Ross Street.
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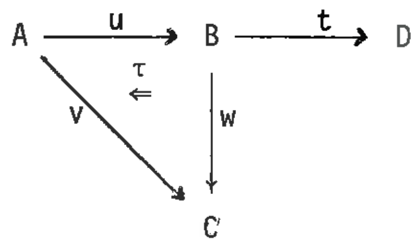
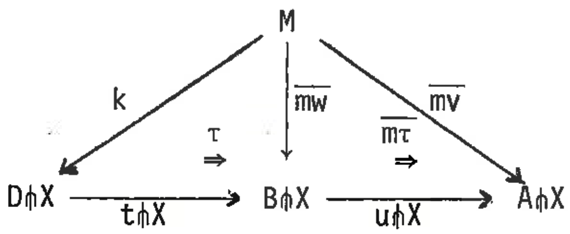
§1. *Complete objects relative to a theory.*

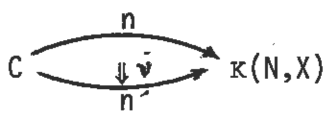
A *theory* τ is a diagram



in Cat. We say that τ is *relevant* to a 2-category $\mathcal{K}$ when the cotensor products $A \pitchfork X$, $B \pitchfork X$, $D \pitchfork X$ exist for all objects X of $\mathcal{K}$. When this is the case, an *object of models for* τ *in an object* X of $\mathcal{K}$ is an object M of $\mathcal{K}$ together with a functor $m: C \to \mathcal{K}(M,X)$ which has the property that the arrow $\overline{mw} : M \to B \pitchfork X$ (corresponding to the functor mw) has an absolute right lifting k, κ (see diagram below) through $t \pitchfork X$ such that the composite 2-cell



exhibits k as an absolute right lifting of $\overline{mv}$ through $tu \pitchfork X$, and, if N is an object of $\mathcal{K}$ and



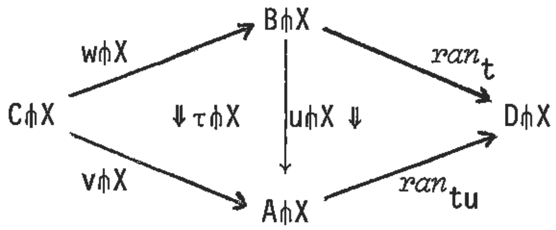
is a natural transformation such that the pair N,n and the pair N,n' both have the property of the pair M,m above, then there exists a unique 2-cell $N \overset{\Downarrow\sigma}{\rightrightarrows} M$ in $\mathcal{K}$ with $\nu = \mathcal{K}(\sigma,X)m$. When it exists, the object M is unique up to isomorphism and denoted by $Mod_{\mathcal{K}}(\tau,X)$, or $Mod(\tau,X)$ when the $\mathcal{K}$ is clear.

Suppose τ is relevant to $\mathcal{K}$. An object X of $\mathcal{K}$ is said to be *complete relative to* τ when the arrows $t \pitchfork X : D \pitchfork X \to B \pitchfork X$, $tu \pitchfork X : D \pitchfork X \to A \pitchfork X$ have right adjoints ran_t, ran_{tu}, respectively, and an object of models for τ in X exists.

In this case, there is a 2-natural isomorphism of categories

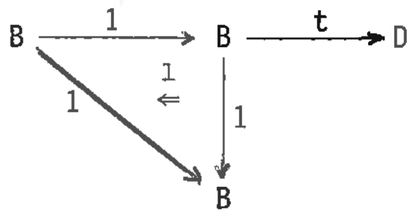
$$\mathcal{K}(K, Mod_{\mathcal{K}}(\mathcal{T},X)) \cong Mod_{\mathbf{Cat}}(\mathcal{T},\mathcal{K}(K,X)) .$$

If $C \pitchfork X$ exists, there is a 2-cell



where the 2-cell in the right-hand triangle corresponds to the identity 2-cell $tu \pitchfork X . 1_{D \pitchfork X} \Rightarrow u \pitchfork X . t \pitchfork X$. It is clear then that $Mod(\mathcal{T},X)$ amounts precisely to an inverter for the composite 2-cell of the above diagram (in the sense of Street [;§5]). So, provided the 2-category $\mathcal{K}$ is sufficiently complete, completeness of an object X amounts to the existence of right adjoints to arrows of the form $t \pitchfork X : D \pitchfork X \to B \pitchfork X$. We now turn to the question of when the existence of such right adjoints is assured by the existence of certain special kinds.

