SKEW MONADS

ROSS STREET

Contents

1. Introduction

Background for these notes can be found in [2, 9, 7, 12, 5, 14, 8, 10].

2. Bicategories and their morphisms

The basics on bicategories can be found in [2] and [13]; we review some of that. For bicategories $\mathscr{A}$ and $\mathscr{X}$, we write $\mathrm{Bicat}(\mathscr{A}, \mathscr{X})$ for the bicategory of morphisms (= lax functors), transformations (= lax natural transformations), and modifications. To be clear about directions, for a morphism $T : \mathscr{A} \to \mathscr{X}$, we have composition and identity constraints in the directions

$$\phi_{v,u} : Tv \circ Tu \Rightarrow T(v \circ u) \quad \text{and} \quad \phi_0 A : 1_{TA} \Rightarrow T1_A$$

while the data for a transformation $\theta : T \Rightarrow S : \mathscr{A} \to \mathscr{X}$ are as displayed in (2.1).

$$\begin{array}{ccc} TA & \xrightarrow{\;Tu\;} & TB \\ {\scriptstyle \theta_A}\downarrow & \overset{\theta_u}{\Longrightarrow} & \downarrow{\scriptstyle \theta_B} \\ SA & \xrightarrow[\;Su\;]{} & SB \end{array} \tag{2.1}$$

We call a morphism $T : \mathscr{A} \to \mathscr{X}$ *normal* when all $\phi_0 A$ are invertible. We call $T : \mathscr{A} \to \mathscr{X}$ a *homomorphism* (or *pseudofunctor*) when all $\phi_{v,u}$ and $\phi_0 A$ are invertible. We write $\mathrm{Hom}(\mathscr{A}, \mathscr{X})$ for the full subbicategory of $\mathrm{Bicat}(\mathscr{A}, \mathscr{X})$ consisting of the homomorphisms.

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A transformation $\theta : T \Rightarrow S$ is called *strong* (or a *pseudonatural transformation*) when all the θ_u are invertible. We write $\mathrm{Hom_s}(\mathscr{A}, \mathscr{X})$ for the subbicategory of $\mathrm{Hom}(\mathscr{A}, \mathscr{X})$ on restricting the morphisms to the strong transformations. Each bicategory $\mathscr{A}$ has a *Yoneda homomorphism*

$$\mathrm{Y}_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathrm{Hom_s}(\mathscr{A}^{\mathrm{op}}, \mathrm{Cat})$$

taking A to $\mathscr{A}(-, A)$; it is an equivalence on hom categories. Indeed, the bicategorical Yoneda Lemma asserts a pseudonatural equivalence

$$\mathrm{Hom_s}(\mathrm{Y}_{\mathscr{A}}A, T) \simeq TA$$

taking θ to $\theta_A(1_A)$. Morphisms of bicategories compose yielding a category of bicategories and morphisms. However, taking transformations as 2-cells does not yield a 2-category: applying a morphism $F : \mathscr{X} \to \mathscr{Z}$ to a transformation $\theta : T \Rightarrow S : \mathscr{A} \to \mathscr{X}$ does not define a transformation. That is, we do not have post-whiskering. Yet we do have pre-whiskering: for any morphism $H : \mathscr{C} \to \mathscr{A}$, we have a homomorphism

$$(H, 1) : \mathrm{Bicat}(\mathscr{A}, \mathscr{X}) \longrightarrow \mathrm{Bicat}(\mathscr{C}, \mathscr{X})$$

defined in the obvious way, taking T to TH and θ to θ_H.

We write $\mathscr{A}^{\mathrm{op}}$ for the bicategory with the same objects as $\mathscr{A}$ while $\mathscr{A}^{\mathrm{op}}(A, B) = \mathscr{A}(B, A)$. We write $\mathscr{A}^{\mathrm{co}}$ for the bicategory with the same objects as $\mathscr{A}$ while $\mathscr{A}^{\mathrm{co}}(A, B) = \mathscr{A}(A, B)^{\mathrm{op}}$. So we write $\mathscr{A}^{\mathrm{coop}}$ for the bicategory with the same objects as $\mathscr{A}$ while $\mathscr{A}^{\mathrm{coop}}(A, B) = \mathscr{A}(B, A)^{\mathrm{op}}$.

Let us put

$$\mathrm{Bicat}^{\mathrm{op}}(\mathscr{A}, \mathscr{X}) = \mathrm{Bicat}(\mathscr{A}^{\mathrm{op}}, \mathscr{X}^{\mathrm{op}})^{\mathrm{op}} \;\text{ and }\; \mathrm{Bicat}^{\mathrm{co}}(\mathscr{A}, \mathscr{X}) = \mathrm{Bicat}(\mathscr{A}^{\mathrm{co}}, \mathscr{X}^{\mathrm{co}})^{\mathrm{co}} \; .$$

Objects of $\mathrm{Bicat}^{\mathrm{op}}(\mathscr{A}, \mathscr{X})$ are still just the morphisms $T : \mathscr{A} \to \mathscr{X}$ but the morphisms of $\mathrm{Bicat}^{\mathrm{op}}(\mathscr{A}, \mathscr{X})$ are called *optransformations* (or *oplax natural transformations*) $T \Rightarrow S : \mathscr{A} \to \mathscr{X}$. Objects of $\mathrm{Bicat}^{\mathrm{co}}(\mathscr{A}, \mathscr{X})$ are called *comorphisms* (or *oplax functors*) from $\mathscr{A}$ to $\mathscr{X}$.

