Johnstone, Peter T.
Sketches of an elephant. A topos theory compendium. II. (English) Zbl 1071.18002
Oxford Logic Guides 44; Oxford Science Publications. Oxford: Clarendon Press (ISBN 0-19-852496-X/set; 0-19-851598-7/v.2). xii, 469-1089, 71 p./bibliogr. (2002).

This is the joint review of Parts I and II (for Part I see [Zbl 1071.18001]).

The importance mathematicians put on proving time-honoured conjectures often obscures the conceptual advances. Two such major advances are confused by the ambiguous term topos.

The original precise definition of topos by Alexander Grothendieck was: category of sheaves on a small site. A site is a category with a notion of cover for its objects; this generalizes the ordered set of open subsets of a topological space with the usual notion of open cover. A sober space can be recaptured from its category of sheaves, however, any generating small set of objects of a topos determines a site for the topos. Consequently, Grothendieck emphasized the topos itself as the generalized space. The idea was a major advance that indeed led to proofs of the Weil conjectures by Grothendieck and Deligne. However, the mathematical community has been slow to warm to the concept itself. As *A. Grothendieck* ["Sketch of a programme", `http://www.math.jussieu.fr/~leila/EsquisseEng.pdf`, subsection 20, p. 253; cf. Lond. Math. Soc. Lect. Note Set. 242, 5–48 (1997; Zbl 0901.14001)] remarks in a footnote: "In writing this, I am aware that rare are the topologists, even today, who realize the existence of this conceptual and technical generalization of topology, and the resources it offers."

L. Illusie ["What is ...a topos?", Notices Am. Math. Soc. 51, 1060–1061 (2004; Zbl 1071.18003)] explains well, in the space available, Grothendieck topos theory. Yet, I take issue somewhat with his last sentence describing elementary toposes as a variant notion and implying their main application is to logic.

The concept of elementary topos, due to F. W. (Bill) Lawvere and Myles Tierney, was a major conceptual advance. It is true that Jean Giraud had characterized Grothendieck toposes in terms of categorical properties based on a given universe of small sets. Cutting these properties down to an elementary (universe independent) level, we are led to the notion of pretopos which, while of some application, is not the Lawvere-Tierney concept. It is also true that Grothendieck had noted the presence of a subobject classifier in his toposes and used the symbol Ω for it. Lawvere recognized the logical role of Ω as an object of truth values.

An elementary topos is defined to be a category with a terminal object, pullbacks, and power objects. That this is equivalent to the definition by Lawvere-Tierney involves nice work by Anders Kock, Christian Mikkelsen, and Robert Paré. The power object of the terminal object is Ω while other power objects are obtained by Cartesian exponentiation. Elementary topos is a proper generalization of Grothendieck topos and the analysis of the former includes and enriches the analysis of the latter.

The remarkable nature of the elementary topos axioms is their power (pun intended). It is possible to do with elementary toposes all you would want to do with Grothendieck toposes. This was confirmed in the original Lawvere-Tierney work where they showed that sheaf theory could be internalized to elementary toposes. Moreover, they showed that forcing, as used to prove independence results in set theory, was an example of sheaf theory in their form.

So it is indeed true that elementary topos theory, as intuitionistic set theory, has contributed to logic. As explained in Peter Johnstone's Preface, the reference to an elephant in the title pays homage to the many other aspects of topos theory. The Preface lists thirteen views of what a topos "really is": one is a generalized space while an intuitionistic higher-order theory is just one other.

There are many books by now that treat some aspects of topos theory but they are mostly in conjunction with other subjects. *P. T. Johnstone*'s original book ["Topos theory", Lond. Math. Soc. Monogr. 10 (1977; Zbl 0368.18001)] remains a valuable scholarly masterpiece yet, naturally, misses the advances of the last three decades. Johnstone has maintained his encyclopedic knowledge of the subject through these advances and mathematicians should be grateful that he has tackled admirably the immense task of ameliorating a bibliography of 1262 publications into a trilogy.

The two volumes are divided into two parts each: Part A: Toposes as categories, Part B: 2-Categorical aspects of topos theory, Part C: Toposes as spaces, and Part D: Toposes as theories. Predicted for Volume 3 are: Part E: Homotopy and cohomology and Part F: Toposes as mathematical universes.

These volumes include fuller and more modern treatments of the topics in "Topos theory". Where necessary, prerequisite background material is provide: Section A1 is a review of regular and Cartesian closed categories, Section B1.1 is a review of 2-categories, and Section B2.1 is a review of enriched categories. An example of new material is the theorem of André Joyal and Myles Tierney that every bounded $\mathcal{S}$-topos consists of the continuous actions of a localic groupoid in $\mathcal{S}$ (Section C5).

What will be the next generation of topos theory? Perhaps it will be at the interface between higher-dimensional category theory and topos theory in serving the needs of homotopical algebraic geometry [see *B. Toën* and *G. Vezzosi*, "Homotopical algebraic geometry. I: topos theory", Adv. Math. 193, 257–372 (2005; Zbl 1120.14012)]. In this respect, Section E2.1 on Quillen model structures, foreshadowed by Johnstone to be in Volume 3, is of particular interest.

With Grothendieck's footnote in mind, we find no excuse now for mathematicians to be ignorant of topos theory. These volumes contain the technicalities, let us use the resource.

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MSC:

18-02 Research exposition (monographs, survey articles) pertaining to category theory

18B25 Topoi

Keywords:

topos; sheaf; site; subobject classifier; Cartesian closed category; allegory; geometric morphism