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★**Higher operads, higher categories.**

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Higher operads were introduced by Michael Batanin in 1996 as a framework for his definition of higher categories; this was reported in ["On the definition of weak ω-category", Rep. No. 96/207, Macquarie Univ., North Ryde, 1996; per revr.]. The book under review presents an exposition of this basic strategy while incorporating some insights and modifications of the author. (Tom Leinster is the author, not the editor, contrary to what appears on the cover.)

The notes at the end of each chapter are quite helpful in providing some perspective of previous work. Here, the reviewer would like to describe in some detail how the subject evolved as further motivation for reading the book.

The operad and category story goes back over forty years. The phenomena typifying this subject emerged in both algebraic topology and category theory. James Stasheff studied associativity up to higher homotopy in topology by constructing polytopes called associahedra which exemplify what we would now call a topological operad. Saunders Mac Lane looked at abstract tensor products on categories, involving associativity only up to isomorphism; we now call these monoidal categories.

There were many connections between these endeavours. The tensor's associativity isomorphisms themselves were subject to conditions, one of which was the fourth associahedron (a pentagon). Mac Lane proved that commutativity of the higher associahedra followed: a so-called coherence theorem.

As usual, category theory was working at two levels. At one, categories and isomorphisms were analogous to spaces and homotopies. At the other, special kinds of monoidal categories, called PROPs, were recognized as the abstract theories encapsulating the correct kinds of operations and axioms. It was further recognized by J. Michael Boardman, Rainer Vogt, Peter May and G. Max Kelly that, in many important cases, the PROP was determined by morphisms with codomain a single generating object, so that composition of morphisms could be recaptured from substitution. In topology this led to the first use of the word operad by May (for what we would now call symmetric operads) and in category theory this led to what Kelly called clubs. Models for these kinds of theories were called algebras for the operad or the club (since they were actually Eilenberg-Moore algebras for an associated monad). It is relevant here to mention that related machinery was created by Graeme Segal.

Originally monoidal categories were not particularly thought of as higher categories; this happened after Jean Benabou generalized them to bicategories which we might now also call weak 2-categories. The challenge was then out to define weak n-categories for all n. These would be structures with objects, 1-morphisms between objects, 2-morphisms between 1-morphisms with the same source and target objects, and so on; there would be various compositions which would be weakly associative, weakly unital, and weakly compatible.

With this background, it was natural (as pointed out by Todd Trimble in the early 1990s) to try to use operads to precisely define weak n-category. John Baez and James Dolan successfully defined weak n-category using coloured symmetric operads; however, the algebras for the operads were diagram shapes (opetopes), not the weak n-categories themselves.

Fundamental to Batanin's work was his discovery of an explicit model for the free strict n-category on a directed n-graph (or globular set). An ordinary operad is, in the first instance, a sequence of sets or spaces; that is, objects indexed by the natural numbers. Natural numbers can be viewed as the morphisms in the free category on the terminal directed graph. Batanin's higher operads were families (collections) indexed by the morphisms of all dimensions in the free strict higher category on the terminal higher graph; these morphisms he codified as plane trees. A higher operad is such a collection equipped with substitution operations.

Also fundamental to Batanin's work was his idea that a notion of contractibility should take care of the coherence requirements of a weak higher category. His definition of weak higher categories was that they were algebras for the initial contractible higher operad.

It should be noted that around the same time Baez-Dolan and Batanin were announcing their definitions, Zouhair Tamsamani produced another definition based on multi-simplicial sets and the Segal machinery.

Leinster's book opens with motivating remarks for topologists; indeed, they would also be useful for anyone with a standard postgraduate mathematics training. He then provides a readable background on monoidal categories, bicategories, enriched categories, strict higher categories, and classical operads. To this experienced category theorist there seems to be an over emphasis on technicalities the author calls "algebraic versus non-algebraic" and "biased versus unbiased"; however, he nicely explains the subtleties for any reader new to the field.

In [Theory Appl. Categ. **10** (2002), 1–70 (electronic); MR1883478] Leinster presented ten different definitions of higher category. Here, he chooses to emphasise just one, with fleeting mention of others. There are some subtle departures from Batanin's original version. One is the definition of contractibility of an operad. The other is that the author does not need to include basic operations in his operad for weak higher categories since even these come from contractibility. The result is that every Leinster weak higher category is an example of a Batanin weak higher category, while the equivalence remains conjectural.

Several people studying Batanin's work noticed that the set-based version of higher operad required for the definition of weak higher categories could be explained more directly than in Batanin's paper [Adv. Math. **136** (1998), no. 1, 39–103; MR1623672]. This simpler description avoids monoidal globular categories while exploiting Batanin's model of the free strict higher category on a globular set: a Cartesian monad is generated. However, monoidal globular categories are by no means superseded: as with classical operads, the set-based version is insufficient for many applications not covered by the book.

Other features of the book are the imaginative explanations (see for example, the description of opetopes), the beautiful diagrams, and the amusing quotations that head each chapter.

To some extent this book can be seen as providing closure to an era concentrating on new definitions of weak n-category. The next era will primarily be aimed at understanding what it really means for definitions to be equivalent.

Addendum (June, 2005): This listing was revised to correct the bibliographic information, which was incorrectly published on the cover of the book (where the author was listed as the editor).

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