by universality of p among such arrows, there exists a unique morphism $c : G/N \to H$ such that $cp = ap$. As a and b both satisfy this condition, the uniqueness of c yields $a = b$. So p is epi.

(a) The only result given is that for $M \lhd G$, the morphism $p : G \to G/M$ is a universal object of the functor $H_M : \mathbf{Grp} \to \mathbf{Set}$ given by $K \mapsto \{f : G \to K \mid f(M) = \mathbf{1}\}$.

Let consider the arrow $p_N : G \to G/N$, as $N \subset M$, $p_N \in \{f : G \to G/N \mid f(M) = \mathbf{1}\}$ then the universality of H_M gives the existence of $u : G/M \to G/N$ such that $up_M = p_N$.

Now let prove $N/M \lhd G/M$. Indeed : $p_M^{-1}(u^{-1}(\{1\})) = p_N^{-1}(\{1\})$ so by composing with p_M one gets $\ker u = p_M(N) = N/M$. N/M is therefore a normal subgroup of G/M, so the projection $\pi : G/M \to (G/M)/(N/M)$ is universal for the functor $H_{N/M}$.

Moreover one has shown that $u \in \{f : G/M \to G/N \mid f(N/M) = \mathbf{1}\}$ so there exists $v : (G/M)/(N/M) \to G/N$ such that $v\pi = u$. Then the following diagram commutes :

$$
\begin{tikzcd}
G \arrow[d, "p_M"] \arrow[dr, "p_N"] & \\
G/M \arrow[r, "u"] \arrow[d, "\pi"] & G/N \\
(G/M)/(N/M) \arrow[ur, "v"] &
\end{tikzcd}
$$

Finally $\pi(p_M(N)) = \pi(N/M) = \mathbf{1}$ so $\pi \circ p_M \in \{f : G \to (G/M)/(N/M) \mid f(N) = \mathbf{1}\}$. Therefore the universality of p_N for H_N gives the existence of $v' : G/N \to (G/M)/(N/M)$ such that $v'p_N = \pi p_M$.

Then $vv'p_N = v\pi p_M = up_M = p_N$ and as p_N is a projection, it is epi so $vv' =$Id. And on the other side, $\pi p_M = v'p_N = v'up_M = v'v\pi p_M$ and as both p_M and π are epis, $v'v =$Id. Then v is a group isomorphism.

8.1.5 Universality of the quotient K-module

Exercise

Show that the quotient of K-module A/S has a description by universality. Derive isomorphism theorems.

Given two K-modules A, S, with S sub-module of A, one has the following diagram in K-**Mod** for any M:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[d, "!"] & A \arrow[d, dotted, "f"] \\ \mathbf{0} \arrow[r, "!"] & M \end{tikzcd} \qquad (\alpha)$$

Let then consider the functor $H : K\text{-}\mathbf{Mod} \to \mathbf{Set}$ given by $M \mapsto \{f : A \to M \mid (\alpha) \text{ commutes.}\}$ (it is defined on arrows by left-composition).

Now a universal element for the functor H is of the form $(M, p : A \to M)$ with $pi = 0$ and satisfy the universality of a infimum for all the couple that makes (α) commutes.

Let's prove that $(A/S, p : A \to A/S)$ is universal : It satisfy by definition the property $pi = 0$. Moreover given $M, f : A \to M)$ such that $fi = 0$, then f respects the S-equivalence on A so it factorises through p into a $u : A/S \to M$.

This is exactly saying that A/S is universal.

8.1.6 Universality of the quotient ring

Exercise

Describe quotients of a ring by a two-sided ideal by universality.

Merely : Let I be a two sided ideal of R a ring. Then the quotient $A = R/I$ a ring given with a map $p : R \to A$ such that $pI = 0$. And which satisfy the universal property :

For all $f : R \to B$ ring morphism, if $fI = 0$ then there exists a unique ring morphism $f' : A \to B$ such that $f = f'p$.

This states that $(A, p : R \to A)$ is universal among the morphisms $R \to B$ which kills I. Therefore let T be the functor from the category of rings to the sets given by $T(B) = \{f : R \to B \mid fI = 0\}$ and if $u : B \to C$ is a ring homomorphism, T maps u to the left composition with u from $T(B)$ to $T(C)$. One has (A, p) is a universal element for T

8.1.7 Universality of the polynomial ring

Exercise

Show that the construction of the polynomial ring $K[x]$ in an indeterminate x over a commutative ring K is a universal construction.

The usual construction for $K[x]$ is the following :

$$K[x] = \bigcup_{n \in \mathbb{N}} j_n(K^n)$$

where j_n is the injection $K^n \hookrightarrow K^{\mathbb{N}} : (a_1, \cdots, a_n) \mapsto (a_1, \cdots, a_n, 0, \cdots)$.

The sequence $(1, 0, 0, \cdots)$ is noted 1, and this gives an injection $i : K \to K[x]$, while $(0, 1, 0, \cdots)$ is noted x, and x^k for the $k+1$th vector of the basis : $(\delta_{i,k})_i$. It then inherits a structure of K-module as sub-module of $K^{\mathbb{N}}$.

Moreover, one may define the product $\mu : K[x] \times K[x] \to K[x]$ as a bilinear application defined on the basis by $\mu(x^k, x^q) = x^{k+q}$. It is then a ring.

It can be seen as a kind of "free ring on K and x". Because it as the following universal property :

Given a ring R and a ring homomorphism $l : K \to R$, for all $r \in R$ there exists a unique morphism of rings $f : K[x] \to R$ such that $f(x) = r$ and $\forall k \in K, f(k) = l(k)$.

This exactly states that $(i : K \to K[x], x)$ is a universal element for the forgetful functor $U : (K \downarrow \mathbf{Rng}) \to \mathbf{Set}$ which maps an arrow to the underlying set of its codomain.

Indeed, given a ring under K, R with an arrow $f : K \to R$, and an element r in the underlying set of R, one can associate the arrow in $(K \downarrow \mathbf{Rng})$ given by $f' : K[x] \to R$ which satisfy $f'i = f$ and $Uf' : x \mapsto r$. It is unique as the last property totally define f' as a ring homomorphism.

8.2 Exercises on section 2 : Yoneda's Lemma p.62

8.2.1 Universality of representation

Exercise

> Let $K, K' : D \to \mathbf{Set}$ be functors having representations (r, ψ) and (r', ψ'), respectively. Prove that to each natural transformation $\tau : K \dot{\longrightarrow} K'$, there is a unique morphism $h : r' \to r$ such that
> $$\tau \cdot \psi = \psi' \cdot D(h, \_) : D(r, \_) \dot{\longrightarrow} K'$$

One has that $(\psi')^{-1} \cdot \tau \cdot \psi$ is a natural transformation $D(r, \_) \dot{\longrightarrow} D(r', \_)$.

But Yoneda lemma gives a bijection $y : D(r', r) \cong \mathrm{Nat}(D(r, \_), D(r', \_))$ that maps $h : r' \to r$ to $D(h, \_)$. Then the above natural transformation is of the form $D(h, \_)$ for a unique $h : r' \to r$.

