A MATHEMATICAL ANALOGY

by

Mark Weber

Mathematics Honours Essay
Macquarie University
November 1996

CONTENTS

I.	BACKGROUND	3
II.	SOME INTRODUCTORY CATEGORY THEORY	4
III.	ADJUNCTIONS	6
IV.	LIMITS The Standard Definitions Weighted Limits and Colimits	10
V.	COMMA CATEGORIES AND KAN EXTENSIONS Comma Categories Kan Extensions	17
VI.	SHEAVES ON TOPOLOGICAL SPACES	24
VII.	GROTHENDIECK TOPOSES Defining Grothendieck Toposes The Associated Sheaf Functor Cover Preserving Functors	30
VIII.	THE DUALITY BETWEEN GEOMETRY AND LOGIC The Main Theorem An Illustrative Example	38
IX.	BIBLIOGRAPHY	41

BACKGROUND

It seems that difficult problems in Mathematics are often solved by analogy. In the early 1960's Grothendieck and then Deligne used an analogy with topological sheaf theory to prove the famous Weil conjectures of algebraic geometry. At a similar time Cohen used a seemingly similar series of analogies in proving the independence of the continuum hypothesis from the other axioms of set theory.

What these pieces of mathematics have in common is that they exploit a type of duality between spatial (topological, geometric) intuition and logic.

Perhaps this duality is unsurprising, since geometry and logic have been bedfellows for as long as mathematics has been studied. In the axiomatic geometry of Euclid and the algebraic technique initiated by Descartes, we see what appears to be the application of some form of logic to the study of (Euclidean) geometry. Taking the logic further and going beyond purely "spatial" systems resulted in the discovery of non-Euclidean geometry in the early 19^{th} century. So this evolution continued, with logic being used to generalise our notion of space, resulting eventually in the theory topological spaces that is in use today.

This line of enquiry sparked another. People began to consider logical mathematical systems independently of any apparent spatial relevance. In this vein axiomatic set theory and abstract algebra (including mathematical logic) came to pass.

These competing viewpoints are most certainly complementary. For example with algebraic topology we see the logical (algebraic) perspective being used to shed light on questions of a topological nature. Conversely, the study of Lie groups imposes a topological flavour to investigations of the general linear groups.

An interesting feature of the evolution of mathematical understanding is the role of analogy. What has tended to happen is that an area of mathematics is advanced by the use of some analogy of it with some other seemingly unrelated area of mathematics (eg Grothendieck/Deligne and Cohen). The next step in this evolution is that this analogy is made more precise, as the connections with this "seemingly unrelated area" become understood (and it ceases to be seemingly unrelated).

The goal of this essay is to give a precise statement of this duality between geometry and logic that has arisen by analogy with topological sheaf theory.

The vehicle for this statement is the theory of Grothendieck toposes, which provide a setting capable for the expression of such a result. However in order to understand this theory we require some considerable background. First general category theory will be discussed, because it is indispensable in the description of the mathematics that follows. Then, an understanding of topological sheaf theory is developed, so that we can understand what we are making an analogy with. Once this material has been traversed, we are finally in a position to complete our program.

SOME INTRODUCTORY CATEGORY THEORY

For categories C and D write [C,D] for the category of functors $C \to D$ and natural transformations between them. Also we let Set denote the category of sets and Cat denote the category of categories.

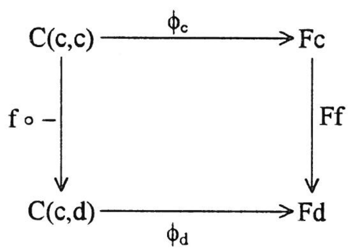
Let $c \in \text{obj}(C)$. Then $C(c,-)$ denotes the functor that maps $d \in \text{obj}(C)$ to the set $C(c,d)$. Then $f : d \to d'$ is mapped to $C(c,f) : C(c,d) \to C(c,d')$ where $g \mapsto f \circ g$. Another notation for $C(c,f)$ is $f \circ -$. By these definitions it follows that $C(c,-) \in [C,Set]$. Dually we define $C(-,c) \in [C^{op},Set]$. Such functors are often called <u>hom functors</u>.

Let $F : C \to D$ be a functor. Then the arrow $u : d \to Fr$, for $d \in \text{obj}(D)$ and $r \in \text{obj}(C)$, is <u>universal from d to F</u> when $C(r,c) \to D(d,Fc)$ given by $f \mapsto Ff \circ u$ is a bijection natural in c. That is, we have the bijection $C(r,c) \cong D(d,Fc)$ natural in c. Dually, $u : Fr \to d$ is <u>universal from F to d</u> when $C(c,r) \to D(Fc,d)$ given by $f \mapsto u \circ Ff$ is a bijection natural in c. That is we have a bijection $C(c,r) \cong D(Fc,d)$ naturally in c.

A functor $F \in [C,Set]$ is said to be <u>representable</u> when $F \cong C(c,-)$ for some $c \in \text{obj}(C)$. Many of the most important categorical constructions can be stated in terms of representable functors. For instance, F is universal from d to F means precisely that the functor $D(d,F-)$ is representable. Because of this, the Yoneda Lemma (below) is one of the most powerful tools in category theory.

Yoneda Lemma: Let $F \in [C,Set]$ and $c \in \text{obj}(C)$. Then $[C(c,-),F] \cong Fc$ given by $\phi \mapsto \phi_c(1_c)$.
Proof:

Let $\phi : C(c,-) \bullet\!\!\to F$. Then for $f \in C(c,d)$ we see that the formula $\phi_d(f) = Ff(\phi_c(1_c))$ follows since by the naturality of ϕ the following diagram commutes:



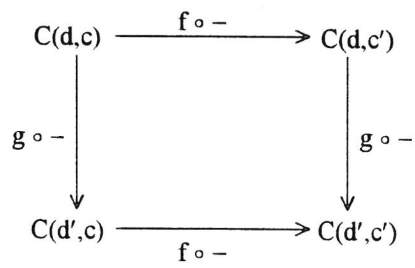
Thus, ϕ is uniquely determined by its value at 1_c in Fc.
Conversely each $x \in Fc$ determines a natural transformation $\phi : C(c,-) \to F$ by $\phi_d(f) = Ff(x)$.
ϕ so defined is natural since for $g : d \to d'$, $Fg(\phi_d(f)) = Fg(Ff(x)) = F(g \circ f)(x) = \phi_{d'}(g \circ f)$. □

The dual version of this lemma says that for $F \in [C^{op},Set]$, $[C(-,c),F] \cong Fc$. It is worth mentioning that the yoneda isomorphism given above is natural in c and F.

We define the <u>Yoneda imbedding</u> to be the functor $Y : C \to [C^{op},Set]$ given on objects by $c \mapsto C(-,c)$. This definition is justified in the following:

Proposition IC1: The Yoneda imbedding is a full and faithful functor.
Proof:

First we need to give the arrow mapping for Y and prove that it determines a functor.
For $f \in C(c,c')$, for $d \in \text{obj}(C)$ the maps $C(d,f) : C(d,c) \to C(d,c')$ defined above, that is $f \circ -$, provide the components for a natural transformation $C(-,c) \bullet\!\!\to C(-,c')$ since the following diagram commutes for each $g \in C(d,d')$:

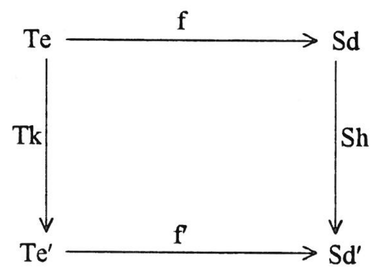


Putting $f = 1_c$ in the above we see that $C(-,1_c) = 1_{C(-,c)}$ and for $f' \in C(c',c'')$ it follows by definition that $C(-,f) \circ C(-,f') = C(-,f \circ f')$, so that Y is indeed a functor.
Let $c,d \in obj(C)$, then by the Yoneda lemma $\phi \in [C(-,c),C(-,d)]$ is determined by its value at 1_c, but the natural transformation $C(-,\phi_c(1_c))$ has this same value, that is, $\phi = C(-,\phi_c(1_c))$. That is, each $\phi \in [C(-,c),C(-,d)]$ is the same as $C(-,f)$ for a unique $f \in C(c,d)$. □

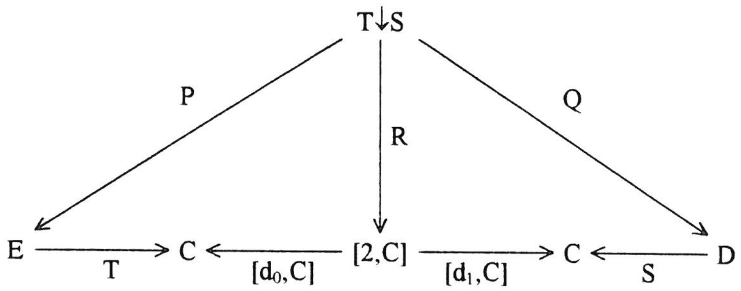
This proposition says that $C(d,c) \cong C(d',c)$ naturally in c iff $d \cong d'$.

We define the category 2 as having two objects and one non-identity arrow and the category 1 that has one object and no non-identity object (a one point set). For an arbitrary category C, [2,C] has arrows of C as objects, and commutative squares of C as arrows. Let $d_0 : 1 \to 2$ be the functor that maps the unique object of 1 to the domain of the non-identity arrow of 2, and let $d_1 : 1 \to 2$ map to the codomain. Then $[d_0,C] : [2,C] \to C$ is the induced functor that takes domains of arrows, and similarly $[d_1,C]$ takes codomains of arrows.

Let $T : E \to C \leftarrow D : S$ be functors. The <u>comma category of T over S</u>, written as $(T{\downarrow}S)$, has as objects triples <e,d,f> where $f : Te \to Sd$ and arrows as pairs <k,h> : <e,d,f> → <e′,d′,f′> where $k : e \to e'$ and $h : d \to d'$ such that the following diagram commutes:



We describe this construction by the following commutative diagram of functors:



R is just the inclusion of $T{\downarrow}S$ as a subcategory of [2,C]. P is the projection described on arrows as:

<k,h> : <e,d,f> → <e′,d′,f′> ↦ $k : e \to e'$

and Q is the projection into D defined similarly. We call P the <u>first projection</u> of the comma category $(T{\downarrow}S)$ and Q the <u>second projection</u>.

ADJUNCTIONS

Adjunctions: Let $F : C \to D$ and $G : D \to C$ be functors. An <u>adjunction</u> is a triple $\langle F, G, \varphi\rangle : C \rightharpoonup D$ where $\varphi : D(Fc,d) \cong C(c,Gd)$ natural in c and d. We write $F \dashv G$ to denote that F is the left adjoint to G.

Proposition A1: Every adjunction $\langle F, G, \varphi\rangle : C \rightharpoonup D$ determines the following:

(i) A natural transformation $\eta : 1_C \stackrel{\bullet}{\to} GF$ where $\eta_c = \varphi(1_{Fc})$ is universal from c to G.

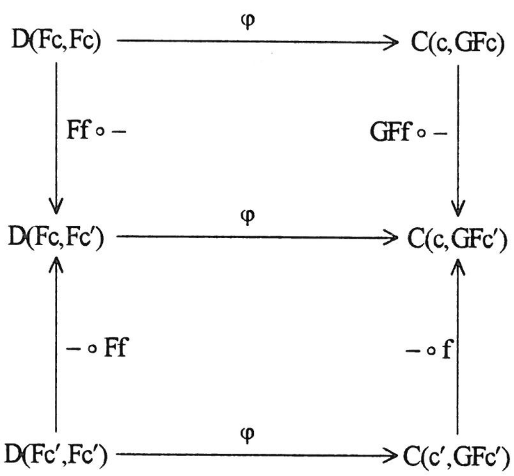
(ii) A natural transformation $\varepsilon : FG \stackrel{\bullet}{\to} 1_D$ where $\varepsilon_d = \varphi^{-1}(1_{Gd})$ is universal from F to d.

where the following laws are satisfied:

(iii) $G\varepsilon \circ \eta G = 1_G$. (iv) $\varepsilon F \circ F\eta = 1_F$.

Proof:

By definition η_c is universal from c to G iff $D(Fc,d) \cong G(c,Gd)$ naturally in d which is true by φ. η is natural iff $\forall c,c' \in obj(C)$, $\forall f \in G(c,c')$, $GFf \circ \eta_c = \eta_{c'} \circ f$. To prove this equality we require the following construction:



By the naturality of φ, the top and bottom squares are commutative. The top square takes $1_{Fc} \in D(Fc,Fc)$ to $GFf \circ \eta_c \in C(c,GFc')$. The bottom square takes $1_{Fc'}$ to $\eta_{c'} \circ f \in C(c,GFc')$. Since the images of 1_{Fc} and $1_{Fc'}$ in $D(Fc,Fc')$ agree as Ff, we see that they must both be mapped to the same element of $C(c,GFc')$. That is, $GFf \circ \eta_c = \eta_{c'} \circ f$.

The proof of (ii) is the dual of (i). Specifically, we have an isomorphism $\varphi^{op} : C^{op}(Gd,c) \cong D^{op}(d,Fc)$ natural in c and d, where for $h \in C(c,Gd)$, $\varphi^{op}(h^{op}) = \varphi^{-1}(h)$. Then $\varepsilon^{op} = \varphi^{op}(1_{Gd}{}^{op})$.

By Yoneda's lemma, the mappings φ and φ^{-1} are determined by (the components of) η and ε. Specifically, for $h \in D(Fc,d)$, $\varphi(h) = Gh \circ \eta_c$ and for $k \in C(c,Gd)$, $\varphi^{-1}(k) = \varepsilon_d \circ Fk$.

Thus, $\forall d \in obj(D)$, $G\varepsilon_d \circ \eta_{Gd} = \varphi(\varepsilon_d) = 1_{Gd}$ and $\forall c \in obj(C)$, $\varepsilon_{Fc} \circ F\eta_c = \varphi^{-1}(\eta_c) = 1_{Fc}$.

That is, $G\varepsilon \circ \eta G = 1_G$ and $\varepsilon F \circ F\eta = 1_F$. □

We call η and ε the <u>unit</u> and <u>counit</u> of the adjunction respectively, and it is the convention to write $\langle F,G,\varphi,\eta,\varepsilon\rangle$ to specify the above adjunction.

Proposition A2: Every adjunction $\langle F,G,\varphi,\eta,\varepsilon\rangle : C \rightharpoonup D$ is determined by:

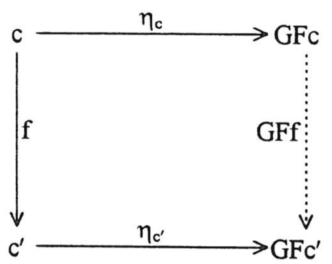
(i) The functor G, the object mapping of F, and $\forall c \in obj(C)$, $\eta_c : c \to GFc$ universal from c to G.

(ii) The functor F, the object mapping of G, and $\forall d \in obj(D)$, $\varepsilon_d : FGd \to d$ universal from F to d.

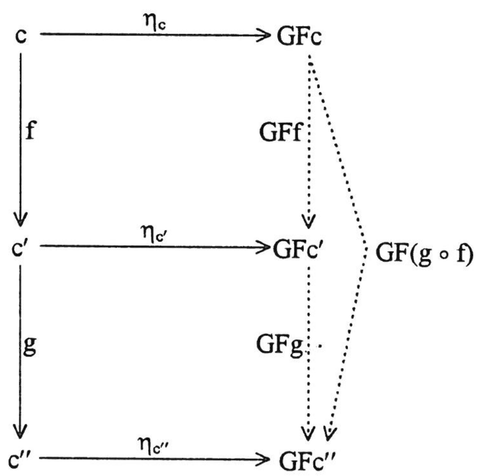
(iii) The functors F and G, the natural transformations η and ε, such that $G\varepsilon \circ \eta G = 1_G$ and $\varepsilon F \circ F\eta = 1_F$.

Proof:

Given the universality of η_c for each $c \in obj(C)$, we have isomorphisms $\varphi_c : D(Fc,d) \cong C(c,Gd)$ natural in d, given by $\varphi(h) = Gh \circ \eta_c$ for $h \in D(Fc,d)$. By the following construction for any $f \in C(c,c')$, we see that the universality of η_c uniquely defines Ff and makes η natural:



F so defined is a functor. Taking $c = c'$ and $f = 1_c$ in the above diagram we see that $F1_c = 1_{Fc}$. The universality of η_c allows us make the identification $F(g \circ f) = Fg \circ Ff$ in the following diagram, so that F preserves composition:



Finally, we need to show that φ is natural in c. For $f \in C(c',c)$ and $h \in D(Fc,d)$ this amounts to showing that $\varphi(h) \circ f = \varphi(h \circ Ff)$. But, $\varphi(h) \circ f = Gh \circ \eta_c \circ f = Gh \circ GFf \circ \eta_{c'} = \varphi(h \circ Ff)$ by the naturality of η. This proves (i).