We identify a functor $t : B \to D$ with the theory



Provided t is relevant to $\mathcal{K}$, an object X of $\mathcal{K}$ is complete relative to t if and only if $t \pitchfork X : D \pitchfork X \to B \pitchfork X$ has a right adjoint ran_t. We identify a category B with the functor $! : B \to 1$, and instead of $! \pitchfork X, ran_!$, we write *diag* $: X \to B \pitchfork X$, *lim* $: B \pitchfork X \to X$, respectively. An object of $\mathcal{K}$ is said to have:

(a) *a terminal object;* (b) *finite products;*

(c) *pullbacks;* (d) *equalizers;*

when it respectively is complete relative to:

(a) the set 0; (b) the sets 0 and 2;

(c) the category $* \rightarrow ? \leftarrow *'$; (d) the category $* \rightrightarrows *'$.

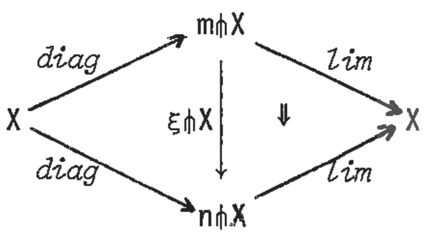
Terminology (b) is justified by the simple result:

Lemma 1. *An object with finite products is complete relative to any finite set* n.

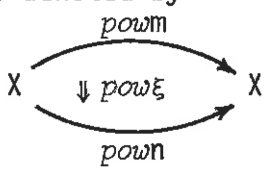
Proof. The inductive formula is:

$$\big((n+1)\pitchfork X \xrightarrow[\mathit{lim}]{} X\big) = \big((n\pitchfork X) \times X \xrightarrow[\mathit{lim}\times X]{} X \times X \xrightarrow[\mathit{lim}]{} X\big). \quad \square$$

Suppose $\xi : n \rightarrow m$ is a function between sets n, m, and that X is complete relative to n, m. The 2-cell

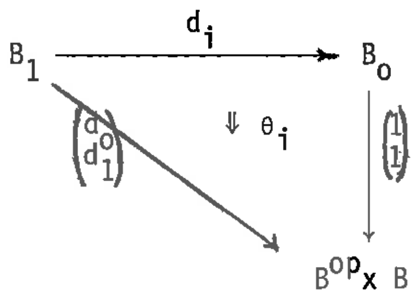


(where the 2-cell in the right-hand triangle corresponds to the identity 2-cell $\xi\pitchfork X\mathit{diag} \Rightarrow \mathit{diag}$) is denoted by

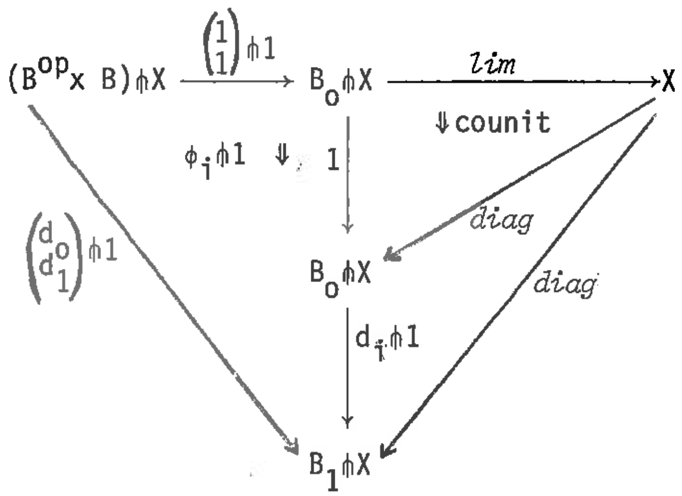


So *pow* is a contravariant functor from the category of sets relative to which X is complete to the category $\kappa(X,X)$.

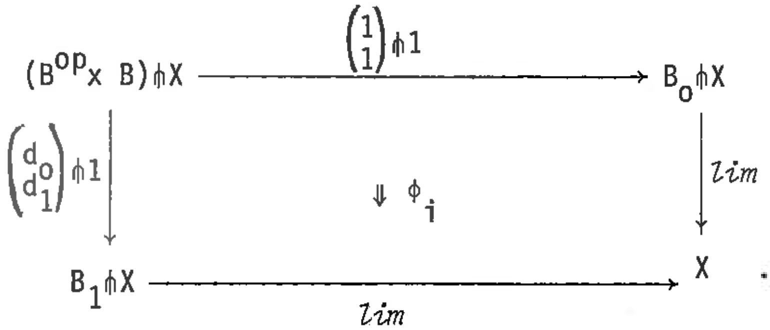
For any category B and $i = 0, 1$, we have a natural transformation