It states exactly that $(K', D(r', \_))$ is universal from the contravariant functor $D(\_, \bullet) : D^{op} \to \mathbf{Set}^D$.

8.2.2 Dual of Yoneda lemma

Exercise

> State the dual of the Yoneda lemma (replacing D by D^{op}).

Let $K : D^{op} \to \mathbf{Set}$ and $r \in D$, then there is a bijection $y : \mathrm{Nat}(D^{op}(r, \_), K) \cong Kr$.

So the contravariant functor $\overline{K}$ satisfy : $y' : \mathrm{Nat}(D(\_, r), \overline{K}) \cong Kr$.

8.2.3 Kan; the coyoneda lemma

Exercise

For $K : D \to \mathbf{Set}$, $(\star \downarrow K)$ is the category of elements $(d, x \in Kd)$. One may define the projection $Q : (\star \downarrow K) \to D$ which maps (d, x) to d, and the constant functor $a : (\star \downarrow K) \to D$, $(d, x) \mapsto a$ for a fixed $a \in D$. Establish a natural isomorphism

$$\mathrm{Nat}(K, D(a, \_)) \cong \mathrm{Nat}(a, Q)$$

One has :

$$(\star \downarrow K) \underset{Q}{\overset{a}{\rightrightarrows}} D \underset{D(a,\_)}{\overset{K}{\rightrightarrows}} \mathbf{Set}$$

Now let's take $\tau : K \dot{\longrightarrow} D(a, \_)$ a natural transformation lying in $\mathrm{Nat}(K, D(a\_))$. Then for all $d \in D$, the map $\tau_d : Kd \to D(a, d)$ so for all $r \in Kd$, $\tau_d(r) \in D(a, d)$ i.e. $\tau_d(r) : a \to d$.

But $x = (d, r)$ is an element of $(\star \downarrow K)$, and with it one can rewrite the arrow $\tau_d(r)$ as follows : $ax = a \to d = Qx$, and call this one σ_x.

Then I claim that σ is a natural transformation lying in $\mathrm{Nat}(a, Q)$. Indeed remains to show the naturality in $x = (d, r)$ expressed in the following diagram :

$$\begin{tikzcd}
a \arrow[r, "\tau_d(r)"] \arrow[d, "id = af"'] & Qx = d \arrow[d, "Qf"] \\
a \arrow[r, "\tau_c(s)"] & Qy = c
\end{tikzcd}$$

Where $f : (d, r) \to (c, s)$ in the comma category (where $y = (c, s)$) which yields $f : Kd \to Kc$ with $f(r) = s$. So $Qf = f$ and moreover, as τ is natural, $\tau_c \circ f = f \circ \tau_d$ so $Qf\sigma_x = f \circ \tau_d(r) = \tau_c f(r) = \tau_c(s) = \sigma_y$: the above diagram is commutative.

Conversely, given a natural transformation $(\sigma_{(d,r)})_{(d,r) \in (\star \downarrow K)} : a \dot{\longrightarrow} Q$, it yields a family of arrows $\tau_d : Kd \to D(a, d)$ with $r \mapsto \sigma_{(d,r)}$ for $d \in D$; which has a kind of "pointwise naturality" (namely the naturality in $r \in Kd$). So τ is a natural transformation lying in $\mathrm{Nat}(K, D(a, \_))$.

One may easily remark that those two construction are inverse of each other, moreover there are defined the same way for all $a \in D$; this gives us the required natural isomorphism.

8.2.4 Naturality is not changed by enlarging the codomain category

Exercise

Let E be a full subcategory of E'. For functors $K, L : D \to E$, with $J : E \to E'$ the inclusion, prove that $\mathrm{Nat}(K, L) \cong \mathrm{Nat}(JK, JL)$.

One sense is obvious : any natural transformation τ between K and L extends to a natural transformation between JK and JL, namely $J \circ \tau$.

Conversely given any natural transformation $\sigma : JK \dot{\longrightarrow} JL$, as J is fully faithful, for every $d \in D$, σ_d is an arrow in E from Kd to Ld, and is obviously natural in d. So one has proved that σ can be seen as a natural transformation in E.

8.3 Exercises on section 3 : Coproducts and Colimits p.68

8.3.1 Tensor product in commutative rings

Exercise

> In the category of commutative rings, show that $R \to R \otimes S \leftarrow S$, with maps $r \mapsto r \otimes 1$ and $1 \otimes s \leftarrow s$, is a coproduct diagram.

The tensor product of rings is the tensor product of $\mathbb{Z}$-algebras : as the $\mathbb{Z}$-algebra structure of any ring is given by the unique ring homomorphism $\mathbb{Z} \to R$. (This is in fact an injection – not full – of categories).

Let U be a ring with two homomorphisms $a : R \to U$ and $b : S \to U$, then by definition $a \otimes b : R \otimes S \to U$ is associated with the bilinear function $\mu_U(a\_, b\_) : R \times S \to U \times U \to U$.

Then this function satisfy $(a \otimes b)(r \otimes 1) = \mu_U(a(r), b(1)) = a(r)$ and $(a \otimes b)(1 \otimes s) = \mu_U(a(1), b(s)) = b(s)$. So we got this factorisation :

$$\begin{tikzcd} R \arrow[r] \arrow[dr, "a"'] & R \otimes S \arrow[d, "a \otimes b"] & S \arrow[l] \arrow[dl, "b"] \\ & U & \end{tikzcd}$$

Moreover the uniqueness of this function as ring homomorphism, result of the uniqueness as $\mathbb{Z}$-linear function.

8.3.2 Coproducts and coequalizers yields pushouts

Exercise

> If a category has (binary) coproducts and coequalizers, show that it also has pushouts. Apply to **Set**, **Grp**, and **Top**.

Let's take a, b, c objects of D category having coproducts and coequalizers, and $u : c \to a$, $v : c \to b$. One wants to construct the pushout of the pair (u, v).

First one may construct the coproduct $a \amalg b$, with the injections $i : a \to a \amalg b$ and $j : b \to a \amalg b$.

Then one has two arrows $iu, jv : c \to a \amalg b$, so let's construct the coequalizer $f : a \amalg b \to e$ of those two.

I claim that

$$\begin{tikzcd} c \arrow[r, "u"] \arrow[d, "v"] & a \arrow[d, "fj"] \\ b \arrow[r, "fi"] & e \end{tikzcd}$$

is a pushout of (u, v).

Indeed by construction of the coequalizer, the above diagram commutes. Moreover given a commutative diagram :

$$
\begin{tikzcd}
c \arrow[r, "u"] \arrow[d, "v"] & a \arrow[d, "k"] \\
b \arrow[r, "h"] & d
\end{tikzcd}
$$

the property of the coproduct $a \amalg b$ yields the existence of a unique factorisation $g : a \amalg b \to d$ such that $gi = h$ and $gj = k$.