(ii) is the dual of (i) in exactly the same way as in the previous proposition.

Given the data in (iii) we construct $\varphi : D(Fc,d) \to C(c,Gd)$ by $f \mapsto Gf \circ \eta_c$ and $\varphi' : C(c,Gd) \to D(Fc,d)$ by $g \mapsto \varepsilon_d \circ Fg$. Then $(\varphi \circ \varphi')(g) = G\varepsilon_d \circ GFg \circ \eta_c$, but $GFg \circ \eta_c = \eta_{Gd} \circ g$ since η is natural so that $(\varphi \circ \varphi')(g) = G\varepsilon_d \circ \eta_{Gd} \circ g = 1_{Gd} \circ g = g$. Similarly, $(\varphi' \circ \varphi)(f) = \varepsilon_d \circ FGf \circ F\eta_c$, but $\varepsilon_d \circ FGf = f \circ \varepsilon_{Fc}$ since ε is natural so that $(\varphi' \circ \varphi)(f) = f \circ \varepsilon_{Fc} \circ F\eta_c = f \circ 1_{Fc} = f$.

$\therefore \varphi' = \varphi^{-1} \Rightarrow \varphi$ is an isomorphism $\Rightarrow \eta_c$ is universal from c to G.

$\therefore$ by (i) φ we have determined an adjunction. □

Corollary A3: If $F \dashv G$ and $F' \dashv G$ then F and F′ are naturally isomorphic.

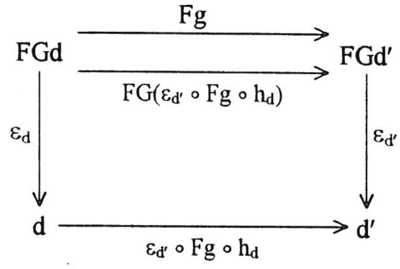
Proof:

$D(Fc,d) \cong C(c,Gd) \cong D(F'c,d)$ naturally in c and d $\Rightarrow Fc \cong F'c$ naturally in c. □

Dually, if two functors have the same left adjoint then they are naturally isomorphic.

Corollary A4: (i) G is faithful iff the components of ε are epi.
(ii) G is full iff the components of ε are split monic.
Proof:

Let $f,g \in D(d,d')$, then $\varphi(f \circ \varepsilon_d) = Gf \circ G\varepsilon_d \circ \eta_{Gd} = Gf$ so that $\varphi^{-1}(Gf) = f \circ \varepsilon_d$ and similarly for g.
$\therefore$ $Gf = Gg \Leftrightarrow f \circ \varepsilon_d = g \circ \varepsilon_d$, from which (i) follows.
Suppose that G is full and consider $d \in obj(D)$. Then $\exists\ h_d \in D(d,FGd)$ such that $\eta_{Gd} = Gh_d$.
$\therefore$ by the naturality of ε, $h_d \circ \varepsilon_d = \varepsilon_{FGd} \circ FGh_d = \varepsilon_{FGd} \circ F\eta_{Gd} = 1_{FGd} \Rightarrow \varepsilon_d$ is split monic.
Suppose that the components of ε are split monic with left inverses h_d and let $g \in C(Gd,Gd')$.



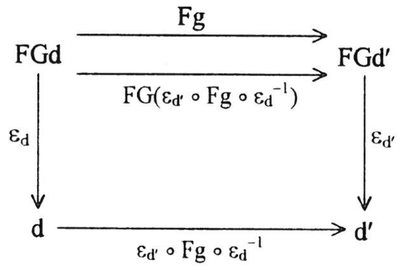
Then the outer square commutes since $\varepsilon_{d'} \circ Fg \circ h_d \circ \varepsilon_d = \varepsilon_{d'} \circ Fg$, and the inner square commutes since ε is natural. But, by the universality of $\varepsilon_{d'}$ we see that $g = G(\varepsilon_{d'} \circ Fg \circ h_d)$. □

Dually, F is faithful iff the components of η are mono, and F is full iff the components of η are split epi. In particular, G is full and faithful iff ε is an isomorphism, and F is full and faithful iff η is an isomorphism.

A functor is an <u>isomorphism of categories</u> when it has a two-sided inverse in Cat. A functor $F : C \to D$ is an <u>equivalence of categories</u> when there is a functor $G : D \to C$ and natural isomorphisms $FG \cong 1_D$ and $GF \cong 1_C$. An <u>adjoint equivalence</u> is an adjunction in which the unit and counit are natural isomorphisms.

Proposition A5: TFSAE:
(i) G is an equivalence of categories.
(ii) G is full and faithful and every $c \in obj(C)$ is isomorphic to Gd for some $d \in obj(D)$.
(iii) G is part of an adjoint equivalence $\langle F,G,\varphi,\eta,\varepsilon\rangle$.
Proof:

(i) $\Rightarrow$ (ii): (i) $\Rightarrow\ \exists\ F : D \to C$, $\eta : 1_C \cong GF$ and $\varepsilon : FG \cong 1_D$. Thus, $\forall c \in obj(C)$, $c \cong GFc$.
Suppose that $Gf = Gg$ for $f,g \in D(d,d')$.
Then by the naturality of ε, $f \circ \varepsilon_d = \varepsilon_{d'} \circ FGf = \varepsilon_{d'} \circ FGg = g \circ \varepsilon_d \Rightarrow f = g$ since ε_d is epi.
$\therefore$ G is faithful, and similarly F is faithful.
Consider $g \in C(Gd,Gd')$.



Then the outer square commutes by definition, and the inner square commutes since ε is natural. Thus, $\varepsilon_{d'} \circ FG(\varepsilon_{d'} \circ Fg \circ \varepsilon_d^{-1}) = \varepsilon_{d'} \circ Fg \Rightarrow G(\varepsilon_{d'} \circ Fg \circ \varepsilon_d^{-1}) = g$ since $\varepsilon_{d'}$ is mono and F is faithful. Hence G is full.

(ii) $\Rightarrow$ (iii): Since $\forall c \in obj(C)$, $\exists\, d_c \in obj(D)$ such that $c \cong Gd_c$ we have an object mapping $F : C \to D$ where $Fc = d_c$ and for each c an isomorphism $\eta_c : c \cong GFc$. Suppose that $g \in C(c,Gd)$, then $(g \circ \eta_c^{-1}) \circ \eta_c = g$ and since G is full and faithful $\exists!\ f \in D(Fc,d)$ such that $g \circ \eta_c^{-1} = Gf$. Thus, η_c is universal from c to G for each $c \in obj(C) \Rightarrow$ F is the object mapping for a left adjoint to G with the components of the unit of this adjunction given by η_c. Since G is full and faithful the counit of the adjunction is also an isomorphism.

(iii) $\Rightarrow$ (i): The unit and counit of the adjoint equivalence are isomorphisms. □

It is important to observe that the above definition of an equivalence of categories is symmetrical in F and G. Therefore by the above proposition, it follows that both $F \dashv G$ and $G \dashv F$. Because of the above proposition, we use the term equivalence of categories in place of adjoint equivalence in what follows. Two categories C and D are equivalent if there exists an equivalence of categories between them, and we write $C \approx D$.

Proposition A6: Every adjunction $<F,G,\varphi,\eta,\varepsilon> : C \rightharpoonup D$ restricts to an equivalence of categories $C' \approx D'$, where C′ is the full sub-category of C consisting of objects c such that η_c is an isomorphism, and D′ is the full subcategory of D consisting of objects d such that ε_d is an isomorphism.

Proof:

We need only the check that the restrictions of F and G are well-defined.

Suppose that $c \in obj(C')$. Then $\varepsilon_{Fc} \circ F\eta_c = 1_{Fc} \Rightarrow \varepsilon_{Fc} = F\eta_c^{-1}$ since η_c is an isomorphism.

$\therefore\ \varepsilon_{Fc}$ is an isomorphism $\Rightarrow Fc \in D'$.

Suppose that $d \in obj(D')$. Then $G\varepsilon_d \circ \eta_{Gd} = 1_{Gd} \Rightarrow \eta_{Gd} = G\varepsilon_d^{-1}$ since ε_d is an isomorphism.

$\therefore\ \eta_{Gd}$ is an isomorphism $\Rightarrow Gd \in C'$.

$\therefore\ F|_{C'} : C' \to D'$ and $G|_{D'} : D' \to C'$ are well-defined and so form an equivalence of categories. □

LIMITS

The Standard Definitions

Let J and C be categories. We define the diagonal functor $\Delta : C \rightarrow [J,C]$ where $\Delta c : J \rightarrow C$ is the functor that maps every object of J to c and every arrow to 1_c. The arrow mapping for Δ takes $f : c \rightarrow c'$ to the natural transformation whose components are all f.

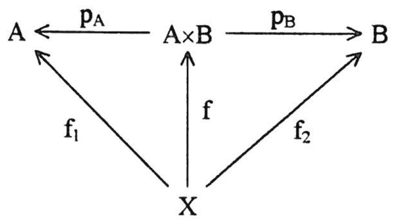
Consider $T \in [J,C]$. Then the <u>limit of T</u> consists of an object lim(T) of C along with an isomorphism

$$C(c,\lim(T)) \cong [J,C](\Delta c,T) \text{ natural in } c.$$

Thus we have a natural transformation $\tau : \Delta\lim(T) \bullet\!\!\rightarrow T$ universal from Δ to T which we call the <u>limiting cone</u> (or universal cone). We will abuse notation and write $\tau : \lim(T) \bullet\!\!\rightarrow T$. A (not necessarily universal) <u>cone</u> from c to T is a natural transformation $\Delta c \bullet\!\!\rightarrow T$ which we will usually write as $c \bullet\!\!\rightarrow T$.

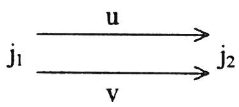
By the Yoneda lemma, any object defined by a natural isomorphism in the way that lim(T) is above is determined up to isomorphism.

When J is a discrete category consisting of two elements j_1 and j_2, $Tj_1 = A$ and $Tj_2 = B$, the limit of T is the <u>product</u> $A \times B$ and the isomorphism above amounts to saying that:

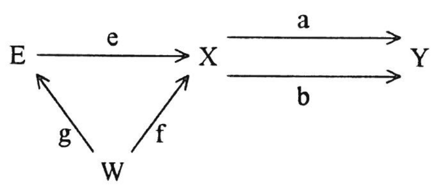


for each pair of arrows f_1 and f_2, $\exists!$ f such that the above diagram commutes where p_A and p_B are the components of the limiting cone. When C = Set we see that $A \times B \cong \{(a,b) : a \in A \text{ and } b \in B\}$ as usual.

When J is the category that has two objects and the following non-identity arrows:



and $Tj_1 = X$, $Tj_2 = Y$, $Tu = a$ and $Tv = g$, then $E = \lim(T)$ is known as an <u>equaliser</u> and the isomorphism amounts to saying that:

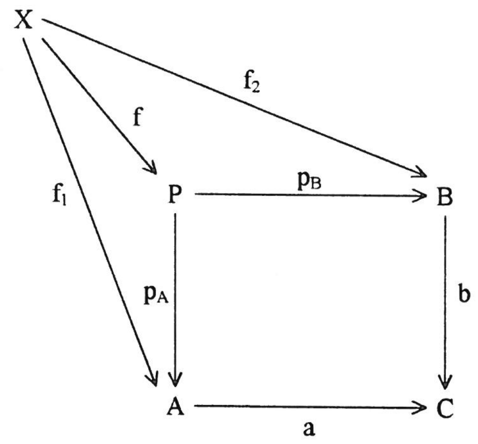


for each f such that $a \circ f = b \circ f$, $\exists!$ g making the triangle commute. Here the components of the limiting cone are e and $a \circ e = b \circ e$. When C = Set, $Z \cong \{x \in X : a(x) = b(x)\}$.

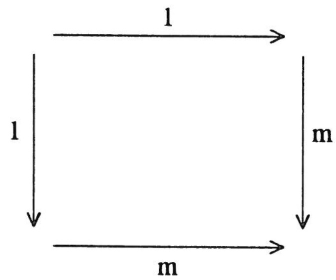
When J is the category that has three objects and the following non-identity arrows:

$$j_1 \xrightarrow{\;u\;} j_3 \xleftarrow{\;v\;} j_2$$

and $Tj_1 = A$, $Tj_2 = B$, $Tj_3 = C$, $Tu = a$ and $Tv = b$, then $P = \lim(T)$ is known as a <u>pullback</u> and the isomorphism amounts to saying that:



for each (f_1,f_2) where $a \circ f_1 = b \circ f_2$, $\exists!\, f$ making the triangles commute. The components of the limiting cone are p_A, p_B and $a \circ p_A = b \circ p_B$. The notation $A \times_C B$ for P is commonly used.
When $C = \text{Set}$, $P \cong \{(x,y) \in A \times B : a(x) = b(y)\}$. Furthermore if m is any arrow of C it follows that m is monic precisely when the following square is a pullback square:



Finally we remark that when J is empty lim(T) is a terminal object of C.

Limits need not always exist. We say that the category C is <u>complete</u> when it has all small limits. That is for any small category J, and any functor $T : J \to C$, the limit of T exists in C. We say that C is <u>finitely complete</u> when it has all finite limits.

It can be shown that if C has all set indexed products (all limits where J is a set) and all equalisers, then it is complete. Similarly if C has finite products and equalisers then it is finitely complete.

The above discussion can be dualised. The <u>colimit of T</u> consists of an object colim(T) of C along with an isomorphism:

$$C(\text{colim}(T),c) \cong [J,C](T,\Delta c) \text{ natural in } c.$$

and we have a <u>limiting cocone</u> (or universal cocone) $\tau : T \bullet\!\!\to \text{colim}(T)$. We can define coproducts (which are disjoint unions in Set), coequalisers and pushouts in the same fashion. The property of epicness can be expressed as a pushout. The empty coproduct is an initial object. If C has all set indexed (finite) coproducts and all coequalisers, then it is (finitely) cocomplete.

Finally we observe that given a functor $T : J \to C$, the universal cocone $T \bullet\!\!\to colim(T)$ is jointly epi, dually the universal cone $lim(T) \bullet\!\!\to T$ is jointly monic. This just amounts to the universality of the (co)cone.

Proposition L1: Limits exist for all $T \in [J,C]$ iff Δ has a right adjoint. Moreover, if we let $\psi : S \bullet\!\!\to T$ for $S, T \in [J,C]$, and σ and τ denote the universal cones of S and T, then $\psi \circ \sigma = \tau \circ \Delta lim(\psi)$.
Proof:

Suppose limits exist for all $T \in [J,C]$. Then we have the functor Δ, the object mapping lim which takes $T \mapsto lim(T)$, and the universal cone $lim(T) \bullet\!\!\to T$ for each $T \in [J,C]$. Thus by proposition A2(i) we have determined an arrow mapping for $lim : [J,C] \to C$ and the adjunction $\Delta \dashv lim$.
Suppose Δ has right adjoint G. Then by definition it follows that $\forall T \in [J,C]$, $lim(T) = GT$.
The second sentence follows from the universality of τ. □

Let $T \in [J,C]$ and $F \in [C,D]$. Then F creates limits for T when:

(i) Each cone $\tau : c \bullet\!\!\to T$ such that $F\tau : Fc \bullet\!\!\to FT$ is a universal cone, is itself universal.

(ii) For each universal cone $\psi : d \bullet\!\!\to FT$, there is a universal cone $\tau : c \bullet\!\!\to T$ where $F\tau = \psi$.

When F satisfies (i) above we say that it reflects limits for T. We say that F preserves the limits of T when each universal cone $\tau : c \bullet\!\!\to T$ yields a universal cone $F\tau$. If F preserves all small limits that exist in C, then we say that F is continuous. It can be shown that if F creates limits for T and FT has a limit, then F preserves the limit of T.

Weighted Limits and Colimits

Consider the equivalence relation on obj(C) generated by $a \sim b$ if $\exists f : a \to b$. Then the connected components of C is the class of equivalence classes under this relation, and is denoted by $\pi_0(C)$.