given by $\theta_o\beta = (1,\beta)$, $\theta_1\beta = (\beta,1)$ for β an arrow in B. Suppose $B_o\pitchfork X$, $B_1\pitchfork X$, $(B^{op}\times B)\pitchfork X$ exist and that X is complete relative to the sets B_o, B_1, The 2-cell



corresponds to a 2-cell



The pair of 2-cells ϕ_o, ϕ_1 correspond to a functor

$$(* \rightrightarrows *') \longrightarrow \mathrm{K}\left((B^{op} \times B) \pitchfork X, X\right),$$

and so yield an arrow

$$(B^{op} \times B) \pitchfork X \longrightarrow (* \rightrightarrows *') \pitchfork X.$$

When X has equalizers, this arrow composes with $lim : (* \rightrightarrows *') \pitchfork X \to X$ to yield an arrow

$$\int_* : (B^{op} \times B) \pitchfork X \longrightarrow X,$$

called the *end arrow* . This arrow can be used to produce right-kan-extension arrows ran_t. For simplicity we look at the finite case, although the same construction gives ran_t under more general conditions.

A theory T (as at the beginning of this section) is called *finite* when A, B, D are finite and C is finitely presented. [Note that we do <u>not</u> call an object X with equalizers and finite products "finitely complete" since this is not the right notion when, for example, κ is the 2-category of 2-categories.]

Theorem 2. *In a finitely complete 2-category, any object with equalizers and finite products is complete relative to any finite theory.*

Proof. It has already been shown that the problem reduces to that of constructing a right adjoint to arrows of the form $t \pitchfork X : D \pitchfork X \to B \pitchfork X$, where $t : B \to D$. The functor

$$D \times B^{op} \times B \longrightarrow \kappa(B \pitchfork X, X)$$

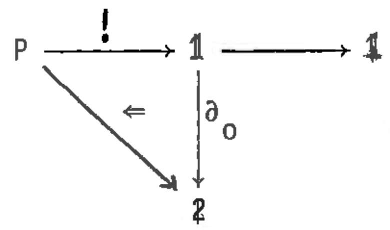
$$(a,b,c) \longmapsto \left(B \pitchfork X \xrightarrow[c \pitchfork X]{} X \xrightarrow[\mathit{pow}\, D(a,tb)]{} X\right)$$

corresponds to an arrow

$$B \pitchfork X \longrightarrow D \pitchfork \left\{(B^{op} \times B) \pitchfork X\right\}$$

which composes with $D \pitchfork \int_*$ to yield ran_t; the verification is left to the reader. □

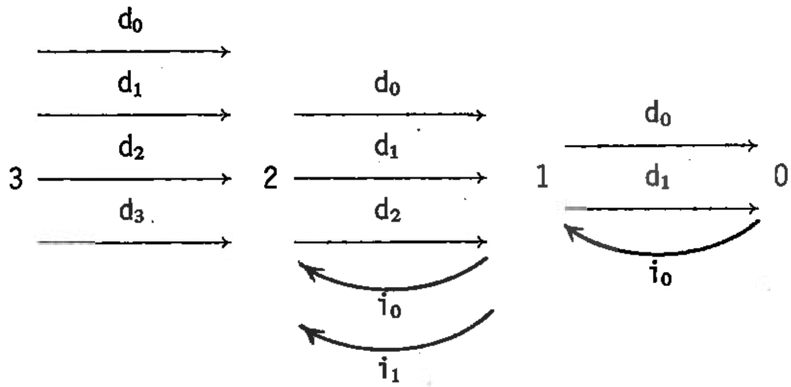
Examples. 1) Let P denote the category $* \to ? \leftarrow *'$. The *theory of monomorphisms* is



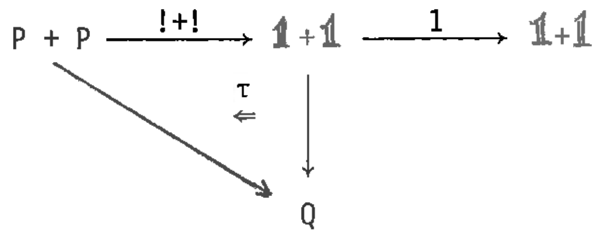
where $P \to 2$ takes both non-identity arrows of P to the non-identity arrow of 2. Write *Mono*(X) for *Mod*(T,X) in this case.