And then as $giu = gjv$ the universal property of the coequalizer of (iu, jv) gives a unique arrow $l : e \to d$ such that $lf = g$. Then it means $l(fi) = h$ and $l(fj) = k$. As all the above construction is given in a unique way, l is unique among such arrows. It is then exactly saying that (fi, fj) is the pushout of (u, v).

Applications :

In Set : The coproduct of two sets a, b is the disjoint union $a \sqcup b$ with the injection arrows.

And the coequalizer of two arrows $f, g : c \to d$ is the quotient of d by the equivalence relation $\sim$ generated by $\forall x \in C,\ f(x) \sim g(x)$.

Therefore a pushout in sets is just a couple of projections onto a quotient set.

In Grp : The coproduct of two groups a, b is the free product $a * b$ with the injection arrows.

And the coequalizer of two morphisms $f, g : c \to d$ is the quotient of d by the normal subgroup N which is the normaliser of the set $\{f(x)g(x)^{-1} \mid x \in c\}$. The universal property follows.

As result, pushouts in the category of groups are the amalgamated sums (quotient of the free product by the normaliser of the set on which the functions agree).

In Top : The coproduct of a, b is as in set, the disjoint union $a \sqcup b$ with the topology generated by the union of the topologies.

And the coequalizer of two continuous maps $f, g : c \to d$ is also the same as in sets : The quotient of d by the smallest equivalence relation which identify f and g together with the quotient topology.

Hence the pushout in **Top** of the following diagram $X_1 \leftarrow X_0 \to X_2$ is the patch of X_1 and X_2 along the image of X_0 on each side.

8.3.3 Coequalizer of two matrices

Exercise

> Describe the coequalizer of $A, B : n \to m$ in $\mathbf{Matr}_K$.

A coequalizer of the pair (A, B) is a matrix $C : m \to k$ such that $CA = CB$ and if one has $DA = DB$ for $D : m \to k'$, then there exist a unique matrix $M : k \to k'$ such that $MC = D$.

Any such matrix satisfy $D(A - B) = 0$ i.e. $\mathrm{Im}(A - B) \subset \ker D$; then C is the cokernel of $A - B$.

Given a basis of $\mathrm{Im}(A-B)$:$(e_1, \cdots, e_d)$ in K^m, it can be extended to a basis of m. Let's now define $c' : K^m \to K^m$ as $e_i \mapsto 0$ for $i \leq d$ and $e_i \mapsto e_i$ for $i > d$, then c' can factorise through an arrow $c : K^m \to K^{m-d+1}$ and an injection $K^{m-d+1} \to K^m$.

Finally the matrix C of c in the canonical basis of K^m is an arrow $m \to m-d+1$ which coequalize A and B. Moreover if D coequalize A and B, given any vector e_i of the image of $A-B$, then $De_i = 0$. In fact one has for all vector e_i of the basis of m, $De_i = Dc'e_i$, hence $D = DC' : m \to k'$. But $C' = JC$ where J is an injection $k \to m$, so $D = (DJ)C$.

Eventually this factorisation is unique as if there are two factorisations $D = MC$ and $D = NC$ then N and M coincident on $\mathrm{Im}C$ but by construction C is surjective so $M = N$.

8.3.4 Existence of coproducts in Cat and such categories

Exercise

Describe coproducts (and show that they exists) in **Cat**, **Mon** and **Grph**.

In Cat : Let A, B be two small categories. Then one can build the category C of objects the union of those of A and B. And with arrows also the reunion of those of A and B (there is no arrow from an object of A to an object of B and symmetrically.)

This construction yields two functors which are the inclusions functors $i : A \to C$ and $j : B \to C$.

Let show that this construction satisfy the universal property of a coproduct.

If $u : A \to D \leftarrow B : v$ then one can construct a functor $w : C \to D$ given by $x \mapsto u(x)$ if $x \in A$, $x \mapsto v(x)$ otherwise ($x \in B$). And the definition on arrows is based on the fact that any arrow f in C is either an arrow in A so is mapped to uf or an arrow in B and then is mapped to vf.

Finally w satisfy $wi = u$ and $wj = v$, and is unique among the arrows satisfying such property as i and j give together all the information of C.

In conclusion, coproducts exists in **Cat** and are disjoint union of categories.

In Mon : Given two monoids M, N, one may construct the free monoid on M and N : $D = M * N$.

Let's construct $A = (M \cup N)^{(\mathbb{N})} = \cup_{n\in\mathbb{N}} i_n (M \cup N)^n$ where i_n is the injection of sets $(M \cup N)^n \hookrightarrow (M \cup N)^{\mathbb{N}}$.

It is a monoid, as free monoid (the set of words) on a set of letters (alphabet). Its multiplication $\mu : A \times A \to A$ is the concatenation of words.

Then two words u, v in A are said to be equivalent if there exists $a, b \in A$ such that $u = au'b$, $v = av'b$, and either

- It exists $p, q \in \mathbb{N}$ such that $u' \in M^p$ and $v' \in M^q$ and their projections to M are equal.
- It exists $p, q \in \mathbb{N}$ such that $u' \in N^p$ and $v' \in N^q$ and their projections to N are equal.

And moreover $e_M \sim e_N \sim ()$ (the empty word which is the neutral element).

This relation can be extended via transitivity into an equivalence on A, moreover it is compatible with the multiplication. Then $A/\sim$ is a monoid; together with two morphisms $N, M \to A/\sim$.

This quotient monoid has the universal property of a coproduct as if a monoid E is the target of two arrows f, g of respective domains N and M, then it gives an arrow from $A \to E$

(obtained by extending f, g as function into a function $f+g : M \cup N \to E$ and then extending it into a map from the free monoid (by, say, universality of the free monoid))

This arrow respects the foregoing equivalence and then factors through an morphism $A/\sim \to E$ which makes the injections commute with f and g.

In Grph : The construction is quite similar to the one in **Cat** : Given two graphs $a : O \rightrightarrows A, b : O' \rightrightarrows B$, one may create a new graph $c : O \cup O' \rightrightarrows A \cup B$. (The existence of coproduct –union– in sets allows us to extend the domain and codomain arrow of the graphs a and b)

Then any graph with functions from a and b, factor through c by the same argument used in **Cat**.

8.3.5 Equivalence relation and coequalizers

Exercise

> If E is an equivalent relation on a set X, show that the usual set X/E can be described by a coequalizer in **Set**

Axioms of equivalence relation : If E is a reflexive relation on X, then it can be seen as a set equipped with two functions $p, q : E \rightrightarrows X$ and an arrow of "diagonal injection" $d : X \to E$ such that $pd = qd = \mathrm{id}_x$.

Moreover an symmetric relation has an endomorphism $s : E \to E$ satisfying $ps = q$ and $qs = p$.

Eventually a transitive relation in the category of sets has a morphism $t : E \times_X E \to E$ such that $p\pi_1 = pt$ and $q\pi_2 = qt$ where π_i is the projection of the product onto the i-th component.