From [13] we have

$$\mathrm{Bicat}^{\mathrm{op}}(\mathscr{A}, \mathrm{Bicat}(\mathscr{B}, \mathscr{C})) \cong \mathrm{Bicat}(\mathscr{B}, \mathrm{Bicat}^{\mathrm{op}}(\mathscr{A}, \mathscr{C})) \; , \tag{2.2}$$

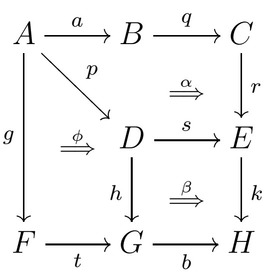
yet we also have

$$\mathrm{Bicat}^{\mathrm{co}}(\mathscr{A}, \mathrm{Bicat}(\mathscr{B}, \mathscr{C})) \cong \mathrm{Bicat}(\mathscr{B}, \mathrm{Bicat}^{\mathrm{co}}(\mathscr{A}, \mathscr{C})) \; . \tag{2.3}$$

3. Sequent notation

Pasting was used in the first paper [2] on bicategories. It was used extensively for 2-categories in [7] and the volume containing it. Pasting in bicategories was put on a firm foundation in the Appendix of [15].

We shall make use of sequent notation for pasting diagrams in a bicategory. For example, we will write the lovely pasting diagram



more mundanely as follows.

$$\begin{array}{l} b\circ t\circ g \\ \hline b\circ h\circ p \\ \hline k\circ s\circ p \\ \hline k\circ r\circ q\circ a \end{array} \quad \begin{array}{l}\phi\\ \beta \\ \alpha\end{array}$$

4. Adjoints for lax natural transformations

This topic was discussed in [11] and provided a basic example for doctrinal adjunction [6].

Let $S, T : \mathscr{A} \longrightarrow \mathscr{X}$ be lax functors from the bicategory $\mathscr{A}$ to the bicategory $\mathscr{X}$. Suppose $p : S \longrightarrow T$ is a transformation: the data are displayed as follows.

$$\begin{array}{ccc} SA & \xrightarrow{\;p_A\;} & TA \\ {\scriptstyle Sf}\downarrow & \overset{p_f}{\Longleftarrow} & \downarrow{\scriptstyle Tf} \\ SB & \xrightarrow[\;p_B\;]{} & TB \end{array} \tag{4.4}$$

Suppose each p_A has a left adjoint $q_A : TA \longrightarrow SA$ in $\mathscr{X}$. Then we have a mate (in the sense of [7])

$$\begin{array}{ccc} TA & \xrightarrow{\;q_A\;} & SA \\ {\scriptstyle Tf}\downarrow & \overset{q_f}{\Longrightarrow} & \downarrow{\scriptstyle Sf} \\ TB & \xrightarrow[\;q_B\;]{} & SB \end{array} \tag{4.5}$$

to (4.4).

Proposition 4.1. *The data* (4.5) *determine an optransformation* $q : T \longrightarrow S$.

In this case we call q the *left adjoint* of p.

5. Lax adjoints for lax functors

This topic reviews work of [4], [3], and Section 1.1 of [15].

Assume $\mathscr{M}$ and $\mathscr{N}$ are bicategories (although we often write as if they were 2-categories).

Suppose $P : \mathscr{N} \longrightarrow \mathscr{M}$ is a lax functor and $Q : \mathscr{M} \longrightarrow \mathscr{N}$ is an oplax functor. We write $\phi : Pg\circ Pf \Rightarrow P(g\circ f)$ and $\phi_0 : 1_{PA} \Rightarrow P1_A$ for the lax structure of P, and $\psi : Q(v\circ u) \Rightarrow Qv\circ Qu$ and $\psi_0 : Q1_X \Rightarrow 1_{QX}$ for the oplax structure of Q.

We shall define what it means for Q to be a *lax left adjoint* for P.

The *lax unit* data are as displayed in the square (5.6).

$$\begin{array}{ccc} X & \xrightarrow{\;n_X\;} & PQX \\ {\scriptstyle u}\downarrow & \overset{n_u}{\Longrightarrow} & \downarrow{\scriptstyle PQu} \\ Y & \xrightarrow[\;n_Y\;]{} & PQY \end{array} \tag{5.6}$$

The *lax counit* data are as displayed in the square (5.7).

$$\begin{array}{ccc} QPA & \xrightarrow{\;e_A\;} & A \\ {\scriptstyle QPf}\downarrow & \overset{e_f}{\Longrightarrow} & \downarrow{\scriptstyle f} \\ QPB & \xrightarrow[\;e_B\;]{} & B \end{array} \tag{5.7}$$

There are six axioms so far for these (where $\alpha : u \Rightarrow v$ and $\sigma : f \Rightarrow g$):

$$\begin{array}{ll} n_Y \circ u & \\ \hline n_Y \circ v & \alpha \\ \hline PQv \circ n_X & n_v \end{array} \quad =_{(1)} \quad \begin{array}{ll} n_Y \circ u & \\ \hline PQu \circ n_X & n_u \\ \hline PQv \circ n_X & PQ\alpha \end{array} \ ;$$

$$\begin{array}{ll} e_B \circ QPf & \\ \hline f \circ e_A & e_f \\ \hline g \circ e_A & \sigma \end{array} \quad =_{(2)} \quad \begin{array}{ll} e_B \circ QPf & \\ \hline e_B \circ QPg & QP\sigma \\ \hline g \circ e_A & e_g \end{array} \ ;$$

