Braids among the groups

Ross Street
Macquarie University, N. S. W. 2109 AUSTRALIA
email: street@mpce.mq.edu.au

Dedicated to Dieter Pumplün

The purpose of this short note is to give a categorical view of some connections among braids, permutations and group automorphisms; it was prompted by the interesting survey [V].

The braid group $\mathbb{B}_n$ of Artin has a familiar presentation in terms of generators $\beta_1, \ldots, \beta_{n-1}$ and relations

$$\beta_r \beta_s = \beta_s \beta_r \quad \text{for } r < s-1, \quad \text{and} \quad \beta_r \beta_{r+1} \beta_r = \beta_{r+1} \beta_r \beta_{r+1}.$$

Also $\mathbb{B}_n$ can be identified with a subgroup of the group of automorphisms of the free group $\mathbb{F}_n$ on n generators $x_1, \ldots, x_n$. The inclusion $\mathbb{B}_n \longrightarrow \mathrm{Aut}(\mathbb{F}_n)$ identifies the generator β_r with the automorphism given by

$$\beta_r(x_i) = \begin{cases} x_i & \text{for } i \neq r, r+1 \\ x_{r+1} & \text{for } i = r \\ x_{r+1} x_r x_{r+1}^{-1} & \text{for } i = r+1 \end{cases} .$$

A good reference is [B; Section 1.4].

We write Gp_* for the category of pointed groups. The objects are pairs (A, a_0) where A is a group and $a_0 \in A$. An arrow $f : (A, a_0) \longrightarrow (B, b_0)$ is a group homomorphism $f : A \longrightarrow B$ such that $f(a_0) = b_0$. The monoidal category Gp of groups will be identified with the full subcategory of Gp_* consisting of the objects (A, a_0) where $a_0 = 1$ is the unit of A.

We refer now to the terminology of [JS]. There is a monoidal (= tensor) structure on Gp_* given by

$$(A, a_0) \otimes (B, b_0) = (A + B, a_0 b_0)$$

where $A+B$ is the coproduct (= free product = amalgamated sum) of the groups A, B with A, B identified with subgroups of $A+B$.

Proposition 1 *The monoidal category* Gp_* *admits a braiding*

$$c_{A,B} = c_{(A,a_0),(B,b_0)} : (A + B, a_0 b_0) \longrightarrow (B + A, b_0 a_0)$$

which takes $a \in A$ *to* $b_0 a b_0^{-1} \in B+A$ *and* $b \in B$ *to* $b \in B+A$.

Proof First notice that the braiding candidate $c_{A,B}$ preserves basepoints since it takes $a_0b_0 \in A+B$ to $(b_0\, a_0\, b_0^{-1})\, b_0 = b_0\, a_0 \in B+A$. Invertibility and naturality of $c_{A,B}$ are clear. Looking at the effects on elements, we see that the composite

$$\begin{array}{ccccc}
A+B+C & \xrightarrow{1+c_{B,C}} & A+C+B & \xrightarrow{c_{A,C}+1} & C+A+B \\
a & \longrightarrow & a & \longrightarrow & c_0\, a\, c_0^{-1} \\
b & \longrightarrow & c_0\, b\, c_0^{-1} & \longrightarrow & c_0\, b\, c_0^{-1} \\
c & \longrightarrow & c & \longrightarrow & c
\end{array}$$

is equal to $c_{A+B,C} : A + B + C \longrightarrow C + A + B$, and that the composite

$$\begin{array}{ccccc}
A+B+C & \xrightarrow{c_{A,B}+1} & B+A+C & \xrightarrow{1+c_{A,C}} & B+C+A \\
a & \longrightarrow & b_0\, a\, b_0^{-1} & \longrightarrow & b_0c_0\, a\, c_0^{-1}b_0^{-1} \\
b & \longrightarrow & b & \longrightarrow & b \\
c & \longrightarrow & c & \longrightarrow & c
\end{array}$$

is equal to $c_{A,B+C} : A + B + C \longrightarrow B + C + A$. **Q.E.D.**

Remark Instead of pointed groups we could take monoids pointed by invertible elements. This would give an even bigger braided monoidal category.