Result (using elements): Now lets show that X/E is the coequalizer of p and q. Indeed if one calls $r : X \to X/E$ the projection, then $rp = rq$. Moreover given a morphism $f : X \to Y$ such that $fp = fq$

Given an element $\xi \in X/E$, one can find an element $x \in X$ such that $rx = \xi$. Then any element $y \in X$ such that $rx = \xi$ satisfy the existence of an element $a \in E$ such that $pa = x$ and $qa = y$, so for any of theses elements y, $f(y) = fqa = fpa = f(x)$. Then one may denote this element $f'(\xi)$. The map f' is everywhere defined and satisfy $f'r = f$.

This show the universality of $p : X \to X/E$ among the arrows equalising p and q.

8.3.6 Existence of coproducts and reprensentability

Exercise

> Show that a and b have a coproduct in C iff the following functor is representable : $C(a, \_) \times C(b, \_) : C \to$ **Set**.

The coproduct of (a,b) is a universal arrow from (a,b) to the diagonal functor $\delta : C \to C\times C$.
So this means that there is an isomorphism

$$C(a \amalg b, d) \cong \mathrm{Hom}_{C\times C}((a,b),(d,d)) = C(a,d) \times C(b,d)$$

which is natural in d. So the functor $C(a,\_) \times C(b,\_)$ is representable.

Conversely if r represents that functor, then r satisfy the universal property of $a \amalg b$ and therefore it exists.

8.3.7 Every abelian group is a colimit of its finitely generated subgroups

Exercise

> If A is an abelian group and J_A is the preorder of its finitely generated subgroups ordered by inclusion, show that A is the colimit of the evident functor $J_A \to \mathbf{Ab}$. Generalise.

Lets call $F : J_A \to \mathbf{Ab}$ the functor which maps the inclusion to the injection morphisms.

One has the following cone to A :

$$\begin{tikzcd}
F_1 \arrow[dr, "i_1"] & \cdots & F_j \arrow[dl, "i_j"] & \cdots \\
& A &
\end{tikzcd}$$

Given a cone in $\mathbf{Ab}$, of base F to the vertex c : $\tau : F \dot{\longrightarrow} \Delta c$.

Construction using (excessively) elements : We will denote for a group G (in $\mathbf{Ab}$ or J_A), G' its underlying set, and similarly for group homomorphisms.

One wants to construct an arrow $f : A \to c$ such that $\tau = f\sigma$ where $\sigma : F \dot{\longrightarrow} \Delta A$ is the cone of the above diagram.

Given $x \in A'$, the subgroup G_x of A generated by x is in J_A. Therefore there exists an arrow $\tau_x : G_x \to c$ which underlying function $\tau'_x : x \mapsto y \in c'$. One may then define $f' : A' \to c'$ by $f'(x) = y = \tau'_x(x)$.

First one may notice that if ever $x \in H'$ where $H \in J_A$, then there is an arrow in J_A : $G_x \subset H$. Therefore as τ is a natural transformation the following diagram commutes :

$$\begin{tikzcd}
F(G_x) \arrow[r, hook] \arrow[d, "\tau_x"] & F(H) \arrow[dl, "\tau_H"] \\
c &
\end{tikzcd}$$

And then $\tau'_H : x \mapsto f'(x)$.

One still has to show that f' can be extended as a group morphism $f : A \to c$. But as the group generated by e_A is $\{e_A\}$, necessarily $f'(e_A) = e_c$. Moreover given two elements $a, b \in A'$, then if $G = \langle a, b\rangle_A \in J_A$, one has $G_a, G_b \subset G$ and moreover $G_{ab} \subset G$, so as stated in the above remark $\tau'_G(a) = f'(a)$, $\tau'_G(b) = f'(b)$ and $\tau'_G(ab) = f'(ab)$. Then in c, as τ_G is a group morphism, $f'(a)f'(b) = f'(ab)$.

Finally f is a group homomorphism $A \to c$ and moreover the construction of f obviously makes $\tau = f\sigma$ as for all $i \in J_A$, σ_i is the inclusion map.

Generalisation : Well... wide question...

It seems I haven't used the hypothesis A abelian. Moreover it seems that the construction holds for monoids, with the category of finitely generated sub-monoids...

The only point where the construction may not be generalised is showing that the given arrow f' can be extended into a morphism.

The concept of finitely generated sub-something also needs to be extended : Given a category C with a forgetful (faithful) functor $U : C \to \mathbf{Set}$. Given $x \in Uc$ the sub-C-object generated by x may be an element $b \in C$ such that $x \in Ub$ and there exists an arrow $i : b \to c$ in C such that Ui is the inclusion $Ub \subset Uc$. And moreover given any $(a, j : a \to c)$ with such property, there exist a $k : b \to a$ such that Uk is also an inclusion.

So the sub-C-object generated by $x \in Uc$ is universal from ... to Uc.

8.4 Exercises on section 4 : Products and Limits p.70

8.4.1 Pullbacks in Sets

Exercise

> In **Set**, show that the pullback of $f : X \to Z$ and $g : Y \to Z$ is given by the set of pairs $\{(x, y) \mid x \in X, y \in Y, fx = gy\}$. Describe pullbacks in **Top**.

In sets : Let's call $A = \{(x, y) \mid x \in X, y \in Y, fx = gy\}$ and p, q its projections onto respectively the first (X) and the second coordinate (Y).

By definition of A, it is obvious that $gq = fp$. Now let's take a set B and two arrows u, v such that the following diagram commutes :

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[d, "v"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Then for $b \in B$, one has $(ub, vb) \in A$ because $fub = gvb$. It gives us a function $(u, v) = h : B \to A$ which maps b to (ub, vb). It follows that $ph = u$ and $qh = v$.

Moreover if another such arrow k exists, then $pkb = ub$ and $qkb = vb$ so $k = h$.

One has thus shown that A is the pullback of f, g in the category of sets.

In the category of topological spaces : We define the underlying set of the pullback in the same way as in sets : $A = \{(x, y) \mid x \in X, y \in Y, fx = gy\}$. But has p and q needs to be morphisms of topological spaces, one has to equip A with the least fine topology making those two continuous; (which is the induced topology as a subset of the catesian topological product $X \times Y$).

With the notation above (replacing functions by continuous functions), one has got a unique function $h : B \to A$ such that $ph = u$ and $qh = v$. Then one may show h is continuous : because the two functions ph and qh are continuous and by definition of the product topology.

8.4.2 The usual cartesian is the categorical product in sets

Exercise

> Show that the usual cartesian product over an indexed set J, with its projections, is a (categorical) product in **Set** and in **Top**.

In the category of sets : Let's denote A the usual cartesian product : $A = \{(x_j)_{j\in J} \mid \forall j \in J,\ x_j \in X_j\}$ One then may define for $j \in J$ a function $p_j : A \to X_j$ given by $(x_i)_{i\in J} \mapsto x_j$.

Now let's show that A has the universal property of the product : given a family of functions $u_j : B \to X_j$, one may consider the function $\phi : B \to A$ given by $x \mapsto (u_j x)_{j\in J}$.