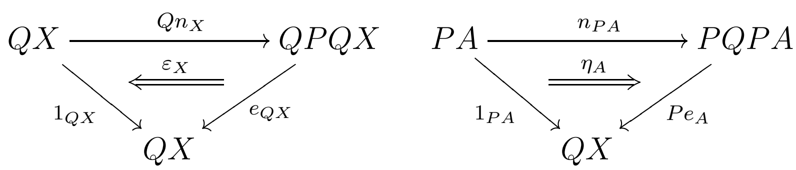
$$\begin{array}{ll} n_X & \\ \hline PQ1_X \circ n_X & n_{1_X} \\ \hline P1_{QX} \circ n_X & P\psi_0 \end{array} \quad =_{(3)} \quad \begin{array}{ll} n_X & \\ \hline P1_{QX} \circ n_X & \phi_0 \end{array} \ ;$$

$$\begin{array}{ll} e_A \circ Q1_{PA} & \\ \hline e_A \circ QP1_A & Q\phi_0 \\ \hline e_A & e_{1_A} \end{array} \quad =_{(4)} \quad \begin{array}{ll} e_A \circ Q1_{PA} & \\ \hline e_A & \psi_0 \end{array} \ ;$$

$$\begin{array}{ll} n_Z \circ v \circ u & \\ \hline PQv \circ n_v \circ u & n_v \\ \hline PQv \circ PQu \circ n_X & n_u \\ \hline P(Qv \circ Qu) \circ n_X & \phi \end{array} \quad =_{(5)} \quad \begin{array}{ll} n_Z \circ v \circ u & \\ \hline PQ(v \circ u) \circ n_X & n_{v \circ u} \\ \hline P(Qv \circ Qu) \circ n_X & P\psi \end{array} \ ;$$

$$\begin{array}{ll} e_C \circ Q(Pg \circ Pf) & \\ \hline e_C \circ QPg \circ QPf & \psi \\ \hline g \circ e_B \circ QPf & e_g \\ \hline g \circ f \circ e_A & e_f \end{array} \quad =_{(6)} \quad \begin{array}{ll} e_C \circ Q(Pg \circ Pf) & \\ \hline e_C \circ QP(g \circ f) & Q\phi \\ \hline g \circ f \circ e_A & e_{g \circ f} \end{array} \ .$$

The *adjunction constraint* data are displayed in diagram (5.8).



(5.8)

There are four more axioms.

$$\begin{array}{ll} 1_{PQX} \circ n_X & \\ \hline Pe_{QX} \circ n_{PQX} \circ n_X & \eta_{QX} \\ \hline Pe_{QX} \circ PQn_X \circ n_X & n_{n_X} \\ \hline P(e_{QX} \circ Qn_X) \circ n_X & \phi \\ \hline P1_{QX} \circ n_X & P\varepsilon_X \end{array} \quad =_{(7)} \quad \begin{array}{ll} 1_{PQX} \circ n_X & \\ \hline P1_{QX} \circ n_X & \phi_0 \end{array}$$

$$\begin{array}{ll} e_A \circ Q1_{PA} & \\ \hline e_A \circ Q(Pe_A \circ n_{PA}) & Q\eta_A \\ \hline e_A \circ QPe_A \circ Qn_{PA} & \psi \\ \hline e_A \circ e_{QPA} \circ Qn_{PA} & e_{e_A} \\ \hline e_A \circ 1_{QPA} & \varepsilon_{PA} \end{array} \quad =_{(8)} \quad \begin{array}{ll} e_A \circ Q1_{PA} & \\ \hline e_A \circ 1_{QPA} & \psi_0 \end{array}$$

$$\begin{array}{ll} e_{QY} \circ Q(n_Y \circ u) & \\ \hline e_{QY} \circ Q(PQu \circ n_X) & Qn_u \\ \hline e_{QY} \circ QPQu \circ Qn_X & \psi \\ \hline Qu \circ e_{QX} \circ Qn_X & e_{Qu} \\ \hline Qu & \varepsilon_X \end{array} \quad =_{(9)} \quad \begin{array}{ll} e_{QY} \circ Q(n_Y \circ u) & \\ \hline e_{QY} \circ Qn_Y \circ Qu & \psi \\ \hline Qu & \varepsilon_Y \end{array}$$

$$
\begin{array}{ll}
Pf & \\ \hline
Pe_B \circ n_{PB} \circ Pf & \eta_B \\ \hline
Pe_B \circ PQPf \circ n_{PA} & n_{Pf} \\ \hline
P(e_B \circ QPf) \circ n_{PA} & \phi \\ \hline
P(f \circ e_A) \circ n_{PA} & Pe_f
\end{array}
\quad =_{(10)} \quad
\begin{array}{ll}
Pf & \\ \hline
Pf \circ Pe_A \circ n_{PA} & \eta_A \\ \hline
P(f \circ e_A) \circ n_{PA} & \phi
\end{array}
$$

Define functors $p_{X,A} : \mathscr{N}(QX, A) \to \mathscr{M}(X, PA)$, $q_{X,A} : \mathscr{M}(X, PA) \to \mathscr{N}(QX, A)$ to be the composites

$$p_{X,A} : \mathscr{N}(QX, A) \xrightarrow{P} \mathscr{M}(PQX, PA) \xrightarrow{\mathscr{M}(n_X,1)} \mathscr{M}(X, PA)\ , \tag{5.9}$$

$$q_{X,A} : \mathscr{M}(X, PA) \xrightarrow{Q} \mathscr{N}(QX, QPA) \xrightarrow{\mathscr{N}(1,e_A)} \mathscr{N}(QX, A)\ . \tag{5.10}$$

Also define natural families

$$\tilde{\varepsilon}_f : q_{X,A}p_{X,A}(f) \Longrightarrow f \quad \text{and} \quad \tilde{\eta}_u : u \Longrightarrow p_{X,A}q_{X,A}(u) \tag{5.11}$$

to be the following respective composites.

$$
\begin{array}{ll}
e_A \circ Q(Pf \circ n_X) & \\ \hline
e_A \circ QPf \circ Qn_X & \psi \\ \hline
f \circ e_{QX} \circ Qn_X & e_f \\ \hline
f & \varepsilon_X
\end{array}
\qquad
\begin{array}{ll}
u & \\ \hline
Pe_A \circ n_{PA} \circ u & \eta_A \\ \hline
Pe_A \circ PQu \circ n_X & n_u \\ \hline
P(e_A \circ Qu) \circ n_X & \phi
\end{array}
$$