Proposition L2: Consider $\Delta 1 : J \to Set$. Then $\pi_0(J) \cong colim(\Delta 1)$.
Proof:

Consider any cocone $\psi : \Delta 1 \bullet\!\!\to x$. Then each component of ψ distinguishes an element of x. Suppose $f \in J(a,b)$, then the naturality of ψ dictates that ψ_a and ψ_b distinguish the same elements of x. That is, ψ can be considered as a function $obj(J) \to x$ that respects the equivalence classes of $\pi_0(J)$. Thus, $\tau : \Delta 1 \bullet\!\!\to \pi_0(J)$ where $\tau_a(0)$ = equivalence class of a in $\pi_0(J)$, must be universal. □

Consider $S \in [J,Set]$, then el(S), the category of elements of S, is defined as follows:

$obj(el(T)) = \{(a, x) : a \in obj(J), x \in Ta\}$.

arrows are given by $f_x : (a,x) \to (b,y)$ where $f \in J(a,b)$ and $Tf(x) = y$.

We have a projection functor $P(S) : el(S) \to J$ given by $f_x \mapsto f$.

Consider $S \in [J,Set]$ and $T \in [J,C]$. Then the limit of T weighted by S consists of an object lim(S,T) of C along with an isomorphism $C(c,lim(S,T)) \cong [J,Set](S,C(c,T))$ natural in c.

Part of the usefulness of weighted limits springs from the fact that they are a more expressive general concept than ordinary limits although they can be constructed in terms of them. We will be more precise about this below, and then use this fact to prove a general statement about colimits of set valued functors.

Lemma L3: $[J,C](\Delta c,T) \cong [J,Set](\Delta 1,C(c,T))$ naturally in c.
Proof:

$[J,C](\Delta c,T) \cong \{(c \to Ta)_{a\in obj(J)} : \text{natural in } a\} \cong \{(1 \to C(c,Ta))_{a\in obj(J)} : \text{natural in } a\}$
$\cong [J,Set](\Delta 1, C(c,T))$. It is easy to check that these bijections are natural in c. □

Lemma L4: $[el(S),Set](\Delta 1,C(c,TP(S)) \cong [J,Set](S,C(c,T))$ naturally in c.
Proof:

$$[el(S),Set](\Delta 1,C(c,TP(S)) \cong \{(1 \to C(c,TP(S)(a,\alpha))_{a\in obj(J),\alpha\in Sa} : \text{natural in } a\}$$
$$\cong \{(1 \to C(c,Ta))_{a\in obj(J),\alpha\in Sa} : \text{natural in } a\} \cong \{(Sa \to C(c,Ta))_{a\in obj(J)} : \text{natural in } a\}$$
$$\cong [J,Set](S,C(c,T)) \text{ naturally in } c.$$
□

Proposition L5: (Ordinary limits are weighted limits) $\lim(T) \cong \lim(\Delta 1,T)$.
Proof:

$C(c,\lim(T)) \cong [J,C](\Delta c,T) \cong [J,Set](\Delta 1,C(c,T)) \cong C(c,\lim(\Delta 1,T))$ naturally in c. □

Proposition L6: (Weighted limits are ordinary limits) $\lim(S,T) \cong \lim(TP(S))$.
Proof:

$C(c,\lim(S,T)) \cong [J,Set](S,C(c,T)) \cong [el(S),Set](\Delta 1,C(c,TP(S)) \cong C(c,\lim(\Delta 1,TP(S)))$ naturally in c, so that $\lim(S,T) \cong \lim(\Delta 1,TP(S)) \cong \lim(TP(S))$. □

Let $S \in [J^{op},Set]$ and $T \in [J,C]$. Then the <u>colimit of T weighted by S</u> is an object colim(S,T) of C and an isomorphism $C(colim(S,T),c) \cong [J^{op},Set](S,C(T,c))$ natural in c.

Now, $C(\lim(S,T^{op}),c) \cong C^{op}(c,\lim(S,T^{op})) \cong [J^{op},Set](S,C^{op}(c,T^{op}) \cong [J^{op},Set](S,C(T,c)) \cong C(colim(S,T),c)$ naturally in c, so that $colim(S,T) \cong \lim(S,T^{op})$. Thus the above results for weighted limits can be translated into results for weighted colimits. That is, $colim(\Delta 1,T) \cong colim(T)$ and $colim(S,T) \cong colim(T(P(S))^{op})$. Next we characterise colimits of set-valued functors.

Proposition L7: Let $S \in [J^{op},Set]$ and $T \in [J,Set]$. Then $colim(S,T) \cong colim(T,S)$.
Proof:

$Set(colim(S,T),x)) \cong [J^{op},Set](S,Set(T,x)) \cong [J^{op},Set](T,Set(S,x)) \cong Set(colim(T,S),x)$ naturally in x, so that $colim(S,T) \cong colim(T,S)$. □

Proposition L8: Let $T \in [J,Set]$. Then $colim(T) \cong \pi_0(el(T))$.
Proof:

$$colim(T) \cong colim(\Delta 1,T) \cong colim(T,\Delta 1) \cong colim((\Delta 1)(P(S)))$$
$$\cong colim(\Delta 1 : el(T) \to Set) \cong \pi_0(el(T)).$$
□

We can think of a cocone $\tau : T \bullet\!\!\to c$ as being a function $obj(el(T)) \to c$ given by $(t,x) \mapsto \tau_t(x)$. Denote the equivalence relation that gives rise to $\pi_0(J)$ as ~. Then a direct corollary of L8, and the fact that universal cocones are jointly epic is the following:

Corollary L9: Let $\tau : T \bullet\!\!\to c$ be a cocone as in the preceding paragraph. Then τ is universal iff it satisfies the following:

(i) The components of τ are jointly surjective.
(ii) The kernel relation of the function $\tau : obj(el(T)) \to c$ is precisely ~.

We do in fact require a special case of this corollary in our section on topological sheaves.

Corollary L10: Let J be a directed set and $T \in [J,Set]$. A cocone $\tau : T \bullet\!\!\to c$ is universal iff it satisfies the following properties:

(i) The cocone is jointly surjective.
(ii) If $t, s \in obj(J)$ and $x \in Tt$ and $y \in Ts$ where $\tau_t(x) = \tau_s(y)$, then $\exists\ f : t \to u$ and $g : s \to u$ such that $Tf(x) = Tg(y)$.

Proof:

Since J is a preorder $(t,x) \sim (s,y)$ in el(T) iff $\exists\ f : t \to u$ and $g : s \to u$ such that $Tf(x) = Tf(y)$. The result follows by the above corollary. □

A nice characterisation for weighted limits of set-valued functors, and an easy proof of the continuity of representable functors springs from the following:

Lemma L11: For S,T ∈ [J,Set], Set(x,[J,Set](S,T)) ≅ [J,Set](S,Set(x,T)).
Proof:

Set(x,[J,Set](S,T)) ≅ $\{(Sa \to Ta)_{a\in obj(J),\, y\in x}$: natural in a$\}$ ≅ $\{(x \times Sa \to Ta)_{a\in obj(J)}$: natural in a$\}$
≅ $\{(Sa \to Set(x,Ta))_{a\in obj(J)}$: natural in a$\}$ ≅ [J,Set](S,Set(x,T))
and these isomorphisms are natural in x. □

Corollary L12: Let S,T ∈ [J,Set]. Then lim(S,T) ≅ [J,Set](S,T).
Proof:

Set(x,[J,Set](S,T)) ≅[J,Set](S,Set(x,T)) ≅ Set(x,lim(S,T)) naturally in x. □

Consider S ∈ [J,Set], T ∈ [J,C] and F ∈ [C,D]. Then F preserves limits for T weighted by S when F(lim(S,T)) satisfies the defining property of lim(S,FT). If F preserves all limits (which means by L6 that is preserves all weighted limits) then we say that F is continuous. Dually, F is cocontinuous when it preserves all (weighted) colimits. We say that F is left exact when it preserves all finite limits. Dually, F is right exact when it preserves all finite colimits.

Corollary L13: Representable functors preserve weighted limits.
Proof:

Suppose S ∈ [J,Set], T ∈ [J,C] and that lim(S,T) exists so that C(c,lim(S,T)) ≅ [J,Set](S,C(c,T)).
Consider the representable functor C(x,–) ∈ [C,Set].
Then, Set(X, C(x,lim(S,T)) ≅ Set(X, [J,Set](S,C(x,T))) ≅ [J,Set](S, Set(X,C(x,T)) naturally in X.
That is, C(x,lim(S,T)) = lim(S,C(x,T)). □

We now investigate weighted limits and colimits for functor categories.

Proposition L14: Suppose that C has limits weighted by S ∈ [J,Set] and that K is some other category. Then [K,C] also has limits weighted by S and they are formed pointwise.
Proof:

Suppose that T : J → [K,C]. Then by the standard adjunction [J × K,C] ≅ [J,[K,C]] natural in J and C, we can consider T : J × K → C. Thus, ∀k ∈ obj(K) we have a functor T(–,k) : J → C.
In fact we have a functor K → [J,C] where k ↦ T(–,k). From L1 we know that ordinary limits are functorial. Thus by L6, lim(S,T(–,k)) ≅ lim(T(–,k) ∘ P(S)) must be functorial in k. We denote this functor as lim(S,T(–,–)).
Now, [K,C](τ,lim(S,T(–,–)) = {τ •→ lim(S,T(–,–))}
≅ $\{(\tau(k) \to lim(S,T(-,k)))_{k\in obj(K)}$ natural in k$\}$
≅ $\{(S \bullet\!\!\to C(\tau(k),T(-,k)))_{k\in obj(K)}$ natural in k$\}$
≅ $\{(Sj \to C(\tau(k),T(j,k)))_{j\in obj(J),\, k\in obj(K)}$ natural in j and k$\}$
≅ $\{(Sj \to [K,C](\tau,T(j,-)))_{j\in obj(J)}$ natural in j$\}$
≅ {S •→ [K,C](τ,T(–,–))} = [J,Set](S,[K,C](τ,T(–,–))).
since ∀k ∈ obj(K), C(c,lim(S,T(–,k))) ≅ [J,Set](S,C(c,T(–,–))) naturally in c.
That is, we have proved that lim(S,T(–,–)) = lim(S,T). □

Since Set is complete and cocomplete, a direct corollary of the above result (and its dual) is that [C,Set] is complete and cocomplete for any category C. Recalling the Yoneda imbedding Y : C → [C^{op},Set], we have a further corollary:

Corollary L15: The Yoneda imbedding preserves limits.
Proof:

Consider $T : J \to C$ such that $\lim(T)$ exists.
For any $c \in obj(C)$, L13 says that $C(c, \lim_J Tj)$ satisifies the defining property of $\lim_J C(c, Tj)$.
Thus by L14 it follows that $C(-, \lim_J Tj)$ satisifies the defining property of $\lim_J C(-, Tj)$.
But, $C(-, \lim_J Tj)$ is just $Y(\lim(T))$ whereas $\lim_J C(-, Tj)$ is just $\lim(YT)$. □

Weighted limits are preserved by adjoint functors. Specifically we have the following:

Proposition L16: Suppose $\langle F,G,\varphi,\eta,\varepsilon\rangle : C \rightharpoonup D$. Then G preserves weighted limits and F preserves weighted colimits.
Proof:

Let $S \in [J^{op},Set]$ and $T \in [J,C]$ and suppose colim(S,T) exists.
Then, $D(F(colim(S,T)),d) \cong C(colim(S,T),Gd) \cong [J^{op},Set](S,C(T,Gd)) \cong [J^{op},Set](S,D(FT,d))$ naturally in d, so that indeed $F(colim(S,T)) = colim(S,FT)$.
By duality, G preserves weighted limits. □

Proposition L17: Suppose $\langle F,G,\varphi,\eta,\varepsilon\rangle : C \rightharpoonup D$, G is full and faithful, and C is cocomplete. Then D is cocomplete.
Proof:

Let $T : J \to D$. Then since G is full and faithful, $[J,C](GT,\Delta Gd) \cong [J,D](T,\Delta d)$.
$\therefore D(F(colim(GT)),d) \cong C(colim(GT),Gd) \cong [J,C](GT,\Delta Gd) \cong [J,D](T,\Delta d)$.
That is, F(colim(GT) satisfies the defining property of colim(T).
Since C is cocomplete, colim(GT) always exists $\Rightarrow$ colim(T) always exists. □

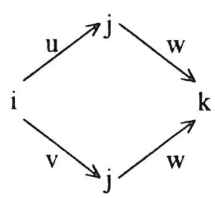
Dually if F is full and faithful and D is complete, then C is complete. Taking these together we see that if $C \approx D$ and C is (co)complete, then D is (co)complete.

Let $T : P \to [J,C]$ be a functor and C be complete. By L14 T has a limit $\lim_P(T)$ which is formed pointwise. Since C is complete we can evaluate $\lim_J(\lim_P(T))$ or "reverse the process" by taking $\lim_P(\lim_J \circ T)$ where $\lim_J : [J,C] \to C$ is the limit functor considered in L1. The question here is whether these alternative limits satsify each others' defining properties (does it matter in which order we take limits?). Equivalently, we are asking whether $\lim_J$ preserves limits. This question is easily answered since by L1, $\lim_J$ has a left adjoint, and so by L16 preserves limits. Dually, colimit functors preserve colimits. It follows easily by L6 that the same results hold for weighted limits and colimits. That is, functors of the form lim(S,–) and colim(S,–) preserve (weighted) limits and colimits respectively.

We may now ask whether limit functors preserve colimits and vice versa. In general this is not true however it is true in certain cases.

A category J is <u>filtered</u> when the following conditions are satisfied:

(i) For any $j, j' \in obj(J)$, there is $k \in J$ and arrows $j \to k$ and $j' \to k$.
(ii) For arrows $u,v : i \to j$, there is an arrow $w : j \to k$ such that the following diagram commutes:



For example a directed set is a filtered category (arrows are given by the preorder relation).

Suppose that J has all finite colimits. Then the existence of coproducts is enough to ensure that J satisfies (i). The existence of pushouts is enough to ensure that condition (ii) is satsified. Thus if J has finite colimits, then it is filtered. We state the following standard result without proof:

Proposition L18: Let J be a small filtered category. Then $\mathrm{colim}_J : [J,\mathrm{Set}] \to \mathrm{Set}$ is left exact.

Recall the Yoneda imbedding Y again. We have the following useful consequence of the Yoneda lemma:

Proposition L19: Let $P \in [C^{op},\mathrm{Set}]$. Then $P \cong \mathrm{colim}(P,Y)$.
Proof:

By the definition of weighted colimit $[C^{op},\mathrm{Set}](\mathrm{colim}(P,Y),Q) \cong [C^{op},\mathrm{Set}](P,[C^{op},\mathrm{Set}](Y-,Q))$ naturally in Q. However, by Yoneda's lemma $[C^{op},\mathrm{Set}](Y-,Q) \cong Q$ naturally in Q.
$\therefore$ $[C^{op},\mathrm{Set}](\mathrm{colim}(P,Y),Q) \cong [C^{op},\mathrm{Set}](P,Q)$ naturally in Q.
$\therefore$ by Yoneda's Lemma (again) we have $\mathrm{colim}(P,Y) \cong P$. □

Let $F \in [J,C]$ be a functor for which all colimits exist. It is of interest to consider object-mappings of the form $\mathrm{colim}(-,F) : [J^{op},\mathrm{Set}] \to C$. To this end we have the following result:

Proposition L20: For $F \in [J,C]$ the $\mathrm{colim}(-,F)$ is the object mapping of a functor that has a right adjoint.
Proof:

For $P \in [J^{op},\mathrm{Set}]$ we note that $D(\mathrm{colim}(P,F),d) \cong [J^{op},\mathrm{Set}](P,D(F-,d))$ naturally in d.
Define the functor $D(F-,-) : D \to [J^{op},\mathrm{Set}]$ on arrows as $f \mapsto D(F-,f)$ as in the proof of IC1.
Thus, we have determined a functor $D(F-,-)$, an object mapping $\mathrm{colim}(-,F)$ and $\forall P \in [J^{op},\mathrm{Set}]$ an arrow $\eta_P : P \to D(F-,\mathrm{colim}(P,F))$ universal from P to $D(F-,-)$.
$\therefore$ A2(i) determines the arrow mapping for $\mathrm{colim}(-,F)$ and that $\mathrm{colim}(-,F) \dashv D(F-,-)$. □

This of course means that $\mathrm{colim}(-,F)$ preserves weighted colimits. Dually, the functor $\lim(-,F)$ preserves weighted limits.

COMMA CATEGORIES AND KAN EXTENSIONS

Comma Categories

Now we study comma categories of the form $(1_C\downarrow\Delta x)$ and $(\Delta x\downarrow 1_C)$ for $x \in obj(C)$. We denote these categories as $C\downarrow x$ and $x\downarrow C$ respectively.