The definition immediately yields that $\forall j \in J,\ p_j\phi = u_j$ so A is the categorical product.

In the category of topological spaces : The construction is the same, and the topology taken on A is the least fine topology that makes the p_j continuous.

Now with the same notations as above, one can show that ϕ is continuous as all the functions $p_j\phi$ are.

8.4.3 Existence of limits using an initial object

Exercise

> If the category J has an initial object s, prove that every functors $F : J \to C$ to any category C has a limit, namely $F(s)$. Dualize.

Let call $r = F(s)$, for all $j \in J$, there exists a unique $u_j : s \to j$. Then one may define $\tau_j = F(u_j) : r \to F_j$; I claim this family of morphisms is a natural transformation $\Delta r \dot{\longrightarrow} F$.

Indeed the unicity of the u_j yields that for any $v : j \to k$, one has $vu_j = u_k$, by applying F, one gets the naturality $F(v)\tau_j = \tau_k F(1_s)$.

Moreover τ is universal among such natural transformations $\sigma : \Delta c \dot{\longrightarrow} F$. Because given any such natural transformation σ, there is a morphism $\sigma_s : c \to r$ and then the naturality of σ makes the following diagram commutes:

$$\begin{tikzcd} c \arrow[r, "\sigma_j"] \arrow[d, "\sigma_s"'] & F_j \\ r \arrow[ur, "\tau_j = F(u_j)"'] & \end{tikzcd}$$

Therefore one has $\sigma = \sigma_s \circ \tau$, which shows the naturality of τ.

Then $F(s) = \varprojlim F$

Dual statement : Given a category J with a terminal object t, then for any functor $J \to C$, F has a colimit : $F(t)$.

8.4.4 Epi is a pushout condition

Exercise

> In any category, prove that $f : a \to b$ is epi iff the following square is a pushout :
>
> $$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "f"'] & b \arrow[d, "\mathbf{1}"] \\ b \arrow[r, "\mathbf{1}"'] & b \end{tikzcd}$$

If f is epi, then any commutative diagram of the kind :

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "f"'] & b \arrow[d, "u"] \\ b \arrow[r, "v"'] & c \end{tikzcd}$$

yields $uf = vf$ then as f is epi, $u = v$. Therefore any factorisation of this diagram through the one in the wording implies the existence of a unique $w = u = v$. Therefore it is a pushout.

Conversely given a two arrows $u, v : b \to c$, such that $uf = vf$, then the diagram above is still commutative and thus the universality of the pushout yields the existence of a unique arrow $w : b \to c$ such that $u = w = v$ so $u = v$ and then f is epi.

8.4.5 Pullbacks conserve monics

Exercise

> Given the following pullback square with f monic, in any category, show that q is monic :
>
> $$\begin{tikzcd} c \arrow[r, "q"] \arrow[d, "p"'] & d \arrow[d, "g"] \\ b \arrow[r, hook, "f"'] & a \end{tikzcd}$$

If one has the following pair $u, v : e \rightrightarrows c$ such that $qu = qv$.

Then $qu = qv$ implies $gqu = gqv$ and as the diagram above commutes $fpu = gqu = gqv = fpv$. But f is monic therefore $pu = pv$.

Moreover one has $gqu = fpu = fpv$ so by universality of c, there is a unique arrow $w : e \to c$ such that $pv = pw$ and $qu = qw$. But both u and v satisfy this property, so $u = v$.

8.4.6 Kernel pair in sets

Exercise

> In **Set**, show that the kernel pair of $f : X \to Y$ is given by the equivalence relation $E = \{(x, x') \mid x, x' \in X,\ fx = fx'\}$, with suitable maps $E \rightrightarrows X$.

Let consider the set E as defined above. It comes with two projections $p, q : E \rightrightarrows X$ which maps respectively (x, x') to x and x'. One obviously has $fp = fq$.

Now given another set F with two arrows $u, v : F \to X$ such that $fu = fv$, then let's consider the function $h : F \to E$ defined by $h(a) = (ua, va)$. One therefore has $ph = u$ and $qh = v$, and moreover h is a unique arrow with theses properties as any element b of E is of the form (pb, qb).

Finally, $p, q : E \rightrightarrows X$ is a pair equal under f and universal among such pairs, so it is the kernel pair of f.

8.4.7 Kernel pair via product and equalizers

Exercise

> If C has finite product and equalizers, show that the kernel pair of $f : a \to b$ may be expressed in terms of the projections $p_1, p_2 : a \times a \to a$ as $p_1 e, p_2 e$, where e is the equalizer of fp_1, fp_2.

The dual statement of exercise 8.3.2 gives that as C has finite products and equalizers, then it as pullbacks.

The kernel pair of f is the pushback in the following diagram :

$$\begin{tikzcd}
c \arrow[r, "u"] \arrow[d, "v"] & a \arrow[d, "f"] \\
a \arrow[r, "f"] & b
\end{tikzcd}$$

So c has two arrows $u, v : c \to a$, by universal property of the product $a \times a$, there exists a unique morphism $e = u \times v$ such that $p_1 e = u$ and $p_2 e = v$.

It only remains to show that e is the equalizer of fp_1 and fp_2, but as c is the pushout of (f, f) any equalizer factors through c, so e is the equalizer.

8.4.8 Concatenation of pullbacks

Exercise

> Consider the following diagram :
>
> $$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$
>
> (a) If both little squares are pullbacks, prove the outside rectangle is a pullback.
>
> (b) If the outside rectangle and the right-hand square are pullbacks, so is the left-hand square.

(a) : Let's take the following notations :

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "k"] & \cdot \arrow[r, "u"] \arrow[d, "v"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "w"] & \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

Suppose one has the following diagram commutes:

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "b"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "fw"] & \cdot \end{tikzcd}$$

We want to show that a and b factorise through a unique arrow d such that $a = uhd$ and $b = kd$.

Then one has $fwb = ga$ but (u, v) is the pullback of (f, g) then there exists a unique c such that $uc = a$ and $vc = wb$. So the following square commutes :

$$\begin{tikzcd} \cdot \arrow[r, "c"] \arrow[d, "b"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "w"] & \cdot \end{tikzcd}$$

And as (h, k) is the pullback of (w, v) then there exist a unique d such that $hd = c$ and $kd = b$.

By gathering the above results one gets $a = uhd$ and $b = kd$.

Moreover d is unique : If d' also factorise a and b : $a = uhd'$ and $b = kd'$, then hd' factorise a and wb but as (u, v) is a pullback, then the unicity of the factorisation yields $hd' = c = hd$.

Then using that (h, k) is a pullback, and $wb = vc$ we can prove that $d = d'$ i.e. the unicity of the factorisation.

(b) : With the initial notations above, let suppose one has the following commutative diagram :

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "b"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "w"] & \cdot \end{tikzcd}$$

Therefore one has this one:

$$\begin{tikzcd} \cdot \arrow[r, "ua"] \arrow[d, "b"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "fw"] & \cdot \end{tikzcd}$$

As (fw, g) as a pullback, namely (uh, k), there exists a unique morphism c such that $uhc = ua$ and $b = kc$.