Recall from [JS] that the braid category $\mathbb{B}$ is the free braided monoidal category generated by a single object 1. In the category $\mathbb{B}$ the objects are natural numbers, all arrows are endomorphisms, and the endomorphism monoid $\mathbb{B}(n,n)$ is the braid group $\mathbb{B}_n$. The tensor product is given by addition of natural numbers and addition of braids. The braiding is determined by taking $c_{1,1} : 1 + 1 \longrightarrow 1 + 1$ to be the generator β_1 of $\mathbb{B}_2$.

Consider the free group $\mathbb{F}_1$ on a single generator as an object of $\mathcal{G}p_*$ by choosing the generator x to be the distinguished point. By freeness of $\mathbb{B}$, there exists a strong monoidal functor $T : \mathbb{B} \longrightarrow \mathcal{G}p_*$ (unique up to isomorphism) with $T1 = \mathbb{F}_1$. It follows that there is a canonical isomorphism $Tn \cong \mathbb{F}_n$ where the distinguished point of the free group $\mathbb{F}_n$ is the product $x_1 \dots x_n$ of the generators.

Proposition 2 *The functor* $T : \mathbb{B} \longrightarrow \mathcal{G}p_*$ *is faithful.*

Proof We must see that $T : \mathbb{B}(n,n) \longrightarrow Gp_*(\mathbb{F}_n, \mathbb{F}_n)$ is injective. But it is easy to see that the composite of this group homomorphism with the inclusion $Gp_*(\mathbb{F}_n, \mathbb{F}_n) \longrightarrow \mathrm{Aut}(\mathbb{F}_n)$ is just the above identification $\mathbb{B}_n \longrightarrow \mathrm{Aut}(\mathbb{F}_n)$. See [B; Corollary 1.8.3, page 25]. **Q.E.D.**

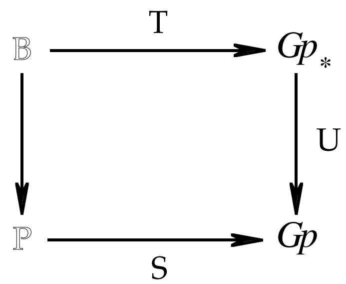
The functor $T : \mathbb{B} \longrightarrow Gp_*$ is not full. A point-preserving automorphism θ of $\mathbb{F}_n$ is in the image of T if and only if there exists a permutation τ of $\{1, 2, \ldots, n\}$ and words $w_1, \ldots, w_n$ in the generators $x_1, \ldots, x_n$ of $\mathbb{F}_n$ such that

$$\theta(x_i) = w_i \, x_{\tau(i)} \, w_i^{-1}$$

for $i = 1, \ldots, n$ (see [B; Theorem 1.9, page 30]).

Let $\mathbb{P}$ denote the free symmetric strict monoidal category on a single generating object 1. Again the objects are the natural numbers and every arrow is an endomorphism; the endomorphism monoid $\mathbb{P}(n,n)$ is the symmetric group $\mathfrak{S}_n$ on n symbols; the tensor product on objects is addition of natural numbers and on arrows is given by the canonical inclusions $\mathfrak{S}_m \times \mathfrak{S}_n \longrightarrow \mathfrak{S}_{m+n}$. There is a canonical braided strict monoidal functor $\mathbb{B} \longrightarrow \mathbb{P}$ which is the identity on objects and forgets braid crossings.

Recall that by Gp we mean the usual category of groups regarded as monoidal using coproduct (= free product of groups) as tensor product. There is a canonical symmetry on Gp. So there exists a strong monoidal functor $S : \mathbb{P} \longrightarrow Gp$ (unique up to isomorphism) with $S1 = \mathbb{F}_1$. The following square of strong monoidal functors does not commute; although it does commute on objects. (Note that right-hand vertical functor is the forgetful functor – which is not braided.)



Let $\mathbb{N}$ denote the free strict monoidal category on a single generating object 1; it is the discrete category whose objects are the natural numbers; the tensor product is addition. Form the following pushout in the category of strict monoidal categories and strict monoidal functors; the functors $\mathbb{N} \longrightarrow \mathbb{P}$, $\mathbb{N} \longrightarrow \mathbb{B}$ are the identity on objects.