Proposition CK1: If C is (finite/small) complete then $C\downarrow x$ is (finite/small) complete.
Proof:

Suppose that C is (finite/small) complete. Let $T : J \to C\downarrow x$ be a functor where J is a (finite/small) category. Let J_t be the category J with a terminal object adjoined to it. That is, $obj(J_t) = obj(J) \cup \{t\}$ and $arr(J_t) = arr(J) \cup \{j \to t : j \in obj(J)\}$. Then J_t is (finite/small) whenever J is, and T can be viewed as a functor $T_t : J_t \to C$ where $\alpha \mapsto TP(\alpha)$ when α was an object or arrow of J (and P is the first projection of the comma category $C\downarrow x$), $t \mapsto x$ and $(j \to t) \mapsto T(j)$. By hypothesis, $lim(T_t)$ exists, and by composition along any component of the universal cone to x, we obtain an arrow $lim(T_t) \to x$ that is the vertex of a cone to T in $C\downarrow x$. The universality of this cone follows directly from the universality of $lim(T_t) \bullet\!\!\to T_t$ in C. □

The dual of this statement says that C is (finite/small) cocomplete $\Rightarrow$ $x\downarrow C$ is (finite/small) cocomplete. By much the same reasoning as in the above proof, we see that $P : C\downarrow x \to C$ creates equalisers and pullbacks, but that products in $C\downarrow x$ correspond to certain pullbacks in C.

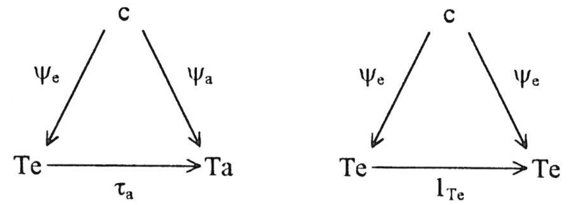
An interesting aspect of comma categories is that they can be used to express various categorical concepts in terms of each other. First we relate initial objects to limits by noting that for $G \in [D,C]$, an arrow u is universal from c to G iff u is initial in $c\downarrow G$, and dually that u is universal from G to c iff u is terminal in $G\downarrow c$.

Lemma CK2: $G \in [D,C]$ has a left adjoint iff $\forall c \in obj(C)$, $c\downarrow G$ has an initial object.
Proof:

$\forall c \in obj(C)$, $c\downarrow G$ has an initial object $\Leftrightarrow$ $\forall c \in obj(C)$, $\exists$ a universal arrow $c \to G(Fc)$ for $Fc \in obj(D)$ $\Leftrightarrow$ F is the object mapping of a left adjoint to G. □

Lemma CK3: Consider $T \in [J,C]$. J has an initial object e $\Rightarrow$ T has limit Te.
Proof:

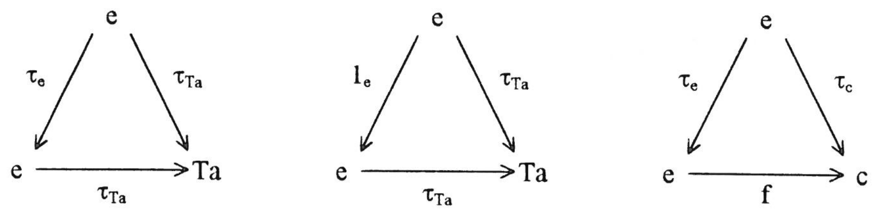
Let e be the initial object of J and consider the cone $\tau : Te \bullet\!\!\to T$ where $\tau_a = Ta_e$ where $a_e : e \to a$ is the unique such arrow. Suppose $\psi : c \bullet\!\!\to T$ is another cone. Then, $\forall a \in obj(J)$



the diagram on the left shows that $\tau \circ \psi_e = \psi$, and the diagram on the right shows that ψ_e is the unique such arrow $c \to Te$, so that τ is a limiting cone. □

Lemma CK3: If $\tau : e \bullet\!\!\rightarrow 1_C$ is a cone and $\tau T : e \bullet\!\!\rightarrow T$ is a limiting cone for $T \in [J,C]$ then e is an initial object in C.
Proof:

For $a \in obj(J)$ and $f \in arr(C)$ the triangles:



commute since τ is natural. Since τT is universal, the first two diagrams prove that $\tau_e = 1_e$, from which it follows by the third diagram that $f = \tau_c$ making e initial. □

In particular, if $J = C$, $T = 1_C$ and $\lim(1_C)$ exists, then $\lim(1_C)$ is initial in C. That is, $\lim(1_C)$ exists iff C has an initial object. Next we obtain a limit criterion for the existence of adjoints.

Lemma CK4: If $G \in [D,C]$ preserves limits then the projection $Q : c{\downarrow}G \rightarrow D$ creates limits.
Proof:

(i) Consider the functor $T : J \rightarrow c{\downarrow}G$ and the cone $\tau : g \bullet\!\!\rightarrow T$ where $Q\tau : Qg \bullet\!\!\rightarrow QT$ is a limiting cone. Then since G preserves limits, $GQ\tau : GQg \bullet\!\!\rightarrow GQT$ is a limiting cone. Consider any other cone $\psi : h \bullet\!\!\rightarrow T$. Then $GQ\psi$ is also a cone and $\exists!$ arrow $f : GQh \rightarrow GQg$ where $GQ\tau \circ f = GQ\psi$. Observe that by the definition of G and Q, $\forall k \in obj(c{\downarrow}G)$ we see that $k : c \rightarrow GQk$. Furthermore, we identify the components τ_a with $GQ\tau_a$ for $a \in obj(J)$.
Thus, $\forall a \in obj(J)$, $\tau_a \circ f = \psi_a$ and the components τ_a are jointly monic (since $GQ\tau_a$ are).
But, $\forall a \in obj(J)$, $Ta = \psi_a \circ h = \tau_a \circ g$ since τ and ψ are cones, so that $\tau_a \circ f \circ h = \tau_a \circ g$.
Thus, $f \circ h = g$ making f the unique arrow in $c{\downarrow}G$ where $\tau \circ f = \psi \Rightarrow \tau$ is a limiting cone.
(ii) Consider the limiting cone $\sigma : w \bullet\!\!\rightarrow QT$. Since G preserves limits, $G\sigma$ is also a limiting cone, but by definition the arrows $q_a : c \rightarrow GQTa$ are components of a cone $q : c \bullet\!\!\rightarrow GQT$ so that $\exists!\ u : c \rightarrow Gw$ such that $q = G\sigma \circ u$. But this says that we have a cone $\tau : u \bullet\!\!\rightarrow T$ given on components by $\tau_a = G\sigma_a : u \rightarrow Ta$ where $Q\tau = \sigma$. □

Proposition CK5: $G \in [D,C]$ has a left adjoint iff:
(i) G preserves all limits that exist in D.
(ii) $\forall c \in obj(C)$, $\lim(Q : c{\downarrow}G \rightarrow D)$ exists in D.
Proof:

Suppose that G has a left adjoint. Then G preserves limits by L16. Furthermore, since $c{\downarrow}G$ has an initial object, we see that $Q : c{\downarrow}G \rightarrow D$ must have a limit.
Suppose G satisfies (i) and (ii). $\forall c \in obj(C)$, by (ii) the composite $Q \circ 1_{c\downarrow G}$ has a limit, and by (i) Q creates limits, so that $1_{c\downarrow G}$ must have a limit $\Rightarrow c{\downarrow}G$ has an initial object. □

We can also express the representability of set-valued functors and the existence of universal arrows in similar terms:

Proposition CK6: $K \in [C,Set]$ is representable iff:
(i) K preserves all limits that exist in C.
(ii) The projection $Q : 1{\downarrow}K \rightarrow C$ has a limit in C.
Proof:

Suppose K is representable. Then K preserves limits by L13. Let $K = C(r,-)$ for $r \in obj(C)$. Then the Yoneda lemma says that 1_r is initial in $1{\downarrow}K \Rightarrow Q$ has limit r.

Suppose that K satisfies (i) and (ii). We use the same argument as in the previous proposition to prove that $1_{1\downarrow K}$ has an initial object. We identify this with $e \in Kr$ for $r \in obj(C)$. Then we have a natural isomorphism $K \cong C(r,-)$ determined by $e \mapsto 1_r$ by the Yoneda lemma. □

Corollary CK7: For $G \in [D,C]$ and $c \in obj(C)$, there is a universal arrow from c to G iff:

(i) $C(c,G-)$ preserves limits.

(ii) $\lim(Q : c\downarrow G \to D)$ exists in D.

Proof:

There is a universal arrow from c to G iff $C(c,G-) \cong D(r,-)$ for $r \in obj(D)$, that is iff $C(c,G-)$ is representable. Furthermore, there is a natural identification between $1\downarrow C(c,G-)$ and $c\downarrow G$. In this way this corollary reduces to the previous proposition. □

Kan Extensions

Consider $K \in [M,C]$ and $T \in [M,A]$.
Consider also the functor $[K,A] : [C,A] \to [M,A]$ given by $[\sigma : S \bullet\!\!\to S'] \mapsto [\sigma K : SK \bullet\!\!\to S'K]$.

A <u>right kan extension of T along K</u> is a functor $Ran_K T \in [C,A]$ along with an isomorphism:

$$[C,A](S,Ran_k T) \cong [M,A](SK,T) \qquad \text{natural in S.}$$

This isomorphism determines and is determined by a natural transformation $\varepsilon : (Ran_k T)K \bullet\!\!\to T$ universal from $[K,A]$ to T, known as the <u>counit</u> of the extension.

Dually, a <u>left kan extension of T along K</u> is a functor $Lan_K T \in [C,A]$ along with an isomorphism:

$$[C,A](Lan_k T,S) \cong [M,A](T,SK) \qquad \text{natural in S.}$$

This isomorprism determines and is determined by a natural transformation $\eta : T \bullet\!\!\to (Lan_k T)K$ universal from T to $[K,A]$, known as the <u>unit</u> of the extension.

By definition, each $T \in [M,A]$ has a (left) right kan extension iff $[K,A]$ has a (left) right adjoint (which justifies the use of the terms unit and counit in the above).

Now we give the construction of the right kan extension as a "pointwise" limit:

Proposition CK8: Adopting the above notation, let $Q_c : c\downarrow K \to M$ denote the second projection for $c \in obj(C)$, and suppose $\forall c \in obj(C)$ that TQ_c has a limit. Then the object mapping $Rc = \lim(TQ_c)$ determines the functor $R \in [C,A]$ that is a right kan extension of T along K, with counit $\varepsilon : RK \bullet\!\!\to T$ as $\varepsilon_m = (\tau_{Km})_1$ where 1 denotes the arrow $1_{Km} \in Km\downarrow K$ and $\tau_c : Rc \bullet\!\!\to TQ_c$ is the limiting cone.

Proof:

First we prove that the object mapping of R yields a functor that commutes with the cones τ_c:

Let $g : c \to d \in arr(C)$. Then g induces a functor $g\downarrow K : d\downarrow K \to c\downarrow K$ where $h \mapsto h \circ g$, so that by definition $Q_c \circ g\downarrow K = Q_d$. That is, we can regard the image of $g\downarrow K$ as a subcategory of $c\downarrow K$ on which the projections Q_c and Q_d agree. Thus τ_c can be restricted to a cone $Rc \bullet\!\!\to TQ_d$ so that $\exists!$ arrow $Rg : Rc \to Rd$ by the universality of τ_d.

Note also that $\tau_d \circ Rg = \tau_c$ so that the R commutes with the limiting cones τ_c.

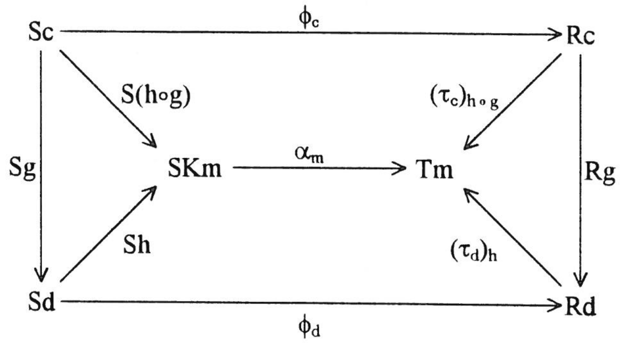
Now, the mapping $C \to Cat$ given by $g \mapsto g\downarrow K$ is a functor and taking restrictions of τ_c as above is also functorial, so that the arrow mapping described here is functorial.

Next we verify that R is indeed a right kan extension with the stated counit:

We wish to prove that $[C,A](S,R) \to [M,A](SK,T)$ given by $\phi \mapsto \varepsilon \circ \phi K$ is an isomorphism. So for $\alpha : SK \bullet\!\!\to T$ we will show $\exists!\ \phi : S \bullet\!\!\to R$ with $\varepsilon \circ \phi K = \alpha$.

First we construct a cone $\sigma_c : Sc \bullet\!\!\to TQ_c$. For $f : c \to Km$ in $c\downarrow K$, let $(\sigma_c)_f = \alpha_m \circ Sf$. This is natural in f since α is natural in m. Since τ_c is a limiting cone, $\exists!\ \phi_c : Sc \to Rc$

such that $(\tau_c)_f \circ \phi_c = (\sigma_c)_f = \alpha_m \circ Sf$. To prove ϕ is natural, let $g \in C(c,d)$, $h : d \to Km$ in $d\!\downarrow\!K$ so that:



the two triangles and the two inner-most quadrilaterals are all commutative. A diagram chase thus reveals that $(\tau_d)_h \circ Rg \circ \phi_c = (\tau_d)_h \circ \phi_d \circ Sg$, and since this equation holds for all $h \in d\!\downarrow\!K$, and the components of τ_c are jointly monic, it follows that $Rg \circ \phi_c = \phi_d \circ Sg$ so that $\phi : S \bullet\!\!\to R$. Setting $d = Km$ and $h = 1_{Km}$ the bottom inner quadrilateral yields the relation $\varepsilon_m \circ \phi_{Km} = \alpha_m$ so that indeed, $\alpha = \varepsilon \circ \phi K$. We have proved the existence of ϕ. Suppose now that we are given $\phi : S \bullet\!\!\to R$ where $\alpha = \varepsilon \circ \phi K$. Let $f : c \to Km$ in $c\!\downarrow\!K$, then let $d = Km$, $h = 1_{Km}$ and $g = f$ in the above diagram. Then $\alpha = \varepsilon \circ \phi K$ means that the bottom inner quadraliteral commutes, the two triangles commute by definition and the outer rectangle commutes since ϕ is natural. A diagram chase reveals that the upper quadrilateral commutes which means that $(\tau_c)_f \circ \phi_c = \alpha_m \circ Sf = (\sigma_c)_f$, which by the universality of τ_c makes ϕ_c unique. □

The dual result yields the formula $Lc = \mathrm{colim}(TP_c)$ for the pointwise construction of the left kan extension of T along K, where $P_c : K\!\downarrow\!c \to M$ is the first projection. The unit is $\eta_m = (\tau_{Km})_1$ where $\tau_c : TP_c \bullet\!\!\to c$ is the limiting cocone. We have a number of corollaries from the above proposition:

Corollary CK9: If M is small and A is complete, then [K,A] has a right adjoint.
Proof:

Since M is small and $K \in [M,C]$ it follows that $c\!\downarrow\!K$ is small $\forall c \in \mathrm{obj}(C)$. Thus, $\forall T \in [M,C]$ and $c \in \mathrm{obj}(C)$ the functors TP_c have limits because of the completeness of A.
Thus, every $T \in [M,C]$ has a right kan extension along K that is constructed pointwise as in the above proposition. Hence we have a functor $[K,A] : [K,C] \to [K,M]$, along with an object mapping $\mathrm{Ran}_K(-) : [K,M] \to [K,C]$, and for each $T \in [K,M]$ an arrow $\varepsilon_T : \mathrm{Ran}_K(T)K \bullet\!\!\to T$ universal from [K,A] to T. By proposition A2(ii) we see that $\mathrm{Ran}_K(-)$ is the object mapping for a functor that is a right adjoint to [K,A]. □

Dually, if M is small and A is cocomplete, then [K,A] has a left adjoint whose object mapping is $\mathrm{Lan}_K(-)$.

Corollary CK10: If the hypotheses of the proposition are satisfied and K is full and faithful, then the counit of the right kan extension is an isomorphism.
Proof:

For each $m \in \mathrm{obj}(M)$ we evaluate $RKm = \lim(TQ_{Km})$. Since K is full, every $g \in C(Km,Km')$ is of the form $g = Kf$ for $f \in M(m,m')$. Thus, $\forall g \in \mathrm{obj}(Km\!\downarrow\!K)$ we see that $g = Kf : 1_{Km} \to g$. Suppose that $f' \in Km\!\downarrow\!K(1_{Km},g)$, then $Kf' \circ 1_{Km} = Kf' = Kf$, but since K is faithful, $f = f'$ which shows that $Km\!\downarrow\!K(1_{Km},g) = \{f\}$. That is, $Km\!\downarrow\!K$ has initial object 1_{Km} so that we evaluate $\lim(TQ_{Km})$ as $TQ_{Km}(1_{Km}) = Tm$ with $(\tau_{Km})_f = Tf$ so that $\varepsilon_m = 1_{TKm}$. □

Dually, if the hypotheses to the dual of the proposition are satisfied and K is full and faithful, then the unit of the left kan extension is an isomorphism. Letting K be a subcategory and $K : M \hookrightarrow C$ in the above corollary, given $T : M \to A$ we obtain genuine extensions of T to C as right and left kan extensions of T along K.