(see next page)

2) Let Q denote the category generated by



satisfying the equations of a truncated simplicial object. The *theory of categories* is the diagram



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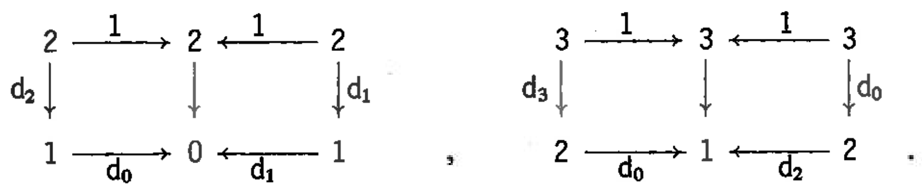
where the natural transformation τ, written as an arrow of diagrams in Q, is



When it exists, *Mod*(T,X) for this T is denoted by *Cat*(X).

3) For the *theory of monoids*, simply replace $\mathbb{1}+\mathbb{1}$ by $\mathbb{1}+\mathbb{1}+\mathbb{1}$ in the theory of categories and send the object of the new $\mathbb{1}$ to 0 in Q. □

It is worth noting that indexed limits which do not generally exist in K may exist when the objects of K involved are complete enough. One such example is suggested by a construction for topoi called "Artin gluing" (see Wraith [ ]).

Theorem 3. Suppose $\mathcal{K}$ *admits the construction of coalgebras* (see Street [ ]) *and also admits products of pairs of objects. If* $f : X \to Y$ *is an arrow in* $\mathcal{K}$ *for which* Y *is complete relative to* 2, *then the comma object* Y/f *exists.*

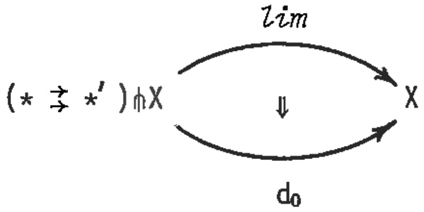
Proof. The arrow $g : X \times Y \to X \times Y$, whose first projection is first projection and whose second projection is the composite

$$X \times Y \xrightarrow[f \times Y]{} Y \times Y \xrightarrow[lim]{} Y,$$

supports a fairly obvious comonad structure. The construction of coalgebras applied to this comonad yields an object $(X \times Y)^g$ which is isomorphic to Y/f. □

It is also worth noting that even more complicated constructions can be obtained in a 2-category by combining the operations of taking objects of models and of forming indexed limits. For example, we can define the *monofier* of a 2-cell $X \underset{}{\overset{}{\Downarrow \sigma}} Y$ to be the universal arrow $k : K \to X$ such that σk is a monomorphism in the category $\mathcal{K}(K,Y)$. If Y has pullbacks then the monofier can be obtained as the pullback in $\mathcal{K}$ of $\mathit{Mono}(Y) \to 2 \pitchfork Y$ along the arrow $X \to 2 \pitchfork Y$ corresponding to σ.

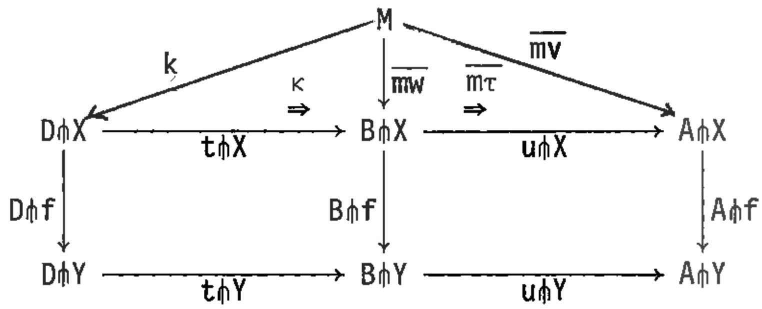
Reversing the direction of all 2-cell in $\mathcal{K}$ in the above, we obtain a discussion of *cocomplete objects relative to a theory.* Note that "right liftings" are turned into "left liftings" and "right adjoints ran_t to $t \pitchfork X$" is turned into "left adjoint lan_t to $t \pitchfork X$"; however, indexed limits in $\mathcal{K}^{co}$ are still indexed limits in $\mathcal{K}$ (<u>not</u> indexed colimits), so that cotensor product in $\mathcal{K}$ is still the relevant operation (not tensor product). An example of a construction which can be performed on a suitably complete and cocomplete object X is the construction of *Regmono*(X), *the object of regular monomorphisms in* X. In order to do this, suppose X has equalizers and pushouts. The obvious 2-cell



induces an arrow $u : (* \rightrightarrows *')\pitchfork X \longrightarrow 2\pitchfork X$ which has a left adjoint f ("cokernel pairs" exist since X has pushouts). Then *Regmono*(X) is the eilenberg-moore object for the monad $t = uf$ on $2\pitchfork X$ (actually $fuf \cong f$ so that t is idempotent and *Regmono*(X) is just the inverter of the unit of the adjunction $f \dashv u$).