Proposition 5.1. *The functor $p_{X,A}$ (5.9) is right adjoint to $q_{X,A}$ (5.10) with counit $\tilde{\varepsilon}$ and unit $\tilde{\eta}$ as in (5.11).*

Proof. This involves two calculations using the axioms. Here is one calculation. The other is dual.

$$
\begin{array}{ll}
Pf \circ n_X & \\ \hline
Pe_A \circ n_{PA} \circ Pf \circ n_X & \eta_A \\ \hline
Pe_A \circ PQ(Pf \circ n_X) \circ n_X & n_{Pf \circ n_X} \\ \hline
P(e_A \circ Q(Pf \circ n_X)) \circ n_X & \phi \\ \hline
P(e_A \circ QPf \circ Qn_X) \circ n_X & P(e_A \circ \psi) \\ \hline
P(f \circ e_{QX} \circ Qn_X) \circ n_X & P(e_f \circ Qn_X) \\ \hline
Pf \circ n_X & P(f \circ \varepsilon_X)
\end{array}
\quad =_{\text{nat}} \quad
\begin{array}{ll}
Pf \circ n_X & \\ \hline
Pe_A \circ n_{PA} \circ Pf \circ n_X & \eta_A \\ \hline
Pe_A \circ PQ(Pf \circ n_X) \circ n_X & n_{Pf \circ n_X} \\ \hline
Pe_A \circ P(QPf \circ Qn_X) \circ n_X & P\psi \\ \hline
P(e_A \circ QPf \circ Qn_X) \circ n_X & \phi \\ \hline
P(f \circ e_{QX} \circ Qn_X) \circ n_X & P(e_f \circ Qn_X) \\ \hline
Pf \circ n_X & P(f \circ \varepsilon_X)
\end{array}
\quad =_{(5)}
$$

$$
\begin{array}{ll}
Pf \circ n_X & \\ \hline
Pe_A \circ n_{PA} \circ Pf \circ n_X & \eta_A \\ \hline
Pe_A \circ PQPf \circ n_{PQX} \circ n_X & n_{Pf} \\ \hline
Pe_A \circ PQPf \circ PQn_X \circ n_X & n_{n_X} \\ \hline
Pe_A \circ P(QPf \circ Qn_X) \circ n_X & \phi \\ \hline
P(e_A \circ QPf \circ Qn_X) \circ n_X & \phi \\ \hline
P(f \circ e_{QX} \circ Qn_X) \circ n_X & P(e_f \circ Qn_X) \\ \hline
Pf \circ n_X & P(f \circ \varepsilon_X)
\end{array}
\quad =_{\phi \text{ assoc}} \quad
\begin{array}{ll}
Pf \circ n_X & \\ \hline
Pe_A \circ n_{PA} \circ Pf \circ n_X & \eta_A \\ \hline
Pe_A \circ PQPf \circ n_{PQX} \circ n_X & n_{Pf} \\ \hline
Pe_A \circ PQPf \circ PQn_X \circ n_X & n_{n_X} \\ \hline
P(e_A \circ QPf) \circ PQn_X \circ n_X & \phi \\ \hline
P(e_A \circ QPf \circ Qn_X) \circ n_X & \phi \\ \hline
P(f \circ e_{QX} \circ Qn_X) \circ n_X & P(e_f \circ Qn_X) \\ \hline
Pf \circ n_X & P(f \circ \varepsilon_X)
\end{array}
\quad =_{\text{nat, funct}}
$$

$$
\begin{array}{ll}
Pf \circ n_X & \\ \hline
Pe_A \circ n_{PA} \circ Pf \circ n_X & \eta_A \\ \hline
Pe_A \circ PQPf \circ n_{PA} \circ n_X & n_{Pf} \\ \hline
P(e_A \circ QPf) \circ n_{PA} \circ n_X & \phi \\ \hline
P(f \circ e_{QX}) \circ n_{PA} \circ n_X & Pe_f \\ \hline
P(f \circ e_{QX}) \circ PQn_X \circ n_X & n_{n_X} \\ \hline
P(f \circ e_{QX} \circ Qn_X) \circ n_X & \phi \\ \hline
Pf \circ n_X & P(f \circ \varepsilon_X)
\end{array}
\quad =_{(10)} \quad
\begin{array}{ll}
Pf \circ n_X & \\ \hline
Pf \circ Pe_{QX} \circ n_{PQX} \circ n_X & \eta_{QX} \\ \hline
P(f \circ e_{QX}) \circ n_{PQX} \circ n_X & \phi \\ \hline
P(f \circ e_{QX}) \circ PQn_X \circ n_X & n_{n_X} \\ \hline
P(f \circ e_{QX} \circ Qn_X) \circ n_X & \phi \\ \hline
Pf \circ n_X & P(f \circ \varepsilon_X)
\end{array}
\quad =_{\text{funct}}
$$

$$\begin{array}{ll} Pf\circ n_X & \\ \hline Pf\circ Pe_{QX}\circ n_{PQX}\circ n_X & \eta_{QX} \\ \hline Pf\circ Pe_{QX}\circ PQn_X\circ n_X & n_{n_X} \\ \hline P(f\circ e_{QX})\circ PQn_X\circ n_X & \phi \\ \hline P(f\circ e_{QX}\circ Qn_X)\circ n_X & \phi \\ \hline Pf\circ n_X & P(f\circ\varepsilon_X) \end{array} \quad =_{\phi\text{ assoc}} \quad \begin{array}{ll} Pf\circ n_X & \\ \hline Pf\circ Pe_{QX}\circ n_{PQX}\circ n_X & \eta_{QX} \\ \hline Pf\circ Pe_{QX}\circ PQn_X\circ n_X & n_{n_X} \\ \hline Pf\circ P(e_{QX}\circ PQn_X)\circ n_X & \phi \\ \hline P(f\circ e_{QX}\circ Qn_X)\circ n_X & \phi \\ \hline Pf\circ n_X & P(f\circ\varepsilon_X) \end{array} \quad =_{\phi\text{ nat}}$$