Continuing to adopt the above notation we say that $G \in [A,X]$ preserves the right kan extension of T along K when $G(\mathrm{Ran}_K T)$ satisfies the defining property of $\mathrm{Ran}_k GT$.

Proposition CK11: Right adjoints preserve right kan extensions.
Proof:

Assume that $\mathrm{Ran}_K T$ exists and that $G \in [A,X]$ has left adjoint $F \in [X,A]$.

Then, $[C,X](S,G(\mathrm{Ran}_K T)) = \{(Sc \to G(\mathrm{Ran}_K T)c)_{c\in obj(C)} : \text{natural in } c\}$
$\cong \{(FSc \to (\mathrm{Ran}_K T)c)_{c\in obj(C)} : \text{natural in } c\}$ since $F \dashv G$
$= [C,A](FS,\mathrm{Ran}_k T) \cong [M,A](FSK,T)$ by the property of $\mathrm{Ran}_K T$
$= \{(FSKm \to Tm)_{m\in obj(M)} : \text{natural in } m\}$
$\cong \{(SKm \to GTm)_{m\in obj(M)} : \text{natural in } m\}$ since $F \dashv G$
$= [M,X](SK,GT)$ and each step is natural in S. □

For a category A, $r \in obj(A)$ and a set X, the copower of a by X is an object $\coprod_{x\in X} a$. We often write $X \bullet a$ to denote this object, and it is clear that $-\bullet a : Set \to A$ given on objects by $X \mapsto X \bullet a$ determines a functor.

Corollary CK12: In the above notation, if A has all copowers and is locally small then representable functors preserve right kan extensions.
Proof:

Consider the representable $A(a,-) : A \to Set$ and define $\mathrm{Ran}_K T$ as above.
Observe that $A(X\bullet a,r) = A(\coprod_{x\in X} a, r) \cong \prod_{x\in X} A(a,r) \cong Set(X,A(a,r))$ naturally in X and r so that $-\bullet a \dashv A(a,-)$ so that $A(a,-)$ preserves right kan extensions. □

A kan extension is pointwise if it is preserved by all representable functors. The following proposition gives the reason for this name:

Proposition CK13: T has a pointwise right kan extension along K iff the limit of TQ_c exists $\forall c \in obj(C)$.
Proof:

Suppose that $\lim(TQ_c)$ exists $\forall c \in obj(C)$. Then $Rc = \lim(TQ_c)$ defines a right kan extension to T along K, and since representables preserve limits, we see that R must be pointwise.
Suppose that we have a pointwise $\mathrm{Ran}_K T \in [C,A]$. Pointwise means that:

$$[C,Set](S, A(a,(\mathrm{Ran}_k T)-)) \cong [M,Set](SK, A(a,T-)) \quad \text{naturally in } S$$

and substituting $C(c,-)$ in this equation gives us:

$$[C,Set](C(c,-), A(a,(\mathrm{Ran}_k T)-)) \cong [M,Set](C(c,K-), A(a,T-)) \quad \text{naturally in } c$$

but the left hand side of this is isomorphic to $A(a,(\mathrm{Ran}_K T)c)$ naturally in c by yoneda's lemma.
Now, $[M,Set](C(c,K-), A(a,T-)) = \{(C(c,Km) \to A(a,Tm))_{m\in obj(M)} : \text{natural in } m\}$
$\cong \{(a \to Tm)_{f\in obj(c\downarrow K)} : \text{natural in } f\} = [c\downarrow K,A](\Delta a,TQ_c)$
and this bijection is natural in c. Thus, $A(a,(\mathrm{Ran}_K T)c) \cong [c\downarrow K,A](\Delta a,TQ_c)$ so that $\mathrm{Ran}_k T$ satisfies the property of $\lim(TQ_c)$. □

We are now in a position to express the limits, weighted limits and adjunctions in terms of certain kan extensions.

Proposition CK14: (limits and kan extensions) Let $T \in [J,C]$ and let $! \in [J,1]$ be the unique such functor. Then lim(T) can be identified with $Ran_!T$.
Proof:

By definition, $[1,C](c,Ran_!T) \cong [J,C](c!,T)$ naturally in c. But [1,C] can be identified with C and c and $Ran_!T$ as elements of C, and c! is the same as Δc. □

Dually, colim(T) can be identified with $Lan_!T$.

Proposition CK15: (adjoints and kan extensions) $G \in [D,C]$ has a left adjoint iff it preserves all limits and 1_D has a pointwise right kan extension along G. In this case the left adjoint is given by $Ran_G 1_D$.
Proof:

We saw in CK5 that G has a left adjoint iff it preserves limits and $lim(Q_c : c\downarrow G \to D)$ exists $\forall c$. In fact by the proof of CK2 the left adjoint is given by $Fc = lim(Q_c)$. The result follows since $Ran_G 1_D(c) = lim(Q_c)$ by CK8. □

Dually, the right adjoint for $F \in [C,D]$ is given by $Lan_F 1_C$.

Proposition CK16: (weighted limits and kan extensions) Let $S \in [J,Set]$ and $T \in [J,C]$. Then if a pointwise right kan extension exists for T along S, then $Ran_ST(1)$ satisfies the defining property of lim(S,T) where 1 denotes the one point set. Conversely, given $K \in [M,C]$ and $T \in [M,A]$, T has a pointwise right kan extension iff $lim(C(c,K-),T)$ exists $\forall c \in obj(C)$, with $Ran_KT(c) = lim(C(c,K-),T)$.
Proof:

First suppose $S \in [J,Set]$ and $T \in [J,C]$ and that a right kan extension exists for T along S. Then by L7, lim(S,T) = lim(TP(S)). However el(S) can be identified with $1\downarrow S$ and P(S) is just the second projection Q_1. That is, $lim(S,T) = lim(TQ_1) = Ran_ST(1)$.
Suppose $K \in [M,C]$ and $T \in [M,A]$. We can identify $1\downarrow C(c,K-)$ with $c\downarrow K$ and their second projections. Thus, we have the following calculation:

$$lim(C(c,K-),T) = lim(1\downarrow C(c,K-) \to J \to C) = lim(c\downarrow K \to J \to C) = Ran_KT(c).$$ □

Dually, the formula for the pointwise left kan extension is given in terms of weighted colimits as:

$$Lan_KT(c) = colim(C(K-,c),T).$$

Let $K : M \to C$ and $P_c : K\downarrow c \to M$ be the first projection. We say that K is <u>dense in C</u> whenever $colim(TP_c) = c$ with colimiting cone made up of all the arrows of $K\downarrow c$. Dually, K is <u>codense in C</u> when $lim(TQ_c) = c$ with limiting cone made up of the arrows of $c\downarrow K$. A subcategory D of C is <u>dense</u> in C when the inclusion $D \hookrightarrow C$ is dense in C. This is important because it means that every element of C is a colimt of elements from D, so that every colimit preserving functor on C is determined by its values on D. By the dual of CK13, K is dense in C iff 1_C is a pointwise left kan extension of K along K.

For example, the Yoneda imbedding is dense in $[C^{op},Set]$ since $Lan_YY(P) = colim([C^{op},Set](Y-,P),Y)$, but by the Yoneda lemma, $[C^{op},Set](Y-,P) \cong P$ and by L19 $colim(P,Y) \cong P$, so that $Lan_YY(P) \cong P$. That is, $1_{[Cop,Set]}$ is a pointwise left kan extension of Y along itself. Therefore every $P \in [C^{op},Set]$ is a colimit of representables.

In what follows we denote the full subcategory of $[[C^{op},Set],D]$ that contains all the colimit preserving (ie cocontinuous) functors as $CoCts[[C^{op},Set],D]$.

Proposition CK17: Let D be cocomplete. Then $[C,D] \approx CoCts[[C^{op},Set],D]$.
Proof:

Since D is cocomplete, $Lan_Y \dashv [Y,D]$ by the dual of CK9. Since Y is full and faithful, $\forall T \in [C,D]$, η_T is an isomorphism, so that Lan_Y is full and faithful and that $1_{[C,D]} \cong [Y,D] \circ Lan_Y$.

We claim that for $F \in [C,D]$, $Lan_Y(F) \cong colim(-,F)$:

Since D is cocomplete and by the dual of CK16, $Lan_Y(F)(P) = colim([C^{op},Set](Y-,P),F)$.

But by the Yoneda lemma, $[C^{op},Set](Y-,P) \cong P$ naturally in P.

$\therefore \forall P \in [C^{op},Set]$, $Lan_Y(F)(P) = colim(P,F)$ naturally in P $\Rightarrow$ $Lan_Y(F) \cong colim(-,F)$.

Recall from L20 that $colim(-,F)$ has a right adjoint and so is cocontinuous.

$\therefore Lan_Y$ restricts to a full and faithful functor $[C,D] \to CoCts[[C^{op},Set],D]$.

To complete the proof we need to prove that for $G \in CoCts[[C^{op},Set],D]$, $G(P) \cong colim(P,GY)$.

But, L19 says that $colim(P,Y) \cong P$ for $P \in [C^{op},Set]$.

Since G preserves weighted colimits, $G(P) \cong G(colim(P,Y)) \cong colim(P,GY)$ naturally in P. □

SHEAVES ON TOPOLOGICAL SPACES

In this section we consider an arbitrary topological space X, and denote its category of open sets (objects are open sets of X, arrows are inclusions) as $\Theta(X)$.

A <u>presheaf</u> of sets on X is a functor $P : \Theta(X)^{op} \to Set$. Elements of P(U) are called <u>sections</u> of P over U. For $x \in X$, consider the full subcategory N(x) of $\Theta(X)$ consisting of the open sets that contain x. Then the <u>stalk</u> of P at x is defined as $P_x = colim(P|_{N(x)})$. For U an neighbourhood of x, and $s \in PU$, we will often write $s_x = \tau_U(s)$ where τ is the universal cocone. The elements of P_x are called <u>germs</u>. We say that x is the <u>base point</u> of the germ $y \in P_x$.

Since colim can be viewed as a functor (dual of L1), for each $x \in X$ we obtain a functor $[\Theta(X)^{op},Set] \to Set$ where $[\psi : P \bullet\!\!\to Q] \mapsto [\psi_x : P_x \to Q_x]$.

Proposition S1: A cocone $\tau : P|_{N(x)} \bullet\!\!\to c$ is universal iff it satisfies the following properties:

(i) The cocone is jointly surjective.

(ii) Suppose U, $V \in N(x)$ and $t \in PU$ and $s \in PV$ where $\tau_U(t) = \tau_V(s)$. Then $\exists\ W \in N(x)$ such that $W \subseteq U \cap V$ and $P(W \subseteq U)(t) = P(W \subseteq V)(s)$.

Proof:

This is a direct translation of L10. □

A presheaf P is called a <u>sheaf</u> when it satisfies the following conditions for any open $U \subseteq X$ and open covering $\cup_i U_i = U$:

(i) The maps $P(U_i \subseteq U) : P(U) \to P(U_i)$ are jointly injective (monopresheaf property).

(ii) Suppose $\forall i$, $s_i \in P(U_i)$ such that $\forall i,j$, $P(U_i \cap U_j \subseteq U_i)(s_i) = P(U_i \cap U_j \subseteq U_j)(s_j)$, then $\exists\ s \in P(U)$ such that $\forall i$, $P(U_i \subseteq U)(s) = s_i$ (glueing condition).

We now give an important example of a sheaf. Firstly let $\iota_X : \Theta(X)^{op} \hookrightarrow Top$ be the inclusion functor and let Z be another topological space. Consider the functor $Top(\iota_X -,Z) : \Theta(X)^{op} \to Set$. Consider U open in X, and suppose $U = \cup_{i\in I} U_i$. Consider f, $g \in Top(U,Z)$. Then if $f|_{U(i)} = g|_{U(i)}$ $\forall i \in I$, then clearly f = g so that the monopresheaf property holds. Suppose that we have an I-indexed family $\{f_i \in Top(U_i,Z) : i \in I\}$ such that $\forall i,j \in I$, $f_i|_{U(i)\cap U(j)} = f_j|_{U(i)\cap U(j)}$. Then we can define the function $f : U \to X$ as $f(x) = f_i(x)$ where $x \in U_i$. Let Y be open in Z and let $y \in f^{-1}(Y) \subseteq U$. Then, $\exists\ U_i$ such that $y \in U_i$ and so $y \in f_i^{-1}(Y) \subseteq f^{-1}(Y)$, and since f_i is continuous, $f_i^{-1}(Y)$ is an open neighbourhood of y. Thus, $f^{-1}(Y)$ is open so that f is continuous. That is, the family $\{f_i \in Top(U_i,Z) : i \in I\}$ determines a unique $f \in Top(U,Z)$ and this is precisely the glueing property. Thus $Top(\iota_X -,Z)$ is indeed a sheaf.

Continuing with this example, let $x \in X$ and suppose that f, $g : U \to Z$ where $U \in N(x)$. We say that f and g belong to the same germ when they agree on some open subset of U. By S1 if follows that this notion of germ corresponds to the general idea defined with presheaves above.

We now present a categorical definition of sheaves.

Proposition S2: A presheaf P is a sheaf iff for any open $U \subseteq X$ and open covering $\cup_{i\in I} U_i = U$, e is an equaliser in the following diagram:

$$P(U) \xrightarrow{\;e\;} \Pi_{i\in I} P(U_i) \underset{b}{\overset{a}{\rightrightarrows}} \Pi_{i,j\in I} P(U_i \cap U_j)$$

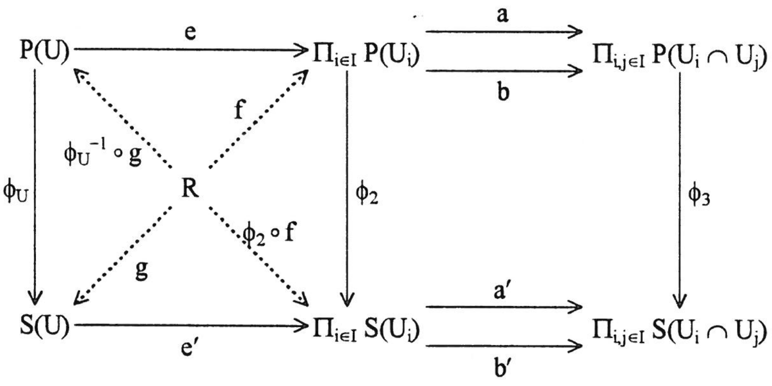
where $(a(u_i))_{i,j\in I} = P(U_i \cap U_j \subseteq U_i)(u_i)$, $(b(u_j))_{i,j\in I} = P(U_i \cap U_j \subseteq U_j)(u_j)$, and $(e(u))_{i\in I} = P(U_i \subseteq U)(u)$.
Proof:

Since e is an equaliser in Set, e is injective and P(U) can be identified with $\{(u_i) \in \Pi_i P(U_i) : a(u_i) = b(u_i)\}$. But the P is a sheaf equivalent to saying that P(U) can be identified with (monopresheaf property) the set of $(s_i) \in \Pi_i P(U_i)$ that satisfy $a(s_i) = b(s_i)$ (glueing condition). □

We can now define Sh(X) to be the full subcategory of $[\Theta(X)^{op},\text{Set}]$ that contains all of the sheaves on X.

