Powerful functors

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This is a slightly extended version of my handwritten note [S] which makes no claim to originality. The main result was obtained by Giraud [G] and later by Conduché [C]. The problem addressed is that of characterizing the *powerful* (or "exponentiable") morphisms in the category **Cat** of (small) categories: that is, those functors $p : E \longrightarrow B$ for which the functor $p^* : \mathbf{Cat}/B \longrightarrow \mathbf{Cat}/E$, given by pulling back along p, has a right adjoint. The reason for the name is that p is powerful if and only if raising to the power p exists in the full slice category $\mathbf{Cat}/B$ (that is, the cartesian internal hom $(A,u)^{(E,p)}$ exists for all objects (A,u) of $\mathbf{Cat}/B$).

We write **Mod** for the bicategory whose objects are (small) categories and for which the hom category $\mathbf{Mod}(A\,,B)$ is the functor category $[B^{op} \times A, \mathbf{Set}]$. The morphisms of **Mod** are called *modules* while the 2-cells are called *module morphisms*. Composition of modules is given by the usual coend formula. We identify **Cat** as a sub-2-category of the bicategory **Mod** by thinking of a functor $f : A \longrightarrow B$ as the module defined by taking $f(b\,,a)$ to be $B(b\,,f(a))$.

For any functor $p : E \longrightarrow B$, the *fibre* over an object b of B is the subcategory E_b of E given by the pullback

$$\begin{array}{ccc} E_b & \longrightarrow & E \\ \downarrow & & \downarrow p \\ \mathbf{1} & \underset{b}{\longrightarrow} & B \end{array} .$$

Each $\beta : b \longrightarrow b'$ determines a module $m_E(\beta) : E_{b'} \longrightarrow E_b$ defined by

$$m_E(\beta)(e,e') = \left\{ \xi : e \to e' \,\middle|\, p(\xi) = \beta \right\}$$

for objects e of E_b and e' of $E_{b'}$. Notice immediately that $m_E(1_b)$ is the identity module of E_b (that is, the hom-functor $E_b(-,-) : E_b^{op} \times E_b \longrightarrow \mathbf{Set}$), and yet, for each composable pair of morphisms $\beta : b \longrightarrow b'$ and $\beta' : b' \longrightarrow b''$ in B, we only have a module morphism

$$\mu_{\beta,\beta'} : m_E(\beta) \otimes m_E(\beta') \longrightarrow m_E(\beta'\beta),$$

which is induced by the composition functions $E(e',\, e'') \times E(e\,,\, e') \longrightarrow E(e\,,\, e'')$. In fact, we have defined a normal lax functor[1]

$$m_E : B^{op} \longrightarrow \mathbf{Mod}\,.$$

Write $\mathbf{Cat}/B$ for the usual slice category of objects $p : E \longrightarrow B$ of **Cat** over B in which the morphisms $f : (E\,,p) \longrightarrow (F\,,q)$ are commutative triangles over B; however, we enrich $\mathbf{Cat}/B$ to become a 2-category by accepting those 2-cells $\theta : f \Rightarrow g : (E,p) \longrightarrow (F,q)$ satisfying $q\,\theta = p$. Write $\mathrm{Bicat}(B^{op}, \mathbf{Mod})$ for the bicategory of lax functors $B^{op} \longrightarrow \mathbf{Mod}$, lax

[1] Lax functors are Bénabou's "morphisms of bicategories" while here "normal" means strictly identity preserving.

transformations, and modifications.

Proposition (Bénabou [B]) *The slice 2-category* $\mathbf{Cat}/B$ *is equivalent to the sub-2-category of the bicategory* $\mathrm{Bicat}(B^{op}, \mathbf{Mod})$ *whose objects are the normal lax functors, whose morphisms are the lax transformations with components at objects* b *of* B *being actual functors, and whose 2-cells are all the modifications.*

Proof (sketch) The value at the object (E, p) of a 2-functor $P : \mathbf{Cat}/B \longrightarrow \mathrm{Bicat}(B^{op}, \mathbf{Mod})$ is defined to be the normal lax functor m_E. For a morphism $f : (E, p) \longrightarrow (F, q)$ over B we define a lax transformation $P(f) : m_E \Rightarrow m_F$ by defining the component $P(f)_b : E_b \longrightarrow F_b$ to be the functor induced by f (meaning that $P(f)_b(e) = f(e)$), and by defining the component

$$\begin{array}{ccc} E_{b'} & \xrightarrow{\;P(f)_{b'}\;} & F_{b'} \\ {\scriptstyle m_E(\beta)}\downarrow & \overset{P(f)_\beta}{\Rightarrow} & \downarrow{\scriptstyle m_F(\beta)} \\ E_b & \xrightarrow[\;P(f)_b\;]{} & F_b \end{array}$$