$$\begin{array}{ll} Pf\circ n_X & \\ \hline Pf\circ Pe_{QX}\circ n_{PQX}\circ n_X & \eta_{QX} \\ \hline Pf\circ Pe_{QX}\circ PQn_X\circ n_X & n_{n_X} \\ \hline Pf\circ P(e_{QX}\circ PQn_X)\circ n_X & \phi \\ \hline Pf\circ P1_X\circ n_X & P\varepsilon_X \\ \hline Pf\circ n_X & \phi \end{array} \quad =_{(7)} \quad \begin{array}{ll} Pf\circ n_X & \\ \hline Pf\circ P1_X\circ n_X & \phi_0 \\ \hline Pf\circ n_X & \phi \end{array} \quad =_{\phi\text{unital}} \quad \begin{array}{ll} Pf\circ n_X & \\ \hline Pf\circ n_X & \text{identity} \end{array}$$

□

Notice that ε_X and η_A can be recovered from the adjunction of Proposition 5.1 via the respective composites

$$\begin{array}{ll} e_A\circ Qn_X & \\ \hline e_A\circ Q(P1_{QX}\circ n_X) & Q(\phi_0\circ n_X) \\ \hline 1_{QX} & \tilde{\varepsilon}_{1_{QX}} \end{array} \qquad \begin{array}{ll} 1_{PA} & \\ \hline P(e_A\circ Q1_{PA})\circ n_X & \tilde{\eta}_{1_{PA}} \\ \hline Pe_A\circ n_X & P(e_A\circ\psi_0) \end{array}$$

Given a lax functor $P:\mathscr{N}\longrightarrow\mathscr{M}$, we obtain a lax functor $P^{\#}$ as the composite

$$\mathscr{N}\xrightarrow{P}\mathscr{M}\xrightarrow{\mathrm{Y}_{\mathscr{M}}}\mathrm{Hom}_{\mathrm{s}}(\mathscr{M}^{\mathrm{op}},\mathrm{Cat})\xrightarrow{\mathrm{incl}}\mathrm{Bicat}^{\mathrm{co}}(\mathscr{M}^{\mathrm{op}},\mathrm{Cat})\ ;$$

so $P^{\#}A=\mathscr{M}(-,PA)$. Given an oplax functor $Q:\mathscr{M}\longrightarrow\mathscr{N}$, we obtain a pseudofunctor $Q_{\#}$ as the composite

$$\mathscr{N}\xrightarrow{\mathrm{Y}_{\mathscr{N}}}\mathrm{Hom}_{\mathrm{s}}(\mathscr{N}^{\mathrm{op}},\mathrm{Cat})\xrightarrow{(Q^{\mathrm{op}},1)}\mathrm{Bicat}^{\mathrm{co}}(\mathscr{M}^{\mathrm{op}},\mathrm{Cat})\ ;$$

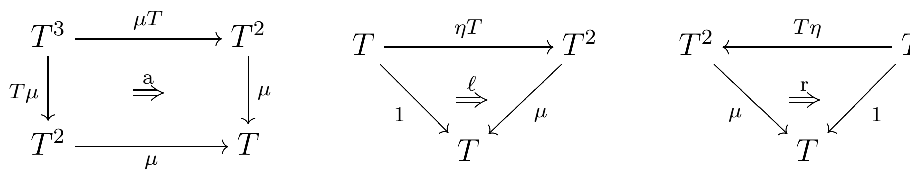
so $Q_{\#}A=\mathscr{N}(Q-,A)$. (Of course, using (2.3), we can also view these as oplax functors (comorphisms) $\mathscr{M}^{\mathrm{op}}\to\mathrm{Bicat}(\mathscr{N},\mathrm{Cat})$.)

Proposition 5.2. *A lax adjunction between P and Q amounts to a lax natural transformation $p:Q_{\#}\Rightarrow P^{\#}$ with a left adjoint $q:P^{\#}\Rightarrow Q_{\#}$ in the sense of Section 4.*

6. Skew monads

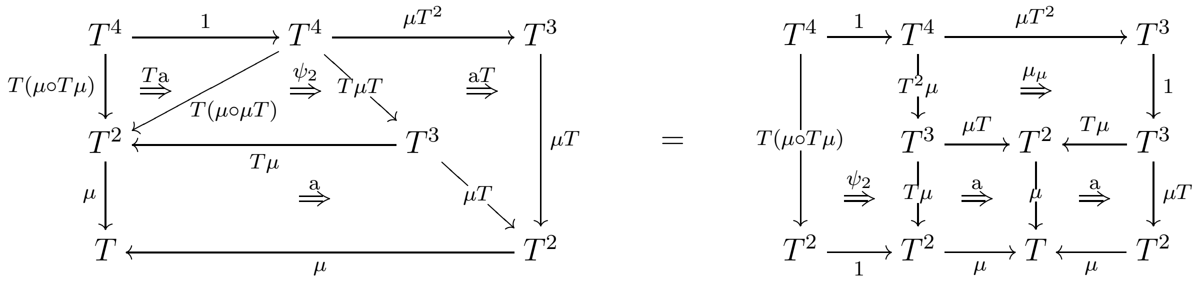
A notion of lax monad on a 2-category $\mathscr{K}$ was defined by Bunge [4]. There is no problem generalizing this to a bicategory $\mathscr{K}$. In any case, we will write as if our bicategory $\mathscr{K}$ were a 2-category. For Bunge, the lax monad involved what we would call an oplax functor $T\colon\mathscr{K}\to\mathscr{K}$: so, for composable morphisms $f\colon X\to Y$ and $g\colon Y\to Z$, we have a morphism $\psi_2\colon T(g\circ f)\Rightarrow Tg\circ Tf$, and for each object X, a morphism $\psi_0\colon T1_X\Rightarrow 1_{TX}$, subject to naturality and coherence conditions.