Now let's prove $a = hc$: One has already $uhc = ua$ moreover $vhc = wkc = wb = va$ so both a and hc are a factorisation of the following diagram by the pullback on the right :

$$\begin{tikzcd} \cdot \arrow[r, "ua"] \arrow[d, "va"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "v"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

So by unicity of the factorisation $a = hc$.

Moreover if d is another morphism such that $a = hd$ and $b = kd$, then $ua = uhd$ and $b = kd$ hence by unicity of the factorisation of (ua, b) by (uh, k), one gets $d = c$.

The existence and the unicity of the factorisation proves that the left square is a pullback.

8.4.9 Equalizers via product and pullbacks

Exercise

Show that the equalizer of $f, g : b \to a$ may be constructed as the pullback of

$$(1_b, f) : b \to b \times a \leftarrow b : (1_b, g)$$

Let's consider the following pullback :

$$\begin{tikzcd} c \arrow[r, "u"] \arrow[d, "v"] & b \arrow[d, "{(1_b, g)}"] \\ b \arrow[r, "{(1_b, f)}"] & b \times a \end{tikzcd}$$

Then the commutativity yields $(u, fu) = (v, gv)$ ie $u = v$ and $fu = gu$. So u is a morphism of C which equalize f and g.

Now given an arrow $w : d \to b$ which equalize f and g, one has the following commutative diagram :

$$\begin{tikzcd} d \arrow[r, "w"] \arrow[d, "w"] & b \arrow[d, "{(1_b, g)}"] \\ b \arrow[r, "{(1_b, f)}"] & b \times a \end{tikzcd}$$

which factorise by the universality of (c, u) through an arrow $h : d \to c$ such that $uh = w$. Moreover this h is unique because any other such morphism $h' : d \to c$ would give another factorisation of w in the pullback, which is forbidden.

It exactly means that $u : c \to b$ is the equalizer of f and g.

8.4.10 Pullbacks and terminal object yields products and equalizers

Exercise

If C has pullbacks and a terminal object, prove that C has all products and equalizers.

Products : Let c with (p,q) denote the pullback of $a \to t \leftarrow b$. Given a d with two arrows $u : d \to a$ and $v : d \to b$, then the following diagram commutes :

$$\begin{array}{ccc} d & \xrightarrow{v} & b \\ \Big\downarrow{\scriptstyle u} & & \Big\downarrow{\scriptstyle !} \\ a & \xrightarrow{!} & t \end{array}$$

So by universality of c there exist a unique arrow $w : d \to c$ such that $pw = u$ and $qw = v$. This is exactly the universal property of the product, so as c satisfy this property, the finite products exist.

Remark : The existence of pullbacks and a terminal object does not necessarily yields all products. A counter-example could be the category **Fin** of small finite sets, with morphisms the usual functions. Then it has pullbacks as in **Set**, a terminal object (namely **1**) therefore finite products but obviously not all infinite product.

Equalizers : The existence of equalizers derive from exercise 8.4.9.

8.5 Exercises on section 5 : Categories with Finite Product p.74

8.5.1 Naturality of the diagonal injection

Exercise

> Prove that the diagonal $\delta_c : c \to c \times c$ is natural in c.

Let C be a category with finite products, the function $\Delta : C \to C$ defined by $c \mapsto c \times c$ is the object function of a functor which maps an arrow $f : c \to d$ to $\Delta f : c \times c \to d \times d$ the unique function satisfying $p'_i \Delta f = f p_i$ for $i = 1, 2$ where the p_i (resp. p'_i) are the projections onto c (resp. d).

One wants to prove that δ is a natural transformation from the identity functor. $\delta : I \dot{\longrightarrow} \Delta$.

The universal property of the product $d \times d$ in the following diagram :

$$\begin{array}{ccc} c & \xrightarrow{f} & d \\ \Big\downarrow{\scriptstyle f} & & \Big\uparrow{\scriptstyle p'_2} \\ d & \xleftarrow[p'_1]{} & d \times d \end{array}$$

yields the existence of a unique morphism $w : c \to d \times d$ such that $p'_1 w = f = p'_2 w$.

But $p'_i(\delta_d f) = (p'_i \delta_d) f = f$ and $p'_i(\Delta f \delta_c) = (p'_i \Delta f)\delta_c = f p_i \delta_c = f$. As δ_c satisfy $p_i \delta_c = 1$.

So by the unicity of the morphism factorising the diagram above, one has $\delta_d f = w = \Delta f \delta_c$. Which can be summarised in the following commutative diagram :

$$\begin{array}{ccc} c & \xrightarrow{\delta_c} & c \times c \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \Delta f} \\ d & \xrightarrow{\delta_d} & d \times d \end{array}$$

This exactly means that δ is natural in c.

8.5.2 Pullbacks in Cat

Exercise

(a) Show that **Cat** has pullbacks.

(b) Show that the comma categories $(b \downarrow \mathbf{C})$ and $(\mathbf{C} \downarrow a)$ are pullbacks in **Cat**.

(a) : We already know that **Cat** has bi-products, so it remains to show that it has equalizers. Then by dual proposition of exercise 8.3.2, it would have pullbacks.

So given $f, g : D \rightrightarrows E$, one can construct the category C with object sets $\{x \in D \mid fx = gx\}$ and arrow set $\{\varphi \in D \mid f\varphi = g\varphi\}$ (similarly it is the same definition for all hom-set).

Then there is a inclusion of categories $i : C \to D$ satisfying $fi = gi$. Conversely given a $j : B \to D$ which coequalize f and g (i.e. $fj = gj$), then for all $b \in B$ (object or arrow), $j(b)$ satisfy $fj(b) = gj(b)$ and therefore $j(b) \in C$. So there is an inclusion $k : B \to C$ such that $ik = j$, moreover a such k is an inclusion therefore unique.

(b) :

Lemma : Given three categories and two functors as follows : $A \longrightarrow_F C \longleftarrow_G B$, the comma category $(F \downarrow G)$ is the pullback :

$$\begin{array}{ccc} (F \downarrow G) & \xrightarrow{R} & C^{\mathbf{2}} \\ \Big\downarrow{\scriptstyle (P,Q)} & & \Big\downarrow{\scriptstyle (\text{domain}, \text{codomain})} \\ A \times B & \xrightarrow{F \times G} & C \times C \end{array}$$

This result is given by exercise 7.4.5.

$(b \downarrow \mathbf{C})$ **:** Is therefore the pullback of the following diagram :

$$\mathbf{C} \underset{(b,\mathbf{1})}{\longrightarrow} \mathbf{C} \times \mathbf{C} \underset{(\text{domain,codomain})}{\longleftarrow} \mathbf{C}^{\mathbf{2}}$$

$(\mathbf{C} \downarrow a)$ **:** Is similarly the pullback of the following diagram :

$$\mathbf{C} \underset{(\mathbf{1},a)}{\longrightarrow} \mathbf{C} \times \mathbf{C} \underset{(\text{domain,codomain})}{\longleftarrow} \mathbf{C}^{\mathbf{2}}$$

8.5.3 Cat has small coproducts

Exercise

Prove that **Cat** has all small coproducts.