§2. *Continuous arrows relative to a theory.*

Let T be a theory as at the beginning of the last section; and suppose T is relevant to the 2-category K. An arrow $f : X \to Y$ in K is said to be *continuous relative to* T when, given an object M of K, a functor $m : C \to \mathrm{K}(M,X)$, and the top left triangle in the diagram



such that the top left triangle and the composite of the top two triangles have the absolute right lifting property, then the left half and the whole diagram have the absolute right lifting property.

(see next page)

Proposition 4. (a) *An arrow with a left adjoint is continuous relative to any relevant theory.*

(b) *For a relevant functor* $r : E \to F$, *the arrow* $r \pitchfork X : F \pitchfork X \to E \pitchfork X$ *is continuous relative to any relevant theory.* □

An arrow is said to *preserve finite products* (or, respectively, *preserve equalizers*, etc.) when it is continuous relative to 0 and 2 (or, to $* \rightrightarrows *'$, etc.).

Theorem 5. *For an arrow* $f : X \to Y$ *in a finitely complete 2-category, if* X *has finite products and equalizers and* f *preserves them then* f *is continuous relative to any finite theory.* □

Let K_T denote the locally full sub-2-category of K consisting of the objects which are complete and the arrows which are continuous relative to the relevant theory T. We have an inclusion 2-functor $I : K_T \to K$ and a 2-functor $Mod(T,-) : K_T \to K$. The next result explains when an indexed limit of objects in K_T is again in K_T.

Theorem 6. *Suppose* T *is a theory relevant to* K *and that* $J : A \to \mathrm{Cat}$, $S : A \to K_T$ *are 2-functors such that* $lim(J,IS)$ *exists. The object* $lim(J,IS)$ *of* K *is complete relative to* T *if and only if* $lim(J,Mod(T,S))$ *exists; and, in this case, there is an isomorphism*

$$Mod(T,lim(J,IS)) \cong lim(J,Mod(T,S)).$$

Proof. The following isomorphisms are 2-natural in $\mathcal{K}$:

$$\mathcal{K}\left(K, Mod(T, lim(J, IS))\right) \cong Mod_{\mathbf{Cat}}\left(T, \mathcal{K}(K, lim(J, IS))\right)$$

$$\cong Mod_{\mathbf{Cat}}\left(T, [A,\mathbf{Cat}](J, \mathcal{K}(K, IS))\right) \cong [A,\mathbf{Cat}]\left(J, Mod_{[A,\mathbf{Cat}]}(T, \mathcal{K}(K, IS))\right)$$

$$\cong [A\ \mathbf{Cat}]\left(J, \mathcal{K}(K, Mod(T, S))\right) \cong \mathcal{K}\left(K, lim(J, Mod(T, S))\right). \qquad \square$$

Theorem 7. *If an object* X *of a 2-category* $\mathcal{K}$ *is complete relative to a relevant theory* T *and a relevant functor* $r : E \to F$ *then* $Mod(T,X)$ *is complete relative to* r.

Proof. Consider the adjunction $r \pitchfork X \dashv ran_r$ between $E \pitchfork X$ and $F \pitchfork X$. By Theorem 6, the objects $E \pitchfork X$, $F \pitchfork X$ are in $\mathcal{K}_T$; and by Proposition 4, the arrows $r \pitchfork X$, ran_r are in $\mathcal{K}_T$. So the adjunction lives in $\mathcal{K}_T$. Applying the 2-functor $Mod(T,-)$, we obtain an adjunction which, using Theorem 6, is isomorphic to one which involves $r \pitchfork Mod(T,X)$ as the left adjoint. $\square$

Theorem 8. *Suppose* X *is an object with equalizers and finite products in a finitely complete 2-category. If* T, T' *are finite theories then there is an isomorphism*

$$Mod(T, Mod(T', X)) \cong Mod(T', Mod(T, X)). \qquad \square$$