Lemma S3: Suppose $P \in [\Theta(X)^{op},\text{Set}]$ and $S \in Sh(X)$ and $P \cong S$. Then $P \in Sh(X)$.
Proof:

Suppose $\phi : P \cong S$. Let $U \in \Theta(X)$ and $U = \cup_{i\in I} U_i$ for $U_i \in \Theta(X)$. Consider the diagram:



where:
$(a(p_i))_{i,j\in I} = P(U_i \cap U_j \subseteq U_i)(p_i)$, $(b(p_j))_{i,j\in I} = P(U_i \cap U_j \subseteq U_j)(p_j)$, and $(e(p))_{i\in I} = P(U_i \subseteq U)(p)$, and similarly:
$(a'(s_i))_{i,j\in I} = S(U_i \cap U_j \subseteq U_i)(s_i)$, $(b'(s_j))_{i,j\in I} = S(U_i \cap U_j \subseteq U_j)(s_j)$, and $(e'(s))_{i\in I} = S(U_i \subseteq U)(s)$, and finally $(\phi_2)_{i\in I} = \phi_{U(i)}$ and $(\phi_3)_{i,j\in I} = \phi_{U(i)\cap U(j)}$.
It follows by the naturality of ϕ that $\phi_2 \circ e = e' \circ \phi_U$, $\phi_3 \circ a = a' \circ \phi_2$ and $\phi_3 \circ b = b' \circ \phi_2$.
Suppose that $a \circ f = b \circ f$. Then $a' \circ (\phi_2 \circ f) = \phi_3 \circ (a \circ f) = \phi_3 \circ (b \circ f) = b' \circ (\phi_3 \circ f)$.
Since e' is an equaliser (since S is a sheaf), $\exists!g$ such that $e' \circ g = \phi_2 \circ f$.
$\therefore$ $\phi_U^{-1} \circ g$ is the unique arrow $R \to P(U)$ such that $e \circ (\phi_U^{-1} \circ g) = f \Rightarrow$ e is an equaliser. □

A <u>bundle</u> over X is an object of the comma category Top↓X, that is, a continuous function into X. A bundle is <u>etale</u> if it is a local homeomorphism. Let Et(X) be the full subcategory of Top↓X consisting of the etale bundles.

Lemma S4: Suppose $p \in$ Top↓X and $q \in Et(X)$ and $p \cong q$. Then $p \in Et(X)$.
Proof:

Let $p : Y \to X$ and $q : Z \to X$. Since $p \cong q$, $\exists$ a homeomorphism $\phi : Z \to Y$ where $p \circ \phi = q$.
Consider any $y \in Y$. Since q is etale, $\exists\ U \in N(\phi^{-1}(y))$ such that $U \cong q(U)$.
$\therefore$ $y \in \phi(U) \cong U \cong q(U) = p(\phi(U))$. □

Define the functor $I : \Theta(X)^{op} \to Top\downarrow X$ which takes open sets to their inclusions, that is, $U \mapsto U \hookrightarrow X$. The arrow mapping of I corresponds to restrictions of these inclusions.

We now construct a functor $\Gamma : Top\downarrow X \to [\Theta(X)^{op},Set]$ as $\Gamma(p) = Top\downarrow X(I-,p)$. For $p \in Top\downarrow X$ and $U \in \Theta(X)$, it follows that $\Gamma(p)(U)$ can be identified with the set $\{s : ps = i : U \hookrightarrow X\}$. The arrow mapping of Γ is defined by the dual of the Yoneda imbedding.

It follows that $\Gamma(p)$ is in fact a sheaf. Letting $p : Y \to X$, we can identify $Top\downarrow X(I-,p)$ as a subfunctor of $Top(\iota_X-,Y)$ which is itself a sheaf. Thus, the monopresheaf property must hold automatically for $\Gamma(p)$. As for the glueing property, let $\cup_{i\in I} U_i = U$, consider a family $(f_i)_{i\in I}$ and define f, all in the same way as in the example. The only difference here is that $\forall i$, $p \circ f_i : U \hookrightarrow X$ by definition. Clearly, f is continuous by the same argument as before. We need only verify that $p \circ f : U \hookrightarrow X$. Let $x \in U$, then $\exists\ U_i$ such that $x \in U_i$ so that $p(f(x)) = p(f_i(x)) = x$. Thus $\Gamma(p)$ is indeed a sheaf $\forall p \in Top\downarrow X$.

We shall now construct an object mapping $L : [\Theta(X)^{op},Set] \to Top\downarrow X$. Let $P \in [\Theta(X)^{op},Set]$, then we define $LP : \coprod_{x\in X} P_x \to X$ as the mapping that takes each germ to its base point. In order that LP be a bundle we require a topology on $\coprod_{x\in X} P_x$ that makes LP continuous. To this end, for each open U in X and $s \in PU$, define a function $s_U{}^\wedge : U \to \coprod_{x\in X} P_x$ by $x \mapsto s_x$.

Consider $\mathcal{B} = \{s_U{}^\wedge(U) : U$ open in X and $s \in PU\}$. Suppose that $r_x \in s_U{}^\wedge(U) \cap t_V{}^\wedge(V)$, then $s_x = t_x = r_x$ and $U, V \in N(x)$. Then by S1, $\exists\ W \in N(x)$ such that $W \subseteq U \cap V$ and $P(W \subseteq U)(s) = P(W \subseteq V)(t) = r \in PW$. Now let $w \in W$. Since the limiting cocone $P|_{N(w)} \bullet\!\!\to P_w$ is natural, it follows that $s_w = t_w = r_w$. That is, we have proved that $r_x \in r_W{}^\wedge(W) \subseteq s_U{}^\wedge(U) \cap t_V{}^\wedge(V)$, so that $\mathcal{B}$ is a base for some topology on $\coprod_{x\in X} P_x$.

We assign this topology to $\coprod_{x\in X} P_x$. For U open in X, $(LP)^{-1}(U) = \cup\ \{s_V{}^\wedge(V) : s \in PV$ with $V \subseteq U$ open$\}$ is open, so that LP is continuous. In fact on $s_U{}^\wedge(U)$, $s_U{}^\wedge$ is a continuous inverse of LP, making LP etale.

Proposition S5: L given above is the object mapping for a functor that is a left adjoint to Γ.
Proof:

Let $P \in [\Theta(X)^{op},Set]$ and for $U \in \Theta(X)$ define $(\eta_P)_U : PU \to \Gamma LPU$ where $s \mapsto s_U{}^\wedge$. Then the $(\eta_P)_U$ are the components for a natural transformation $\eta_P : P \bullet\!\!\to \Gamma LP$:

Let $V \subseteq U$, $s \in PU$ and $t = P(V \subseteq U)(s) \in PV$. Consider any $v \in V$.
By the naturality of the limiting cocone $P|_{N(v)} \bullet\!\!\to P_v$, it follows that $s_v = t_v$.
That is, $\forall v \in V$, $s_U{}^\wedge(v) = t_V{}^\wedge(v) \Rightarrow s_U{}^\wedge|_V = t_V \Rightarrow \eta_P$ is natural.

Suppose we are given any $\phi : P \bullet\!\!\to \Gamma p$ where $p \in Top\downarrow X$. Let $dom(p) = Y$.
Define $g : \coprod_{x\in X} P_x \to Y$ as $s_x \mapsto \phi_U(s)(x)$ where $s \in PU$ so that $U \in N(x)$. We claim that g is well-defined as a function:

Suppose $s \in PU$ and $t \in PV$ where $s_x = t_x$.
Then by S1, $\exists\ W \in N(x)$ where $W \subseteq U \cap V$ and $P(W \subseteq U)(s) = P(W \subseteq V)(t) = r \in PW$.
By the naturality of ϕ, it follows that $\phi_W(r) = \phi_U(s)|_W = \phi_V(t)|_W$.
$\therefore\ \phi_U(s)(x) = \phi_W(r)(x) = \phi_V(t)(x)$ so that g is indeed well-defined.

We claim that $p \circ g = LP$:

Consider any $s \in PU$ where $U \in N(x)$. Since $\phi_U(s) \in \Gamma p(U)$, we have $p \circ \phi_U(s) : U \hookrightarrow X$.
$\therefore\ g(s_x) = \phi_U(s)(x) \in p^{-1}\{x\} \Rightarrow p \circ g = LP$.

We claim that g is a continuous function:

Let G be open in Y. Then $g^{-1}(G) = \{s_x : \phi_U(s)(x) \in G\} = \{s_x : x \in (\phi_U(s))^{-1}(G)\}$.
Since $\phi_U(s)$ is continuous, $(\phi_U(s))^{-1}(G)$ is open in X so that $g^{-1}(G) \in \mathcal{B}$.

$\therefore$ g can be identified with $\gamma : LP \to p$ in $Top\downarrow X$.
Now, $\forall s \in PU$ and $x \in U$, $(\Gamma\gamma \circ \eta_P)_U(s)(x) = (g \circ s_U{}^\wedge)(x) = g(s_x) = \phi_U(s)(x) \Rightarrow \Gamma\gamma \circ \eta_P = \phi$.
Suppose that $\gamma' : LP \to p$ such that $\Gamma\gamma' \circ \eta_P = \phi$. Then γ' can be identified with $g' : \coprod_{x\in X} P_x \to Y$ where $\forall s \in PU$ and $x \in U$, $g'(s_x) = (g' \circ s_U{}^\wedge)(x) = (\Gamma\gamma' \circ \eta_P)_U(s)(x) = \phi_U(s)(x)$.

$\therefore\ g'(s_x) = g(s_x) \Rightarrow$ g is uniquely determined $\Rightarrow \eta_P$ is universal from P to Γ.
$\therefore$ by A2(i) we see that L extends to a functor such that $L \dashv \Gamma$. □

Lemma S6: If P is a sheaf, U is open in X and $s, s' \in PU$, then $s = s' \Leftrightarrow \forall x \in U,\ s_x = s'_x$.
Proof:

$\Rightarrow$ follows by definition.
Suppose $\forall x \in U,\ s_x = s'_x$. Then $\forall x \in U,\ \exists\ V_x \in N(x)$ where $V_x \subseteq U$, such that $P(V_x \subseteq U)(s) = P(V_x \subseteq U)(s')$. But the V_x form an open cover of U, so that by the monopresheaf property, the maps $P(V_x \subseteq U)$ are jointly injective. Thus, $s = s'$. □

Proposition S7: For $P \in [\Theta(X)^{op}, Set]$, $P \in Sh(X) \Leftrightarrow \eta_P$ (defined in the proof of S5) is an isomorphism.
Proof:

($\Leftarrow$) follows directly from S3 since ΓLP is a sheaf.
Suppose that P is a sheaf and consider $U \in \Theta(X)$.
For $s, t \in PU$, $(\eta_P)_U(s) = (\eta_P)_U(t) \Rightarrow s_U{}^\wedge = t_U{}^\wedge \Rightarrow \forall x \in U,\ s_x = t_x \Rightarrow s = t$ by S6.
$\therefore (\eta_P)_U$ is injective.
Let $f : U \to \coprod_{x\in X} P_x$ be a continuous function where $LP \circ f : U \hookrightarrow X$.
To prove that $(\eta_P)_U$ is surjective we need to prove that $\exists\ s \in PU$ such that $f = s_U{}^\wedge$.
Now, let $t \in PV$ for some $V \subseteq U$, then $f^{-1}(t_V(V)) = \{x \in V : f(x) = t_x\}$ is open in X since f is continuous.
That is, for each $V \subseteq U$ and $t \in PV$ we have an open set $G(V, t) = \{x \in V : f(x) = t_x\} \subseteq U$.
Now, for $x \in U$, $f(x) = s(x)_x$ for some $s(x) \in PU_x$ and $U_x \subseteq U$, so that $x \in U_x$.
$\therefore x \in G(U_x, s(x)) \subseteq U$, so that these sets (for $x \in U$) form an open cover of U.
For $x, y \in X$, $G(U_x, s(x)) \cap G(U_y, s(y)) = \{z \in U_x \cap U_y : f(z) = s(x)_z = s(y)_z\}$.
$\therefore \forall z \in U_x \cap U_y,\ s(x)_z = s(y)_z$.
$\therefore$ by S6, $P(G(U_x, s(x)) \cap G(U_y, s(y)) \subseteq G(U_x, s(x)))(s(x))$
$= P(G(U_x, s(x)) \cap G(U_y, s(y)) \subseteq G(U_x, s(y)))(s(y))$.
$\therefore$ by the glueing condition, $\exists\ s \in PU$ such that $\forall x \in U$ such that $P(G(U_x, s(x)) \subseteq U)(s) = s(x)$.
$\therefore \forall x \in U,\ f(x) = s(x)_x = s_x = s_U{}^\wedge(x) \Rightarrow f = s_U{}^\wedge$. □

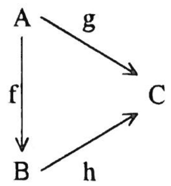
For $p \in Top{\downarrow}X$, say $p : Y \to X$, we claim that the maps $\varepsilon_p : L\Gamma p \to p$ which can be identified with continuous functions $\varepsilon_p : \coprod_{x\in X} (\Gamma p)_x \to Y$ given by $f_x \mapsto f(x)$ are the components of the natural transformation $\varepsilon : L\Gamma \bullet\!\!\to 1_{Top\downarrow X}$ that is the counit of the adjunction $L \dashv \Gamma$. Every element in $\coprod_{x\in X} (\Gamma p)_x$ is of the form f_x for $f \in \Gamma p(U)$ and $U \in N(X)$ since the limiting cocones $\Gamma p|_{N(x)} \bullet\!\!\to (\Gamma p)_x$ are jointly surjective. Suppose that for $U, V \in N(x)$, we have $f \in \Gamma p(U)$ and $g \in \Gamma p(V)$ where $f_x = g_x$. Then the functions f and g agree on some neighbourhood of x so that $f(x) = g(x)$. That is, ε_p is well-defined as a function $\coprod_{x\in X} (\Gamma p)_x \to Y$. Let W be open in Y. Then

$$\varepsilon_p^{-1}(W) = \{f_x \in \textstyle\coprod_{x\in X} (\Gamma p)_x : f(x) \in W\} = \cup_f \{f_x : x \in f^{-1}(W)\}$$

where the union is over all $f \in \Gamma p(U)$ for $U \in \Theta(X)$. But, since $f^{-1}(W)$ is open it follows that $\varepsilon_p^{-1}(W)$ is open so that ε_p is continuous as a function $\coprod_{x\in X} (\Gamma p)_x \to Y$. Taking $f \in \Gamma p(U)$, $f(x) \in p^{-1}\{x\} \Rightarrow \varepsilon_p(f_x) \in p^{-1}\{x\} \Rightarrow p \circ \varepsilon_p = L\Gamma p$ so that ε_p is an arrow $L\Gamma p \to p$ of $Top{\downarrow}X$.

It remains to be shown that ε is the stated counit. To this end we will prove that $\Gamma\varepsilon_p \circ \eta_{\Gamma p} = 1_{\Gamma p}$ so that the result follows by the universality of $\eta_{\Gamma p}$. Let $U \in \Theta(X)$. By definition $(\Gamma\varepsilon_p)_U : Top{\downarrow}X(IU, L\Gamma p) \to Top{\downarrow}X(IU, p)$ is given by $f \mapsto \varepsilon_p \circ f$. Thus, $(\Gamma\varepsilon_p)_U \circ (\eta_{\Gamma p})_U(s) = \varepsilon_p \circ s_U{}^\wedge$ for $s \in \Gamma p(U)$. However, for $x \in U$, $(\varepsilon_p \circ s_U{}^\wedge)(x) = \varepsilon_p(s_x) = s(x)$, so that $(\Gamma\varepsilon_p)_U \circ (\eta_{\Gamma p})_U = (1_{\Gamma p})_U$.

Lemma S8: If g and h are local homeomorphisms, and f is a continuous function that makes the triangle:



commute, then f is a local homeomorphism.
Proof:

Let $a \in A$. Since g is a local homeomorphism, $\exists\ V \in N(a)$ such that $V \cong g(V)$.
Since h is a local homeomorphism, $\exists\ W \in N(f(a))$ such that $W \cong h(W)$.
But, $g(a) = h(f(a))$ so that $g(V) \cap h(W) \neq \varnothing$ and open, where $a \in V$ and $f(a) \in W$.
$\therefore$ we can write $U \cong g(U)$ and $g(U) \cong h(g(U))$ where $g(U) = h(g(U)) = g(V) \cap h(W)$.
$\therefore\ U \cong h^{-1}(g(U))$ so that by the commutativity of the triangle, $f : U \cong f(U)$. □

Proposition S9: For $p \in Top\downarrow X$, $p \in Et(X) \Leftrightarrow \varepsilon_p$ is an isomorphism.
Proof:

($\Leftarrow$) follows directly from S4 since $L\Gamma p$ is etale.
Let p be etale. We need to show that $\varepsilon_p : \coprod_{x\in X} (\Gamma p)_x \to Y$ is a homeomorphism.
Suppose that $f_x, g_y \in \coprod_{x\in X} (\Gamma p)_x$ and that $f(x) = g(y)$, for $f \in \Gamma p(U)$, $g \in \Gamma p(V)$, $U \in N(x)$ and $V \in N(y)$. By definition, $p^{-1}\{x\} \ni f(x) = g(y) \in p^{-1}\{y\} \Rightarrow x = y$.
Consider now the restrictions $f, g \in \Gamma(p)(U \cap V)$. Since p and $(U \cap V) \hookrightarrow X$ are local homeomorphisms, f and g are local homeomorphisms by S8. So, we can assume that $\exists\ W \in N(x)$ such that $p(f(W)) \cong f(W) \cong W \cong g(W)$. But $\forall w \in W$, $p(f(w)) = p(g(w)) = w \Rightarrow f|_W = g|_W$.
$\therefore$ f and g agree on some neighbourhood of x $\Rightarrow f_x = g_x$. That is, ε_p is injective.
Suppose that $y \in Y$. Then since p is etale $\exists\ W \in N(y)$ such that $W \cong p(W)$. Let f be the inverse of p on W considered as $f : p(W) \to X$. Then, $f(p(y)) = y$ so that $\varepsilon_p(f_{p(y)}) = y$. That is, ε_p is surjective.
Finally, p and $L\Gamma p$ are local homeomorphisms $\Rightarrow \varepsilon_p$ is a local homeomorphism by S8. □

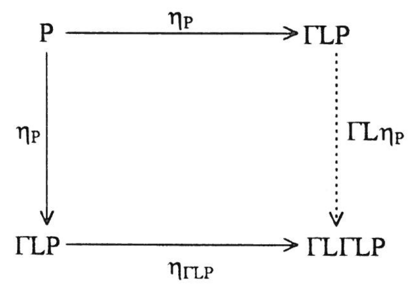
Corollary S10: The adjunction of S5 restricts to an equivalence $Sh(X) \approx Et(X)$.
Proof:

By S7, Sh(X) is the full subcategory of $[\Theta(X)^{op},Set]$ consisting of all the P for which η_P is an isomorphism. By S9, Et(X) is the full subcategory of $Top\downarrow X$ consisting of all the p for which ε_p is an isomorphism. The result follows by A6. □

For the next corollary we distinguish the inclusions $i_1 : Sh(X) \hookrightarrow [\Theta(X)^{op},Set]$ and $i_2 : Et(X) \hookrightarrow Top\downarrow X$.