Let X be a small set seen as a discrete category. Given a functor $F : X \to \mathbf{Cat}$, one may construct a category $\mathbf{C}$ with objects $\cup_{x\in X}\mathrm{Obj}(F(x))$ and with arrows set $\cup_{x\in X}\mathrm{Arrows}(F(x))$ this is the same that defining the hom-set of two elements c, d as the emptyset if those two werent in the same $F(x)$ and their "usual" hom-set if they were. (one can easily verify that those sets are small as X is and all $F(x)$ also are, so $\mathbf{C} \in \mathbf{Cat}$).

It indeed yields a natural transformation (family of injections) from F to the constant functor $\Delta\,\mathbf{C}$.

Then given another category $D \in \mathbf{Cat}$ with a natural transformation $\tau : F \dot{\longrightarrow} \Delta D$ (family of arrows). Then one may define a functor $T : \mathbf{C} \to D$ by for $c \in \mathbf{C}$ as there exists a unique $x \in X$ such that $c \in F(x)$, one may set $Tc = \tau_x(c)$. The same definition works for arrows as well, this then gives the universality of $\mathbf{C}$.

8.5.4 Pointwise product of functors

Exercise

> If B has finite products show that any functor category B^C also has finite products (calculated pointwise).

We are going to prove the stronger result : B^C has all the limits that B has. (I.e. given a category X, if B has X-limits, then so has B^C).

Proof using comma categories :

Existence of limits seen as functor to a comma category : Let X be a category, and suppose that B has all X-limits; there is a functor

$$\varprojlim \_ : [X, B] \longrightarrow (\Delta_B \downarrow B^X).$$

Where $\Delta_B : B \to B^X$ is the diagonal injection : $\Delta_B b = b! : X \to \mathbf{1} \to B$. And B^X in the comma category stands for the identity functor.

Indeed the object function maps a functor $T : X \to B$ to a couple

$$(r = \varprojlim T, \tau : \Delta r \dot{\longrightarrow} T).$$

About the arrow function : if $\phi : T \dot{\longrightarrow} S$, then one has a natural transformation $\phi\cdot\tau : \Delta r \dot{\longrightarrow} S$, where (r, τ) is the universal element associated to the functor T and (s, σ) to the functor S. Precisely, as this last pair is universal, there exist a unique arrow $u : r \to s$ such that $\sigma u = \phi\tau$. Therefore one may define the functor on arrows by

$$\varprojlim \phi = u.$$

This definition gives indeed a functor.

Moreover let $(r, \tau : \Delta_B r \dot{\longrightarrow} T)$ be the image of $T : X \to B$ via the limit functor. Its universal property states that for every cone $(s, \sigma : \Delta_B s \dot{\longrightarrow} T)$ there is a unique arrow $t : s \to r$ such that $\sigma = \tau \circ t$.

This can be rewritten as a bijection $\mathrm{Nat}(\Delta_B s, T) \cong B(s, r)$. But these natural bijections are elements of $(\Delta_B \downarrow T)$ where T is seen as a fixed element in B^X.

What we want to do : Let fix X a category. One wants to construct (this is only the first step) a functor $[X, B^C] \to (\Delta \downarrow [X, B^C])$.

But one already knows that there is a really natural isomorphism $\omega : [X, B^C] \cong [X \times C, B] \cong [C \times X, B] \cong [C, B^X]$, see exercise 7.3.2 for the first (and the last) isomorphism(s).

So by composing with the functor

$$\varprojlim : [X, B] \to (\Delta_B \downarrow [X, B])$$

defined above, one gets a functor $[X, B^C] \to [C, (\Delta_B \downarrow [X, B])]$

Functor between comma categories : Exercise 7.4.5 gives that the comma category $E = (\Delta_{B^C} \downarrow [X, B^C])$ is the following pullback :

$$\begin{tikzcd}
E \arrow[r, "R"] \arrow[d, "{(P,Q)}"] & {[X, B^C]^{\mathbf{2}}} \arrow[d, "{(\mathrm{domain}, \mathrm{codomain})}"] \\
B^C \times [X, B^C] \arrow[r, "\Delta \times 1"] & {[X, B^C] \times [X, B^C]}
\end{tikzcd}$$

But as it also apply to $D = (\Delta_B \downarrow B^X)$, given a functor $T : C \to D$, one has :

$$\begin{tikzcd}
 & & (B^X)^{\mathbf{2}} \arrow[dr, "{(\mathrm{domain}, \mathrm{codomain})}"] & \\
C \arrow[r, "T"] & D \arrow[ur, "R'"] \arrow[dr, "{(P', Q')}"'] & & B^X \times B^X \\
 & & B \times B^X \arrow[ur, "\Delta \times 1"'] &
\end{tikzcd}$$

This gives us two functors, namely $(P' \circ \_, Q' \circ \_) : [C, D] \to [C, B \times B^X]$ and $R \circ \_ : [C, D] \to [C, (B^X)^{\mathbf{2}}]$. Up to isomorphisms, theses are functors from $[C, D]$ to respectively $B^C \times [X, B^C]$ and $[X, B^C]^{\mathbf{2}}$.

$$\begin{tikzcd}
 & & {[X, B^C]^{\mathbf{2}}} \arrow[dr, "{(\mathrm{domain}, \mathrm{codomain})}"] & \\
{[C, D]} \arrow[r, dotted, "V"] \arrow[urr, bend left, "R' \circ _"] \arrow[drr, bend right, "{(P', Q') \circ _}"'] & E \arrow[ur, "R"] \arrow[dr, "{(P, Q)}"'] & & {[X, B^C] \times [X, B^C]} \\
 & & B^C \times [X, B^C] \arrow[ur, "\Delta \times 1"'] &
\end{tikzcd}$$

So the universal property of pullbacks give us a unique functor $V : [C, D] \to E$ such that the above diagram commutes.

Proof by Ross (Somehow 2-categorical argument) : Saying that X-limits exist in B is exactly stating that there is an adjunction φ :

$$B \underset{\lim}{\overset{\Delta_B}{\rightleftarrows}} B^X$$

Given by the bijections :

$$\varphi_{x,F} : \mathrm{Cone}(x, F) = B^X(\Delta_B x, F) \cong B(x, \lim F)$$

Lemma 1 : Given a (small) category C, any adjunction $A \underset{G}{\overset{F}{\rightleftarrows}} X$ in **Cat** is conserved by "homming" from C in the following sense :

$$X(Fa, x) \cong A(a, Gx) \quad \text{becomes} \quad X^C(F \circ \alpha, \xi) \cong A^C(\alpha, G \circ \xi)$$

Let's call $\eta : 1_X \stackrel{\cdot}{\longrightarrow} GF$ the unit of the adjunction and $\varepsilon : FG \stackrel{\cdot}{\longrightarrow} 1_A$ the counit.