at $\beta : b \longrightarrow b'$ to be the function $F_b(x, f(e)) \times m_E(\beta)(e, e') \longrightarrow m_F(\beta)(x, f(e'))$ taking the equivalence class of the pair $(\chi, \xi) \in F_b(x, f(e)) \times m_E(\beta)(e, e')$ to $f(\xi)\,\chi \in m_F(\beta)(x, f(e'))$. It is easy to see that a 2-cell $\theta : f \Rightarrow g : (E, p) \longrightarrow (F, q)$ induces a modification $P(\theta) : P(f) \Rightarrow P(g)$ in an obvious way and that what we have is a 2-functor P landing in the specified sub-2-category of $\mathrm{Bicat}(B^{op}, \mathbf{Mod})$. Every lax transformation $\lambda : m_E \longrightarrow m_F$ having each λ_b a functor is of the form $P(f)$ for a unique $f : (E, p) \longrightarrow (F, q)$. Similarly, each modification $P(f) \Rightarrow P(g)$ is of the form $P(\theta)$ for a unique θ .

The inverse equivalence for P is a generalization of the so-called "Grothendieck construction" of a fibration from a category-valued pseudo-functor (which itself is a generalization of the classical category of elements of a presheaf). Given a normal lax functor $N : B^{op} \longrightarrow \mathbf{Mod}$, we obtain a category $E = \mathrm{coll}\,N$ as the lax colimit (or *collage*) of N and a functor $p : E \longrightarrow B$ induced by the lax cocone

$$\begin{array}{ccccc} Nb' & \longrightarrow & \mathbf{1} & \overset{b'}{\searrow} & \\ {\scriptstyle N(\beta)}\downarrow & & \beta \Uparrow & & B \\ Nb & \longrightarrow & \mathbf{1} & \underset{b}{\nearrow} & \end{array} \quad .$$

Explicitly, the objects of E are pairs (b, x) where b is an object of B and x is an object of Nb ; a morphism $(\beta, \chi) : (b, x) \longrightarrow (b', x')$ consists of a morphism $\beta : b \longrightarrow b'$ in B and an element $\chi \in N(\beta)(x, x')$; and composition uses composition in B and the composition constraints for N. Of course, $p(b, x) = b$ and $p(\beta, \chi) = \beta$. Clearly there is a canonical isomorphism $P(E, p) \cong N$ of lax functors. **q.e.d.**

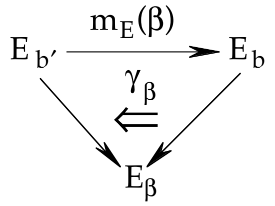
For any functor $p : E \longrightarrow B$ and any morphism $\beta : b \longrightarrow b'$ in B, we can also form the pullback

$$\begin{array}{ccc} E_\beta & \longrightarrow & E \\ \downarrow & & \downarrow p \\ \mathbf{2} & \underset{\beta}{\longrightarrow} & B \end{array} .$$

Notice that E_β contains E_b and $E_{b'}$ as disjoint full subcategories, and $E_\beta(e,e') = m_E(\beta)(e,e')$ and $E_\beta(e',e) = \varnothing$ for $e \in E_b$ and $e' \in E_{b'}$. This means that

$$E_b \hookrightarrow E_\beta \hookleftarrow E_{b'}$$

is a codiscrete cofibration from $E_{b'}$ to E_b and we have the collage (or lax colimit)



in **Mod** .

Now we come to our main business: that of investigating what it means for the functor

$$p^* : \mathbf{Cat}/B \longrightarrow \mathbf{Cat}/E$$

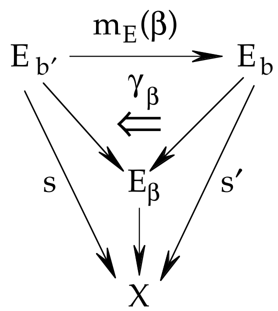
given by pulling back along p, to have a right adjoint. Since the domain functor $\mathbf{Cat}/E \longrightarrow \mathbf{Cat}$ is comonadic (in fact the counit with the right adjoint is a split monomorphism), the functor p^* has a right adjoint if and only if the functor

$$- \times_B E : \mathbf{Cat}/B \longrightarrow \mathbf{Cat}$$

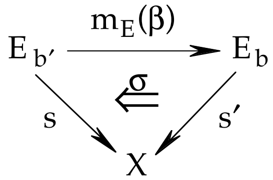
has a right adjoint [D]. Such an adjoint is determined by its value $h : Z \longrightarrow B$ on each object $X \in \mathbf{Cat}$; such an h is called a right lifting of X through $- \times_B E$ and participates in a bijection