Corollary S11: $\Gamma L \dashv i_1$ and $i_2 \dashv L\Gamma$.
Proof:

Identify $\eta : 1_{[\Theta(X)op,Set]} \bullet\!\!\to i_1\Gamma L$ with the η defined above in the $L \dashv \Gamma$ adjunction, and $\varepsilon : \Gamma L i_1 \bullet\!\!\to 1_{Sh(X)}$ with η^{-1} defined above on Sh(X), since η is an isomorphism there.
Thus, $i_1\varepsilon \circ \eta i_1$ is identified with $\eta^{-1} \circ \eta$ on Sh(X) $\Rightarrow i_1\varepsilon \circ \eta i_1 = 1_{Sh(X)}$.
Note that the following square commutes $\forall P \in [\Theta(X)^{op},Set]$ by the naturality of η:



and by the universality of η_P (the horizontal version), it follows that $\Gamma L\eta_P = \eta_{\Gamma LP}$.
Thus $\forall P \in [\Theta(X)^{op},Set]$, $\varepsilon_{\Gamma LP} \circ \Gamma L\eta_P = \eta^{-1}_{\Gamma LP} \circ \eta_{\Gamma LP} = 1_{\Gamma LP}$.
$\therefore$ by A2(iii) it follows that $\Gamma L \dashv i_1$.
The result $i_2 \dashv L\Gamma$ follows almost identically. In this case we use the counit of the $L \dashv \Gamma$ adjunction to define our unit and counit for this adjunction, and the necessary identities (from A2(iii)) follow similarly. □

The functor ΓL above is known as the associated sheaf functor.

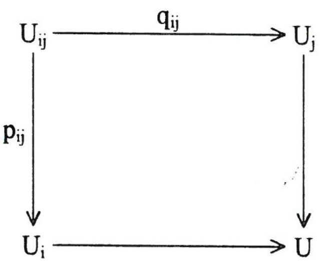
Consider another topological space Y and a continuous function $f : X \rightarrow Y$. Then we see that f induces a functor $f_* : Sh(X) \rightarrow Sh(Y)$ given by $f_*F(U) = F(f^{-1}U)$. One can always construct a left exact left adjoint f^* to f_*. The details of this construction are not important for the next section, but the fact that f_* always has a left exact left adjoint is one vital component of the analogy that we are attempting to build up.

GROTHENDIECK TOPOSES

Defining Grothendieck Toposes

Now we use the results of the previous chapter on topological sheaves to provide a notion of sheaf for categories more general than $\Theta(X)$.

By direct analogy with topological sheaves, we present the notions of topology and sheaf for a category. First we require some notation. Consider the arrows $U_i \to U$ and $U_j \to U$. We adopt the following notation for the pullback diagram:



Let C be a small category with pullbacks. A Grothendieck pretopology for C is a set R(U) of families of arrows $\{U_i \to U : i \in I\}$ for each object U of C such that:

(i) $\forall U \in obj(C)$, $\{1_U\} \in R(U)$.

(ii) If $U_k \to U$ is an arrow of C and $\{U_i \to U : i \in I\} \in R(U)$, then $\{q_{ik} : i \in I\} \in R(U_k)$.

(iii) If $\{U_i \to U : i \in I\} \in R(U)$ and $\forall i \in I$, $\{(U_i)_j \to U_i : j \in J\} \in R(U_i)$, then $\{(U_i)_j \to U_i \to U : i \in I, j \in J\} \in R(U)$.

The elements of R(U) are called covering families for the pretopology.

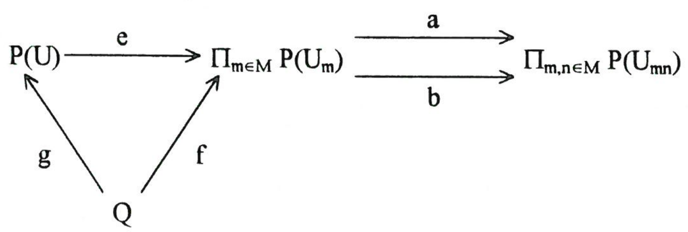
$\Theta(X)$ has an obvious Grothendieck pretopology. For each $U \subseteq X$ open, take R(U) to be all the open coverings of U.

A presheaf on C is a functor $P : C^{op} \to Set$. Proposition S2 motivates us to provisionally define a sheaf on C to be a presheaf F such that for each covering family $\{U_i \to U : i \in I\} \in R(U)$, e is an equaliser in the following diagram:

$$F(U) \xrightarrow{e} \Pi_{i\in I} F(U_i) \underset{b}{\overset{a}{\rightrightarrows}} \Pi_{i,j\in I} F(U_{ij})$$

where $a_{ij} = Fp_{ij} \circ p_i$, $b_{ij} = Fq_{ij} \circ p_j$, and $e_i = F(U_i \to U)$ and where p_i and p_j are product projections.

Suppose that R is some pretopology on C, $\alpha_U = \{U_i \to U : i \in I\} \in R(U)$ and P is a sheaf for R. Consider also $\beta_U = \{\alpha_m : U_m \to U : m \in M\} \supseteq \alpha_U$, that is, $I \subseteq M$. Now, consider the following diagram for P:



Then given f such that $a \circ f = b \circ f$, it follows that $\exists!$ g such that $\forall i \in I$, $f_i = e_i \circ g$. Consider any $m \in M$. Then $\forall i \in I$, $F(q_{im}) \circ f_m = F(p_{im}) \circ f_i = F(p_{im}) \circ e_i \circ g = F(q_{im}) \circ e_m \circ g$.
But, $\{q_{im} : i \in I\} \in R(U_m)$ so that the arrows $F(q_{im})$ are jointly monic.
$\therefore f_m = e_m \circ g$. That is, e is an equaliser so that P satisfies the sheaf property for the covering β_U.

Now suppose that $\beta_U \subseteq \alpha_U$ and that every arrow in α_U factors through an arrow in β_U. Consider again the above diagram for P. Given f such that $a \circ f = b \circ f$, we can extend f to a function $Q \to \Pi_{i\in I} PU_i$ such that the same condition holds for the corresponding diagram with respect to the covering α_U. Thus we can find g making the above diagram commute (since we can do this in the corresponding α_U diagram).

We wish to show that g is the unique such arrow. Suppose that $u,v \in PU$ such that $e(u) = e(v)$. Then $\forall m \in M$, $F(U_m \to U)(u) = F(U_m \to U)(v)$. Consider $i \in I$. Then by hypothesis, $\exists\, m \in M$ such that $U_i \to U$ factors through $U_m \to U$. That is, $F(U_i \to U)(u) = F(U_i \to U_m) \circ F(U_m \to U)(u) = F(U_i \to U_m) \circ F(U_m \to U)(v) = F(U_i \to U)(v)$.
Since this equation holds $\forall i \in I$, and that e in the α_U diagram is injective, it follows that $u = v$.
$\therefore$ e (in the β_U diagram) is injective $\Rightarrow$ g is unique.
Thus P satisfies the sheaf condition with respect to β_U.

The above paragraphs exhibit the sense in which our provisional definitions are redundant, since many pretopologies give identical sheaves. In what follows we are able remove this redundancy and also the assumption that C has pullbacks.

Let C be a small category. Then a <u>sieve on the object U</u> of C is a subfunctor of $C(-,U)$.

Every sieve is a covering family. For any sieve $\alpha_U \hookrightarrow C(-,U)$, we identify the covering family α_U with the objects of the category of elements of the presheaf α_U.
Consider a covering family α_U. Suppose that α_U is in fact a sieve. Let $g : W \to V$ and $f : V \to U \in \alpha_U$. That is, $f \in \alpha_U(V)$. Then considering the map $C(g,U) : C(V,U) \to C(W,U)$, it follows that $\alpha_U(g)(f) = f \circ g$ so that $f \circ g \in \alpha_U(W)$, that is, $f \circ g \in \alpha_U$.
Conversely, if α_U has the property $\forall g : W \to V$ and $\forall f : V \to U \in \alpha_U$, that $f \circ g \in \alpha_U$, then it follows that α_U is a subfunctor of $C(-,U)$.
Given a covering family α_U, we can generate a sieve $\alpha'_U = \{f : V \to U : f \text{ factors through an arrow in } \alpha_U\}$.
Observe that a covering family and the sieve it generates will satisfy the sheaf condition for exactly the same collection of presheaves.

Some notation is required for the next definition. Let R be a presheaf and $f \in C(V,U)$. Then let $f^*(R)$ be the pullback of $R \hookrightarrow C(-,U)$ along $C(-,f)$ in $[C^{op},Set]$. Remember that $[C^{op},Set]$ is complete so that it must have pullbacks, and they are formed pointwise from Set. Translating the pretopology definition into the language of sieves, we obtain the following definition.

A Grothendieck topology on C is a set J(U) of sieves for each $U \in obj(C)$ such that:

(i) $C(-,U) \in J(U)$.

(ii) If $R \in J(U)$ and $f \in C(V,U)$, then $f^*(R) \in J(V)$.

(iii) If $R \in J(U)$ and S is a sieve on U such that $\forall V \in obj(C)$, $\forall f \in R(V)$ we have $f^*(S) \in J(V)$, then $S \in J(U)$.

The elements of J(U) are called covering sieves for the topology.

A category C with a Grothendieck topology defined on it is called a site and is denoted by (C,J).

A sheaf on (C,J) is a presheaf F such that $\forall U \in obj(C)$, $R \in J(U)$, each $R \bullet\!\!\rightarrow F$ has exactly one extension to a natural transformation $C(-,U) \bullet\!\!\rightarrow F$. A presheaf P is a separated presheaf when it satisfies this condition with "exactly one" replaced by "at most one". We denote the full subcategory of $[C^{op},Set]$ consisting of all the sheaves under the topology J as Sh(C,J).

Let $\rho_R = [C^{op},Set](R \hookrightarrow C(-,U),P) : [C^{op},Set](C(-,U),P) \rightarrow [C^{op},Set](R,P)$. That is, ρ_R takes $C(-,U) \bullet\!\!\rightarrow P$ to its restriction on R. Another way to say that F is a sheaf is that $\forall U \in obj(C)$, $R \in J(U)$, ρ_R is an isomorphism. Similarly, F is a separated presheaf when $\forall U \in obj(C)$, $R \in J(U)$, ρ_R is monic.

Consider $R \in J(U)$ and $F \in [C^{op},Set]$. We saw above that we can regard R as a covering family $\{\alpha_i : U_i \rightarrow U : i \in I\}$. Given a natural transformation $\phi : R \bullet\!\!\rightarrow F$, we obtain $(\phi_{U(i)}(\alpha_i))_{i\in I} \in \Pi_{i\in I} FU_i$. Furthermore, the naturality of ϕ says that $\phi_{U(ij)}(\alpha_i \circ p_{ij}) = Fp_{ij}(\phi_{U(i)}(\alpha_i))$.

$\therefore$ for $i,j \in I$, $Fp_{ij}(\phi_{U(i)}(\alpha_i)) = \phi_{U(ij)}(\alpha_i \circ p_{ij}) = \phi_{U(ij)}(\alpha_j \circ q_{ij}) = \phi_{U(ij)}(\alpha_i \circ p_{ij}) = Fp_{ij}(\phi_{U(j)}(\alpha_j))$.

Similarly, $(u_i)_{i\in I} \in \Pi_{i\in I} FU_i$ with $Fp_{ij}(u_i)) = Fp_{ij}(u_j))$ $\forall i,j \in I$, determines $\phi : R \bullet\!\!\rightarrow F$ by $\phi_{U(i)}(\alpha_i) = u_i$.

That is, $[C^{op},Set](R,F) \cong \{(u_i)_{i\in I} \in \Pi_{i\in I} FU_i : \forall i,j \in I,\ Fp_{ij}(u_i)) = Fp_{ij}(u_j))\}$.

By the definition of equaliser for sets, F satisfies the "equaliser sheaf condition" for R $\Leftrightarrow$ $FU \cong \{(u_i)_{i\in I} \in \Pi_{i\in I} FU_i : \forall i,j \in I,\ Fp_{ij}(u_i)) = Fp_{ij}(u_j))\}$.

$\therefore$ F satisfies the "sieve sheaf condition" for R $\Leftrightarrow$ $[C^{op},Set](R,F) \cong [C^{op},Set](C(-,U),F)$ $\Leftrightarrow$ F satisfies the "equaliser sheaf condition" for R, since by the Yoneda lemma $[C^{op},Set](C(-,U),F) \cong FU$.

That is, the above definition of sheaf in terms of sieves is equivalent to the provisional definition made for pretopologies.

The set $\mathcal{J}(C)$ of Grothendieck topologies on C is partially ordered by inclusion. That is, $J \leq J'$ in $\mathcal{J}((C)$ when $\forall U \in obj(C)$, $J(U) \subseteq J'(U)$. Clearly, $Sh(C,J') \subseteq Sh(C,J)$ when $J \leq J'$ in $\mathcal{J}(C)$. In particular we have the minimal topology in which only the representables are covering sieves, so that all presheaves are sheaves. On the other hand, we have the maximal topology in which every sieve is a covering sieve. Suppose that P is a sheaf for the maximal topology. Then $\forall U \in obj(C)$ we take the R to be the empty sieve, which also the initial object of $[C^{op},Set]$. Applying the sheaf condition along with the Yoneda lemma for R, we see that $PU \cong [C^{op},Set](R,P) \cong 1$. Thus, in this case the only sheaf is the terminal object of $[C^{op},Set]$. Consider $\{J_i : i \in I\} \subseteq \mathcal{J}(C)$. Then by definition $\cap_{i\in I} J_i \in \mathcal{J}(C)$. Thus, $\mathcal{J}(C)$ admits all infima. Since $\mathcal{J}(C)$ has a supremum, it follows that $\mathcal{J}(C)$ admits all suprema (so that it is a complete lattice). We are thus able to make the following definition.

The canonical topology on C is the largest topology for which all the representables are sheaves. A topology J on C is said to be subcanonical when it is smaller than the canonical topology in which case all of the representables are in Sh(C,J).

A Grothendieck Topos is a category $\mathcal{E}$ that is equivalent to Sh(C,J) for some site (C,J). Furthermore we define a geometric morphism between Grothendieck toposes $\mathcal{E}$ and $\mathcal{F}$ to be a functor $f_* : \mathcal{E} \rightarrow \mathcal{F}$ that has a left exact left adjoint. We usually denote this left adjoint by f^*.

A very simple example of a Grothendieck topos is Set. Specifically, let $C = 1$ so that J is determined trivially . Then every presheaf is a sheaf so that $[1,\mathrm{Set}] \cong \mathrm{Sh}(1) \cong \mathrm{Set}$. Another example is any category of presheaves $[C^{op},\mathrm{Set}]$ since we can take J to be the minimal topology.