The hypothesis can be written with those commutative diagrams respectively in $\mathrm{Hom}(A, X)$ and $\mathrm{Hom}(X, A)$ (which are categories) :

$$\begin{tikzcd} F1 \arrow[r, "\eta"] \arrow[dr, "1"'] & FGF \arrow[d, "\varepsilon"] \\ & 1F \end{tikzcd} \qquad \begin{tikzcd} G1 \arrow[r, "\varepsilon"] \arrow[dr, "1"'] & GFG \arrow[d, "\eta"] \\ & 1G \end{tikzcd}$$

Therefore since $\mathrm{Hom}(C, \_)$ is a 2-functor, it gives functors $\mathbf{Cat}(A, X) \to \mathbf{Cat}(C, \mathbf{Cat}(A, X))$ and $\mathbf{Cat}(X, A) \to \mathbf{Cat}(C, \mathbf{Cat}(X, A))$. Those two functors map the above commutative diagrams respectively to the followings :

$$\begin{tikzcd} F^C \arrow[r, "\eta^C"] \arrow[dr, "1"'] & (FGF)^C \arrow[d, "\varepsilon^C"] \arrow[r, phantom, "\simeq"] & F^C G^C F^C \\ & F^C \end{tikzcd} \qquad \begin{tikzcd} G^C \arrow[r, "\varepsilon^C"] \arrow[dr, "1"'] & (GFG)^C \arrow[d, "\eta^C"] \arrow[r, phantom, "\simeq"] & G^C F^C G^C \\ & G^C \end{tikzcd}$$

So the existence of the natural transformations $\eta^C : 1 \to G^C F^C$ and $\varepsilon^C : F^C G^C \to 1$ give the required adjunction.

Lemma 2 : $\Delta_B^C = \Delta_B \circ \_ = \Delta_{B^C}$ (the lemma states the second identity since the first is the definition of Δ_B^C).

Given any $b \in B$, $\Delta_B b$ is the arrow

$$X \underset{!}{\longrightarrow} \star \underset{b}{\longrightarrow} B$$

So Δ_B can be seen has left composition with the arrow $! : X \to \star$. Therefore so is Δ_{B^C} and the result follows.

Conclusion : With the data above one can construct an adjunction :

$$B^C \underset{\lim^C}{\overset{\Delta_{B^C}}{\rightleftarrows}} [X, B^C]$$

The right adjoint $\lim^C$ to the functor Δ_{B^C} is therefore a functor giving limits for any functor in $[X, B^C]$

8.6 Exercises on section 6 : Groups in Categories p.76

8.6.1 Monoids

Exercise

> C is a category with finite products and a terminal object t. Describe the category $\mathbf{Mon}_C$ of monoids in C and show it has finite products.

Objects of the category $\mathbf{Mon}_C$ are of the form $(c, \mu : c \times c \to c, \eta : t \to c)$ such that some property are respected (namely associativeness of μ and neutrality of η with μ on both sides). Now given an arrow $f : c \to c'$ in C, it is an arrow in $\mathbf{Mon}_C$ as soon as it makes the following diagrams commute :

$$\begin{tikzcd} c \times c \arrow[r, "\mu"] \arrow[d, "(f,f)"] & c \arrow[d, "f"] \\ c' \times c' \arrow[r, "\mu'"] & c' \end{tikzcd} \qquad \begin{tikzcd} t \arrow[r, "\eta"] \arrow[dr, "\eta'"] & c \arrow[d, "f"] \\ & c' \end{tikzcd}$$

Then one can easily see that the restriction of the composition law of C to $\mathbf{Mon}_C$ stay in $\mathbf{Mon}_C$ as the concatenation of two commutative diagrams as those above is commutative.

Limits (finite products) : Given two objects of $\mathbf{Mon}_C$ as above, then one can equip the product $d = c \times c'$ with a structure of monoid :

First of all, one has :

$$d \times d = (c \times c') \times (c \times c') \cong c \times (c' \times c) \times c' \cong c \times (c \times c') \times c' \cong (c \times c) \times (c' \times c')$$

in a fancy natural way (which was not obvious at first because an isomorphism $c \times c \cong c \times c$ need not be natural).

Then one has $\mu \times \mu' : (c \times c) \times (c' \times c') \to c \times c' = d$. So by composing those two arrows, one may define a multiplication $m : d \times d \to d$.

Similarly there is an arrow $n = (\eta, \eta') : t \to d$.

Those arrows satisfy the properties of a monoids structure because each coordinate satisfy theses properties by hypothesis. Moreover the projections arrows $p : d \to c$ and $q : d \to c'$ are morphisms of monoids in C.

Universality of the product : Now given another monoid in C, say e with two C-monoid morphisms $u : e \to c$ and $v : e \to c'$. One has by universality of the product in C a unique arrow $f : e \to d = c \times c'$ such that $pf = u$ and $qf = v$.

Let now show that f is a morphism of monoids in C, one wants to show that $m(f, f) = fm$ but $pfm = um = m(u, u) = m(pf, pf) = pm(f, f)$ and identically with q so as p and q totally define f, f is a morphism for m and by a simple calculation so it is for n. Finally f is a C-monoid morphism (and unique because of its uniqueness as morphism in C).

Therefore $(d, m, n), p, q$ is the product of (c, μ, η) and (c, μ', η') in $\mathbf{Mon}_C$.

8.6.2 Groups in Grp

Exercise

(a) If A is an abelian group (in **Set**), show that its multiplication $A \times A \to A$, its unit $\mathbf{1} \to A$ and its inverse $A \to A$ are morphism of groups.

(b) Deduce that A is a group in **Grp**.

(c) Prove that every group in **Grp** has this form.

(a) : For any group, $\eta : \mathbf{1} \to A$ and $\mu : A \times A \to A$ are group morphisms. Namely $\mu(\eta, \eta) = \eta\mu$, and $\mu(\mu, \mu) = \mu$.

Moreover for A abelian, one has also $\mu(\xi, \xi) = \xi\mu$ and since $\eta = \xi\eta$ is always true, ξ is a group homomorphism.

(b) : Since μ, η andξ are group homomorphisms, in the category of groups, (A, μ, η, ξ) is a group.

(c) : Let A be a group in **Grp**, then if p, q are the projections $A \times A \to A$, then $p \times q : A \times A \to A \times A$ is the identity. Let's denote $b = q \times p$. It follows that $b^2 = 1$.

An abelian group is a group such that $\mu b = \mu$.

A very very huge commutative diagram (stating $\xi(ab) = \xi(b)\xi(a)$) using the hypotheses may prove us the following result :

$$\mu \circ b \circ (\xi \times \xi) = \xi \circ \mu$$

Therefore since saying ξ is a group morphism yields $\xi \circ \mu = \mu \circ (\xi \times \xi)$, and by regularity of $\xi \times \xi$ in the monoid $\mathrm{End}(A \times A)$, one has $\mu b = \mu$. So A is abelian.

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