A *right skew monad* $\mathbb{T}$ on $\mathscr{K}$ consists of an oplax functor T on $\mathscr{K}$, oplax natural transformations $\mu\colon T^2\to T$ and $\eta\colon 1_{\mathscr{K}}\to T$, and modifications as shown in diagram (6.12).

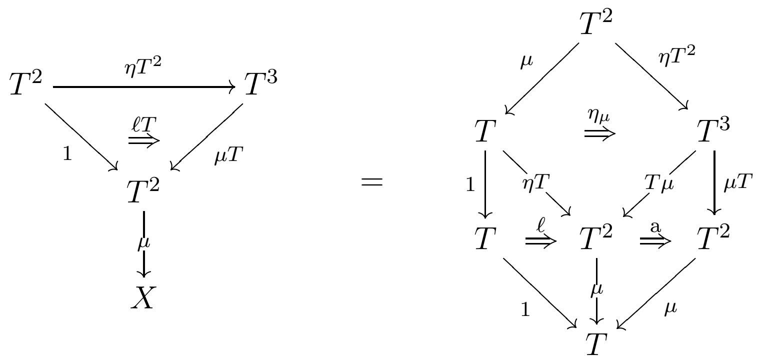


(6.12)

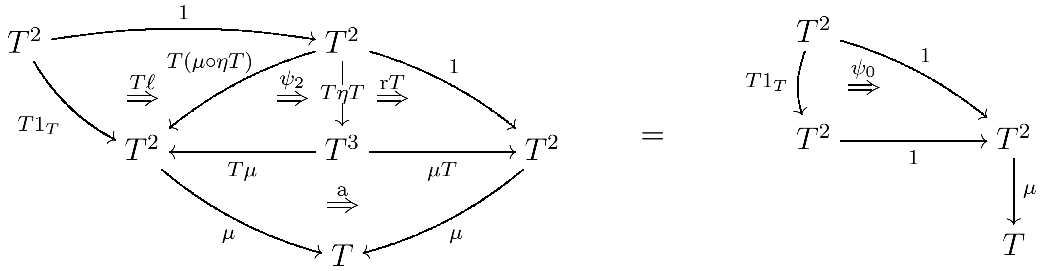
There are five axioms (6.13), (6.14), (6.15), (6.16), (6.17).



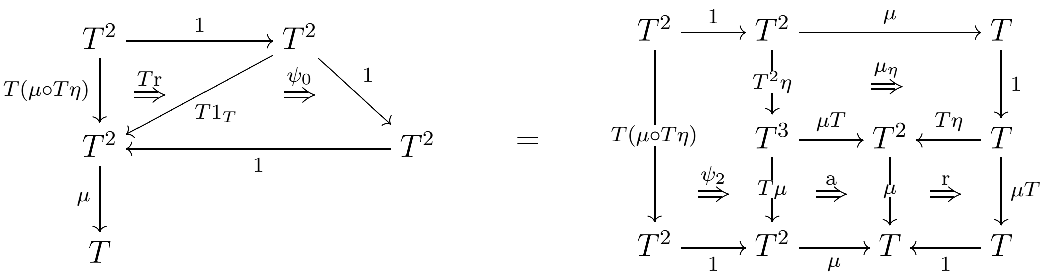
(6.13)



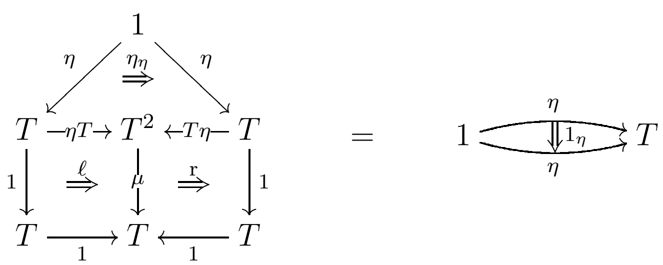
(6.14)



(6.15)



(6.16)



(6.17)

Our later work may be expressed using the bicategory $\mathscr{K}(T)$ defined as follows for any endo-pseudofunctor T on $\mathscr{K}$. The objects are pairs (X, x) where $x\colon TX \to X$ in $\mathscr{K}$. The morphisms

$(f,\xi)\colon (X,x) \to (Y,y)$ are squares (6.18) in $\mathscr{K}$.

$$\begin{array}{ccc} TX & \xrightarrow{\;Tf\;} & TY \\ {\scriptstyle x}\big\downarrow & \overset{\xi}{\Rightarrow} & \big\downarrow{\scriptstyle y} \\ X & \xrightarrow[\;f\;]{} & Y \end{array} \tag{6.18}$$

The 2-cells $\sigma\colon (f,\xi) \Rightarrow (g,\zeta)$ are 2-cells $\sigma\colon f \Rightarrow g$ in $\mathscr{K}$ which are compatible with ξ, ζ in the obvious way.

A morphism $f\colon X \to Y$ in $\mathscr{K}$ is called T-*extendable* when, for all $x\colon TX \to X$, there exists a left extension, denoted $\hat{f}(x)\colon TY \to Y$, of $f \circ x$ along Tf. In this case, we have a functor

$$\hat{f}\colon \mathscr{K}(TX,X) \longrightarrow \mathscr{K}(TY,Y)\ .$$

A morphism $f\colon X \to Y$ in $\mathscr{K}$ is called T-*liftable* when, for all $y\colon TY \to Y$, there exists a right lifting, denoted $\check{f}(y)\colon TX \to X$, of $y \circ Tf$ through f. In this case, we have a functor

$$\check{f}\colon \mathscr{K}(TY,Y) \longrightarrow \mathscr{K}(TX,X)\ .$$

When f is both, we have an adjunction $\hat{f} \dashv \check{f}$.