$$(\mathbf{Cat}/B)((A,u),(Z,h)) \cong \mathbf{Cat}(A \times_B E, X)$$

which is natural in (A,u). As is so often the case with right adjoints, this allows us to discover what the category Z must be. Take $A = \mathbf{1}$ and $u = b : \mathbf{1} \longrightarrow B$ to find that an object of Z over b amounts to a functor $s : E_b \longrightarrow X$. So the <u>objects</u> of Z are pairs (b, s) where $b \in B$ and s is such a functor. Now take $A = \mathbf{2}$ and $u = \beta : \mathbf{2} \longrightarrow B$ to find that a morphism of Z over β amounts to a functor $E_\beta \longrightarrow X$.

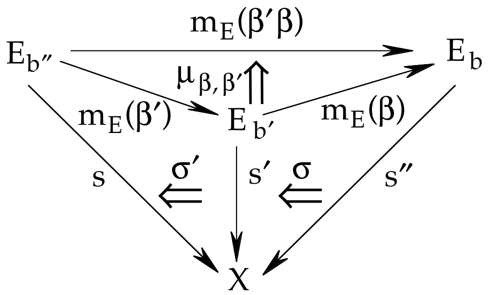


By the collage property, this is the same as a diagram



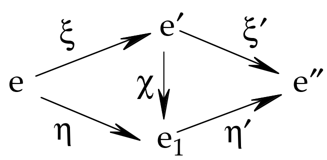
in **Mod**. So a <u>morphism</u> $(\beta, \sigma) : (b, s) \longrightarrow (b', s')$ in Z amounts to a morphism $\beta : b \longrightarrow b'$ in B together with a σ as in the above triangle.

The problem comes when we try to define <u>composition</u> in Z. The appropriate diagram



is <u>not well formed for pasting.</u> However, if each $\mu_{\beta,\beta'}$ is invertible then Z becomes a category and $h : Z \longrightarrow B$, where $h(b, s) = b$ and $h(\beta, \sigma) = \beta$, is a right lifting of X through the functor $- \underset{B}{\times} E$.

To say each $\mu_{\beta,\beta'}$ is invertible is to say $m_E : B^{op} \longrightarrow$ **Mod** is a pseudofunctor (or "homomorphism" in Bénabou's terminology). Yet what does it mean combinatorially for each $\mu_{\beta,\beta'}$ to be invertible? Take a composable pair of morphisms $\beta : b \longrightarrow b'$ and $\beta' : b' \longrightarrow b''$ in B and take $e \in E_b$ and $e'' \in E_{b''}$. Consider the category $M_E(\beta,\beta')(e,e'')$ whose objects are composable pairs of morphisms $\xi : e \longrightarrow e'$ and $\xi' : e' \longrightarrow e''$ in E such that $p\xi = \beta$ and $p\xi' = \beta'$, and whose morphisms are commutative diagrams



in which $\chi : e' \longrightarrow e_1$ is in the fibre $E_{b'}$ over b'. Then $\big(m_E(\beta') \otimes m_E(\beta)\big)(e,e'')$ is the set of path components of the category $M_E(\beta,\beta')(e,e'')$, and, $\mu_{\beta,\beta'}$ has component at (e, e'') induced by

$$M_E(\beta,\beta')(e,e'') \longrightarrow m_E(\beta'\beta)(e,e''), \quad (\xi, \xi') \longmapsto \xi' \xi .$$

With these preliminaries, the following precise statement is easily verified.

Theorem (Giraud [G], Conduché [C]) *For a functor* $p : E \longrightarrow B$ *, the following conditions are equivalent:*

(i) the functor $p^* : \mathbf{Cat}/B \longrightarrow \mathbf{Cat}/E$ *has a right adjoint;*

(ii) the normal lax functor $m_E : B^{op} \longrightarrow \mathbf{Mod}$ *is a pseudofunctor;*

(iii) for all $\beta : pe \longrightarrow b'$ *and* $\beta' : b' \longrightarrow pe''$ *in* B*, and* $\zeta : e \longrightarrow e''$ *in* E *over* $\beta\beta'$*, there exist* $\xi : e \longrightarrow e'$ *and* $\xi' : e' \longrightarrow e''$ *over* β *and* β'*, respectively, with composite* ζ*, and any two such pairs* (ξ, ξ') *are connected by a path in the category* $M_E(\beta,\beta')(e,e'')$.

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