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{B} \\ \downarrow & & \downarrow \\ \mathbb{P} & \longrightarrow & \mathbb{MBP} \end{array}$$

Since obj : Mon(Cat) —> Mon(Set) has a right adjoint, this pushout is preserved by taking object sets. So $\mathbb{MBP}$ has the natural numbers as objects. There is a (unique up to isomorphism) strong monoidal functor R : $\mathbb{MBP}$ —> *Gp* whose restriction to $\mathbb{P}$ is S and to $\mathbb{B}$ is UT.

It is also true that $\mathbb{P}$ is the free strict monoidal category containing an object bearing an involutive Yang-Baxter operator and $\mathbb{B}$ is the free strict monoidal category containing an object bearing a mere Yang-Baxter operator [JS]. From the universal property of pushout, it follows that $\mathbb{MBP}$ is the free strict monoidal category containing an object bearing two Yang-Baxter operators, one of which is involutive. The group $\mathbb{MBP}(n,n)$ has a presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying the braid relations plus the relations $\sigma_i \sigma_i = 1$ $(1 \leq i < n)$ and generators $\beta_1, \ldots, \beta_{n-1}$ satisfying the braid relations (but no mixed relations).

Now factor the functor R : $\mathbb{MBP}$ —> *Gp* into a full bijective-on-objects functor H : $\mathbb{MBP}$ —> $\mathbb{BP}$ followed by a faithful functor K : $\mathbb{BP}$ —> *Gp*. It is easy to see that $\mathbb{BP}$ inherits a unique strict monoidal structure such that H, K become strict, strong monoidal functors. The group $\mathbb{BP}(n,n)$ is the *braid-permutation group* BP_n of [FRR]; it is the quotient of the group $\mathbb{MBP}(n,n)$ obtained by imposing the mixed relations

$$\sigma_r \beta_{r+1} \beta_r = \beta_{r+1} \beta_r \sigma_{r+1}$$
$$\beta_r \sigma_{r+1} \sigma_r = \sigma_{r+1} \sigma_r \beta_{r+1} ;$$

the mixed relations $\sigma_r \sigma_{r+1} \beta_r = \beta_{r+1} \sigma_r \sigma_{r+1}$ are a consequence. The elements of BP_n can be represented geometrically as *welded braids* [FRR].

A related structure is the free symmetric braided strict monoidal category $\mathbb{PB}$. The group $\mathbb{PB}(n,n)$ is the quotient of $\mathbb{BP}(n,n)$ obtained by imposing the further mixed relations

$$\beta_r \beta_{r+1} \sigma_r = \sigma_{r+1} \beta_r \beta_{r+1} .$$

The monoidal categories $\mathbb{N}$, $\mathbb{P}$, $\mathbb{B}$, together with appropriate substitution operations, determine monads on the 2-category *Cat* of categories; see [K] and [JS]. The algebras for these monads are respectively strict monoidal categories, symmetric strict monoidal categories and braided strict monoidal categories. Similarly, $\mathbb{P}\mathbb{B}$ determines a monad on *Cat*; an algebra is a strict monoidal category equipped with a braiding and a symmetry.

Finally, we should mention the *reduced braid-permutation group* $\overline{\mathrm{BP}}_n$ which is the quotient of the group $\mathbb{P}\mathbb{B}(n,n)$ obtained by imposing the further relations

$$\beta_r \sigma_r = \sigma_r \beta_r \, .$$

These groups are the endomorphism monoids for a category

$$\overline{\mathbb{B}\mathbb{P}}$$

which is free symmetric braided strict monoidal in such a way that the symmetry and braiding commute.

References

[B] J.S. Birman, *Braids, Links, and Mapping Class Groups* (Annals of Math Studies Number 82, Princeton University Press 1974).

[FRR] R. Fenn, R. Rimányi and C. Rourke, The braid-permutation group, *Topology* 36 (1997) 123-135.

[JS] A. Joyal and R. Street, Braided tensor categories, *Advances in Math* 102 (1993) 20-78.

[K] G.M. Kelly, An abstract approach to coherence, *Lecture Notes in Math* 281 (Springer-Verlag, 1972) 106-147.

[M] A.A. Markov, Foundations of the algebraic theory of braids, *Trudy Mat. Inst. Steklova* **16** (1945, Russian, English Summary).

[V] V.V. Vershinin, Braid groups and loop spaces, (Preprint, December 1997, Sobolev Institute of Math., Novosibirsk, 630090, Russia).