If a morphism $f\colon X \to Y$ has a right adjoint $f^*\colon Y \to X$ then it is both T-extendable and T-liftable. Indeed, $\hat{f}(x) = f \circ x \circ Tf^* = \mathscr{K}(Tf^*, f)(x)$ and $\check{f}(y) = f^* \circ y \circ Tf = \mathscr{K}(Tf, f^*)(y)$.

7. The Galois connection

Consider bicategories $\mathscr{C}$ and $\mathscr{K}$ and lax morphisms $S, T\colon \mathscr{C} \to \mathscr{K}$. Consider an oplax natural transformation $\theta\colon S \Rightarrow T$ and an object A of $\mathscr{K}$.

Write $\theta \nabla A$ when, for all objects $X \in \mathscr{X}$ and $f\colon SX \to A$, the left extension

$$\begin{array}{ccc} SX & \xrightarrow{\;\theta_X\;} & TX \\ & {\scriptstyle f}\searrow \quad \overset{\kappa_\theta}{\Rightarrow} \quad \swarrow{\scriptstyle \mathrm{lan}_{\theta_X} f} & \\ & A & \end{array} \tag{7.19}$$

of f along θ_X exists, and, for all $u\colon Y \to X$ in $\mathscr{C}$, the diagram

$$\begin{array}{ccc} SY & \xrightarrow{\;\theta_Y\;} & TY \\ {\scriptstyle Su}\big\downarrow & \overset{\theta_u}{\Rightarrow} & \big\downarrow{\scriptstyle Tu} \\ SX & \xrightarrow{\;\theta_X\;} & TX \\ & {\scriptstyle f}\searrow \quad \overset{\kappa_\theta}{\Rightarrow} \quad \swarrow{\scriptstyle \mathrm{lan}_{\theta_X} f} & \\ & A & \end{array} \tag{7.20}$$

exhibits $(\mathrm{lan}_{\theta_X} f) \circ Tu$ as $\mathrm{lan}_{\theta_Y}(f \circ Su)$.

The condition $\theta \nabla A$ might be considered a cocompleteness condition on A given θ, or an exactness condition on θ given A.

Example 7.1. If the oplax natural transformation $\theta\colon S \Rightarrow T$ has a right adjoint (in the sense of Section 4) which is a pseudonatural transformation then $\theta \nabla A$ for every object A.

8. A skew monoidal category

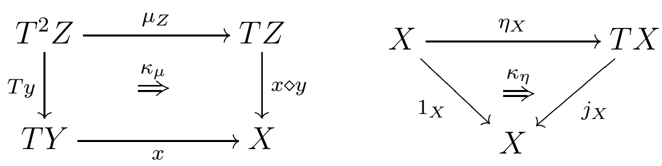
Suppose $\mathbb{T}$ is a right skew monad on a bicategory $\mathscr{K}$ and X is an object of $\mathscr{K}$.

We write $\mathbb{T}\nabla X$ when $\mu\nabla X$ and $\eta\nabla X$.

We define a left skew bicategory $\mathrm{co}\mathscr{K}_{\mathbb{T}}$ called the *coKleisli bicategory* of $\mathbb{T}$. The objects are those objects X of $\mathscr{K}$ such that $\mathbb{T}\nabla X$. The hom categories are defined by

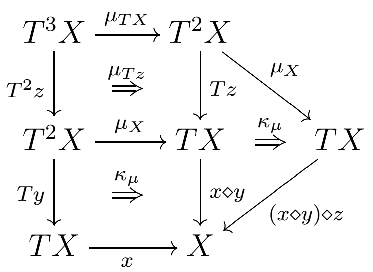
$$\mathrm{co}\mathscr{K}_{\mathbb{T}}(Y,X) = \mathscr{K}(TY,X)\ .$$

The composite of $y\colon TZ \to Y$ and $x\colon TY \to X$, denoted by $x \diamond y$, is defined by $x \diamond y = \mathrm{lan}_{\mu_Z}(x \circ Ty)$. A candidate j_X for skew identity is defined by $j_X = \mathrm{lan}_{\eta_X} 1_X$.



(8.21)

Using $\mu\nabla X$, we see that the pasted composite 2-cell

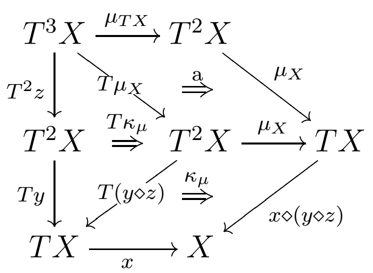


(8.22)

exhibits $(x \diamond y) \diamond z$ as a left extension of $x \circ Ty \circ T^2z$ along $\mu_X \circ \mu_{TX}$. Therefore, by the left extension property, there is a unique 2-cell

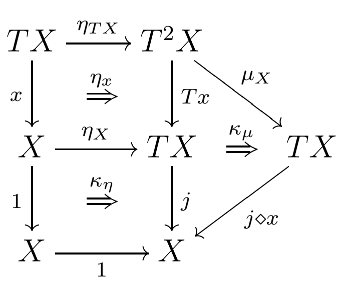
$$\alpha_{x,y,z}\colon (x \diamond y) \diamond z \Longrightarrow x \diamond (y \diamond z) \tag{8.23}$$

which pastes onto (8.22) to yield the pasted 2-cell (8.24).



(8.24)

Using $\eta\nabla X$, we see that the pasted composite 2-cell

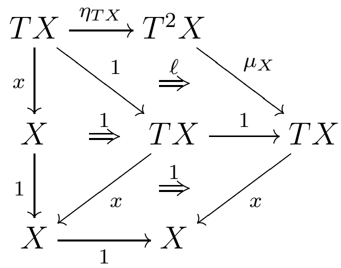


(8.25)

exhibits $j \diamond x$ as a left extension of x along $\mu_X \circ \eta_{TX}$. Therefore, by the left extension property, there is a unique 2-cell

$$\lambda_x \colon j \diamond x \Longrightarrow x \tag{8.26}$$

which pastes onto (8.25) to yield the pasted 2-cell (8.27).

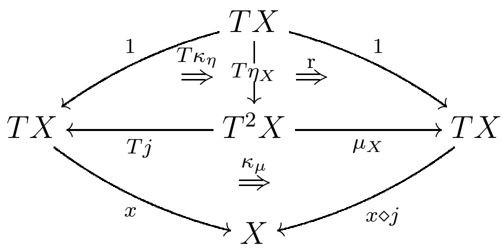


(8.27)

Define the 2-cell

$$\rho_x \colon x \Longrightarrow x \diamond j \tag{8.28}$$

to be the pasted composite (8.29).



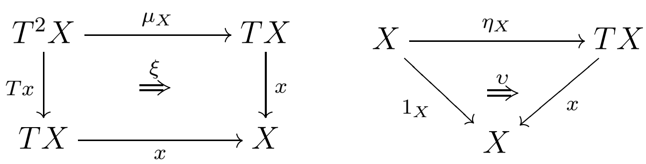
(8.29)

Proposition 8.1. *For any right skew monad* $\mathbb{T}$ *on* $\mathscr{K}$ *and any object* $X \in \mathscr{K}$ *satisfying* $\mathbb{T}\nabla X$*, the tensor product* $\diamond$ *and skew unit* j (8.21) *equipped with constraints* (8.23), (8.26), (8.28), *define a left skew monoidal structure on the category* $\mathscr{K}(TX, X)$. *If* $\mathbb{T}$ *is left normal then so is* $\mathscr{K}(TX, X)$.

9. Lax algebras

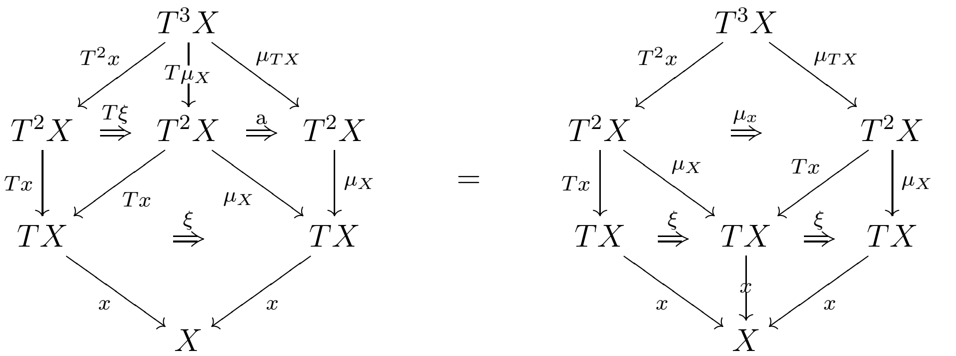
Lax algebras for 2-monads were defined in Section 2 of [12]. Those appearing here are a generalization to skew monads as occurring in [1].

Suppose $\mathbb{T}$ is a right skew monad on the bicategory $\mathscr{K}$. A *lax* $\mathbb{T}$*-algebra structure* on $X \in \mathscr{K}$ consists of 2-cells

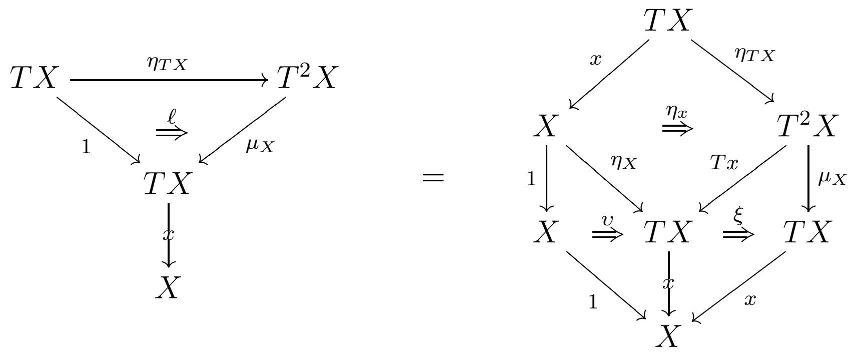


(9.30)

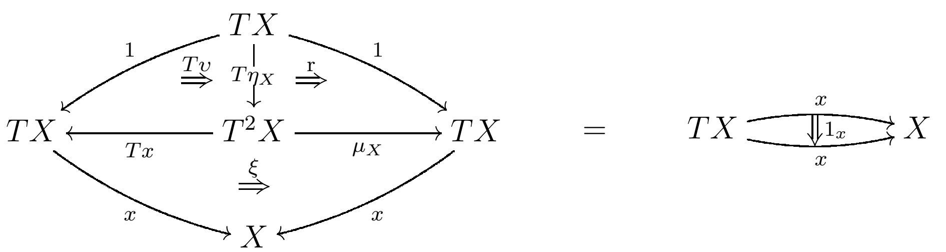
subject to equations (9.31), (9.32) and (9.33).



(9.31)



(9.32)



(9.33)

Proposition 9.1. *The category of lax $\mathbb{T}$-algebra structures on $X \in \mathscr{K}$ is isomorphic to the category of monoids in the skew monoidal category $\mathscr{K}(TX, X)$ of Proposition 8.1. If $f\colon X \to Y$ is T-extendable then $\hat{f}\colon \mathscr{K}(TX, X) \to \mathscr{K}(TY, Y)$ is opmonoidal. If $f\colon X \to Y$ is T-liftable then $\check{f}\colon \mathscr{K}(TY, Y) \to \mathscr{K}(TX, X)$ is monoidal.*

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