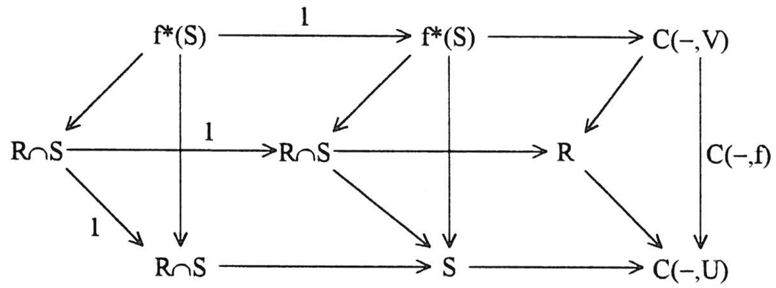
The Associated Sheaf Functor

We now construct a generalisation of the associated sheaf functor from topological sheaves, on the site (C,J).

Lemma GT1: Let $U \in \mathrm{obj}(C)$ and $R, S \in J(U)$. Then $R \cap S \in J(U)$.

Proof:

Take $f : V \to U$ in R. Saying that $f \in R$ is the same as saying that $C(-,f)$ factors through the inclusion $R \hookrightarrow C(-,U)$. In the diagram:



the bottom right, top and bottom left, and the large right squares are pullbacks by definition and the right and left-most triangles commute. The middle triangle commutes since $S \hookrightarrow C(-,U)$ is monic, all the squares on the right commute and the right-most triangle commutes. It follows that the top right square is a pullback (since the big and bottom right squares are pullbacks), so that by axiom (ii) for Grothendieck topologies, $f^*(S) \in J(V)$. Furthermore, the big left square is a pullback (since the little left squares are). Thus the entire diagram is a pullback (since the big left and right squares are). Thus, $f^*(R \cap S) = f^*(S) \in J(V)$. Since this is true $\forall V \in \mathrm{obj}(C)$ and $f \in R(V)$, it follows by axiom (iii) for Grothendieck topologies that $R \cap S \in J(U)$. □

We observe that J(U) can be regarded as a filtered category. Specifically, the objects are sieves, the arrows are reverse inclusions, and the above lemma indicates filteredness. Notice also that J(U) regarded in this way has an initial object $C(-,U)$.

For $U \in \mathrm{obj}(C)$, define $LP(U) = \mathrm{colim}_{R\in J(U)}\, [C^{op},\mathrm{Set}](R,P)$. Since colim is functorial this defines a presheaf. Let $(\tau_P)_R : [C^{op},\mathrm{Set}](R,P) \to LP(U)$ denote the components of the universal cocone. Also, define $y_P : PU \to [C^{op},\mathrm{Set}](C(-,U),P)$ to be the isomorphism given by the Yoneda lemma. Then we can define the arrows $(\eta_P)_U = (\tau_P)_{C(-,U)} \circ y : PU \to LPU$. Clearly the η_U are natural in U, so that $\eta_P : P \to LP$.

Lemma GT2: Let P be a presheaf.

(i) P is separated $\Leftrightarrow$ η_P is monic.

(ii) P is a sheaf $\Leftrightarrow$ η_P is an isomorphism.

Proof:

Fixing $U \in \mathrm{obj}(C)$, P is separated (a sheaf) $\Leftrightarrow$ $\forall R \in J(U)$, ρ_R is a monic (isomorphism).
Since y is an isomorphism, η_P is a monic (isomorphism) $\Leftrightarrow$ $\tau_{C(-,U)}$ is a monic (isomorphism).
Since J(U) is a directed set, we can apply L10(ii) to τ.
$\therefore$ $\forall R,S \in J(U)$, $f : R \to P$ and $g : S \to P$ where $\tau_R(f) = \tau_S(g)$, $\exists\, T \hookrightarrow R\cap S$ such that $f|_T = g|_T$.
In particular, $\tau_{C(-,U)}(f) = \tau_{C(-,U)}(g) \Rightarrow \exists\, T \in J(U)$ such that $\rho_T(f) = \rho_T(g)$.
Since $\forall R \in J(U)$, $\tau_{C(-,U)} = \tau_R \circ \rho_R$, $\tau_{C(-,U)}$ injective $\Rightarrow$ ρ_R is injective $\forall R$.

Conversely, suppose ρ_R is injective $\forall R \in J(U)$. Then $\tau_{C(-,U)}(f) = \tau_{C(-,U)}(g) \Rightarrow \exists\, T \in J(U)$ where $\rho_T(f) = \rho_T(g) \Rightarrow f = g$, so that $\tau_{C(-,U)}$ is injective.
Suppose that $\forall R \in J(U)$, ρ_R is surjective. Then $f \circ \tau_{C(-,U)} = g \circ \tau_{C(-,U)} \Rightarrow \forall R \in J(U)$, $f \circ \tau_R \circ \rho_R = g \circ \tau_R \circ \rho_R \Rightarrow \forall R \in J(U)$, $f \circ \tau_R = g \circ \tau_R \Rightarrow f = g$ since τ_R are jointly surjective.
$\therefore \forall R \in J(U)$, ρ_R epi $\Rightarrow \tau_{C(-,U)}$ epi.
Finally, suppose that $\tau_{C(-,U)}$ is an isomorphism. For $R \in J(U)$, we construct a cocone componentwise as $\psi_S : [C^{op},Set](S,P) \to [C^{op},Set](R,P)$ as $\psi_S = \rho_R \circ \tau_{C(-,U)}^{-1} \circ \tau_S$.
Then, $\forall S \in J(U)$, $\tau_R \circ \psi_S = \tau_S \Rightarrow \psi$ is a limiting cocone since τ is.
$\therefore \tau_R$ is an isomorphism $\Rightarrow \rho_R$ is an isomorphism. □

Lemma GT3: If F is a sheaf and P is a presheaf, then any $\phi : P \bullet\!\!\rightarrow F$ factors uniquely through η.
Proof:

We seek a unique arrow $\psi : LP \bullet\!\!\rightarrow FU$ such that $\psi \circ \eta_P = \phi$. The following diagram makes this construction obvious:

$$\begin{array}{ccccc}
PU & \xrightarrow{\;y_P\;} & [C^{op},Set](C(-,U),P) & \xrightarrow{\;(\tau_P)_{C(-,U)}\;} & LPU \\
\downarrow{\scriptstyle \phi_U} & & \downarrow{\scriptstyle \phi\,\circ\,-} & & \downarrow{\scriptstyle L\phi_U} \\
FU & \xrightarrow[\;y_F\;]{} & [C^{op},Set](C(-,U),F) & \xrightarrow[\;(\tau_F)_{C(-,U)}\;]{} & LFU
\end{array}$$

The left square commutes by the naturality of the Yoneda isomorphism. The right square commutes since colim is functorial. Thus, the large square, whose horizontal arrows are $(\eta_P)_U$ and $(\eta_F)_U$, is commutative. Note that y_P and y_F are isomorphisms (by definition), and that $(\tau_F)_{C(-,U)}$ is an isomorphism since F is a sheaf. Thus we take $\psi = \eta_F^{-1} \circ L\phi$. □

Lemma GT4: For any presheaf P, LP is a separated presheaf.
Proof:

Take $\alpha, \beta \in LPU$. These correspond to $\alpha^\wedge, \beta^\wedge : C(-,U) \bullet\!\!\rightarrow LP$ by the Yoneda lemma. Suppose that $\exists\, Q \in J(U)$ such that $\alpha^\wedge|_Q = \beta^\wedge|_Q$ (ie $\rho_Q(\alpha^\wedge) = \rho_Q(\beta^\wedge)$). By L9 the elements of LPU are equivalence classes of natural transformations $R \bullet\!\!\rightarrow P$ for $R \in J(U)$. Thus we can also represent α and β as natural transformations $\alpha^\vee : R \bullet\!\!\rightarrow P$, $\beta^\vee : S \bullet\!\!\rightarrow P$ for some $R, S \in J(U)$.
That is, $(\tau_P)_R(\alpha^\vee) = \alpha$ and similarly for β.
Let $h : V \to U \in QV$. Then $\alpha^\wedge(h) = \beta^\wedge(h)$ so that $LP(h)(\alpha) = LP(h)(\beta)$.
Then by axioms (ii) and (iii) for Grothendieck topologies it follows that $\exists\, T \subseteq R \cap S$ such that $\alpha^\vee|_T = \beta^\vee|_T$ so that indeed $\alpha = (\tau_P)_R(\alpha^\vee) = (\tau_P)_R(\beta^\vee) = \beta$.
$\therefore \forall Q \in J(U)$, ρ_Q is monic $\Rightarrow$ LP is separated. □

Lemma GT5: For any separated presheaf P, LP is a sheaf.
Proof:

Consider $\phi : R \bullet\!\!\rightarrow LP$ where $R \in J(U)$. We need to extend this to $C(-,U) \bullet\!\!\rightarrow LP$. Given such an extension, uniqueness follows by GT4. Let $f : V \to U \in R$, then $\phi_V(f) \in LPV$ and so corresponds to an equivalence class of natural transformations $S \bullet\!\!\rightarrow P$ for $S \in J(V)$. Thus, we can represent $\phi_V(f)$ by $\sigma_f : S_f \bullet\!\!\rightarrow P$ where $S_f \in J(V)$. Construct a sieve $Q = \{f \circ g : f \in R \text{ and } g \in S_f\}$ thought of here as a covering family. Then by Grothendieck topology axiom (iii) $Q \in J(U)$. We construct the natural transformation $\psi : Q \bullet\!\!\rightarrow P$ as $\psi(f \circ g) = \sigma_f(g)$, for $f : V \to U$ and $g : W \to V$.
First we claim that ψ is well-defined:

Taking $f : V \to U \in R$ and any $h : W \to V$, the naturality of ϕ gives $\phi_W(f \circ h) = LP(h)(\phi_V(f))$. Thus $\sigma_{f \circ h}$ is equivalent to σ_f (under the colimit equivalence relation).

$\therefore \exists\, T_{f,h} \in J(W)$ such that $\forall g \in T_{f,h}$, $\sigma_{f\circ h}(g) = \sigma_f(h \circ g)$.
Suppose that $f \circ g = f' \circ g'$. Taking $k \in T = T_{f,g} \cap T_{f',g'}$ we see that:
$P(k)(\sigma_f(g)) = \sigma_f(g \circ k) = \sigma_{f\circ g}(k) = \sigma_{f'\circ g'}(k) = \sigma_{f'}(g' \circ k) = P(k)(\sigma_{f'}(g'))$.
That is, $\sigma_f(g)|_T = \sigma_{f'}(g')|_T$ when we regard $\sigma_f(g)$ and $\sigma_{f'}(g')$ as $C(-,W) \bullet\!\!\to P$ by the Yoneda lemma. That is, $\sigma_f(g) = \sigma_{f'}(g')$ as elements of LPW.
Furthermore, since ϕ and σ_f are natural, it follows that ψ is natural.

Since $\psi : Q \bullet\!\!\to P$ it represents an element of LPU (under the colimit equivalence relation, namely $\tau_Q(\psi) \in LPU$). Thus ψ corresponds to a natural transformation $C(-,U) \bullet\!\!\to LP$ by the Yoneda lemma. It follows by the definition of ψ that $\psi : C(-,U) \bullet\!\!\to LP$ is an extension of ϕ, that is, that $\psi|_R = \phi$. □

Proposition GT6: L^2 is a left exact left adjoint to the inclusion $i : Sh(C,J) \hookrightarrow [C^{op},Set]$.
Proof:

Firstly, $L^2 : [C^{op},Set] \to Sh(C,J)$ by GT4 and GT5. Then by two applications of GT3 it follows that $\eta_{LP} \circ \eta_P : P \to L^2P$ is universal from P to L^2 for each presheaf P. Thus $L^2 \dashv i$ by A2(i).
Now, consider a functor $T : I \to [C^{op},Set]$ where I is finite, and denote T(i) by P_i and write lim(T) as $\lim_i(P_i)$. Also, let $U \in obj(C)$.

Then, $\lim_i(LP_i)(U) = \lim_i(LP_i(U))$ since limits in $[C^{op},Set]$ are formed pointwise.
$= \lim_i(\text{colim}_{R\in J(U)}\, [C^{op},Set](R,P_i))$
$= \text{colim}_{R\in J(U)}\, \lim_i([C^{op},Set](R,P_i))$ since J(U) is filtered and using L18
$= \text{colim}_{R\in J(U)}\, [C^{op},Set](R,\lim_i(P_i))$ since $[C^{op},Set](R,-)$ preserves lims.
$= L(\lim_i(P_i))(U)$.

That is, $\lim_i(LP_i) = L(\lim_i(P_i))$. That is, L preserves finite limits.
$\therefore L^2$ is a left exact left adjoint to i. □

Thus we see that $i : Sh(C,J) \hookrightarrow [C^{op},Set]$ is our first example of a geometric morphism. Since i is full and faithful, it follows by A4 that $L^2 \circ i \cong 1_{Sh(C,J)}$.

We conclude this section with an important proposition regarding Grothendieck toposes.

Proposition GT7: Any Grothendieck topos $\mathcal{E}$ is complete and cocomplete.
Proof:

It is easy to see that Sh(C,J) is complete for any site (C,J). Let $\lim_i(P_i)$ be some limit of sheaves. Then since $[C^{op},Set]$ is complete $\lim_i(P_i)$ is well-defined as a presheaf. However, the sheaf condition for each covering sieve can be expressed in terms of an equaliser diagram. Since the limit of limits is a limit (see remarks beneath L17) it follows that $\lim_i(P_i)$ must satisfy the equaliser sheaf condition since the individual P_i do.
Thus, $\mathcal{E}$ is equivalent to a complete category and so must be complete.
For some site (D,K), $\mathcal{E} \approx Sh(D,K)$. By GT6, it follows that there is a full and faithful geometric morphism $I_* : \mathcal{E} \to [D^{op},Set]$. Since $[D^{op},Set]$ is cocomplete, the result follows by L17. □

Cover Preserving Functors

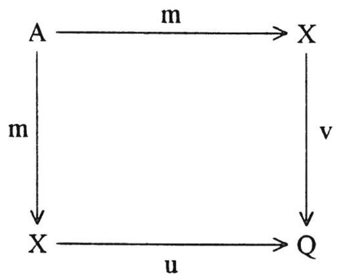
We are interested in geometric morphisms $f_* : \mathcal{E} \to [C^{op},Set]$ and their relationship to $f = f^* \circ Y$ where Y denotes the Yoneda imbedding $C \to [C^{op},Set]$. We say that $f : C \to \mathcal{E}$ is cover preserving when it takes covering families in C to jointly epimorphic families of arrows in $\mathcal{E}$. Realise that since $\mathcal{E}$ is not necessarily small, we cannot define a topology on it using the above definitions directly, so perhaps the term cover preserving is a bit misleading. However, it is possible to define Grothendieck topologies on large categories, but we will not go into this.

The main result here is that f is cover preserving iff f_* factors through $i : Sh(C,J) \hookrightarrow [C^{op},Set]$. Before proving this result we require a number of preliminary results.

Lemma GT8: Every injective function is an equaliser.
Proof:

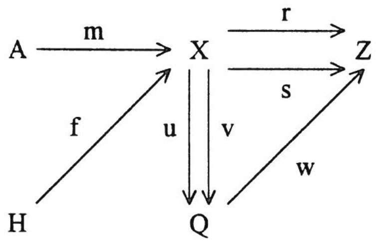
Let $m : A \to X$ be injective. Then f is the equaliser of the functions $u, v : X \to 2$ where these are defined as: $u(x) = 1$ iff $x \in im(m)$ and $v(x) = 1\ \forall x \in X$. □

Lemma GT9: If $m : A \to X$ is an equaliser in some category D then it is the equaliser of u, v where the following is a pushout:



assuming that this pushout exists in D.
Proof:

Assume that m is the equaliser of r and s:



Since m, u, and v form a pushout square and $r \circ m = s \circ m$, $\exists!$ w where $w \circ u = r$ and $w \circ v = s$. Suppose $u \circ f = v \circ f$. Then $r \circ f = s \circ f$ so that f factors uniquely through m. □

In the above situation we call (u,v) the <u>kernel pair</u> of m. A direct corollary of GT8, GT9 and L14 is that for any category D, every monic in $[D^{op},Set]$ is the equaliser of its kernel pair.

Lemma GT10: In any category D and $e \in arr(D)$, e is an equaliser and e is epi $\Rightarrow$ e is invertible.
Proof:

Suppose $e : A \to X$ is the equaliser of $u,v : X \to Y$. Since e is epi and $u \circ e = v \circ e$, it follows that $u = v \Rightarrow u \circ 1_X = v \circ 1_X$. Thus it follows that $\exists!\ m : X \to A$ such that $e \circ m = 1_X$.
$\therefore e \circ m \circ e = e \Rightarrow m \circ e = 1_A$ since e is monic since e is an equaliser. □

Lemma GT11: Every monic in $\mathcal{E}$ is an equaliser.
Proof:

Let $m : E \to E'$ be a monic in $\mathcal{E}$ and recall the full and faithful geometric morphism $I_* : \mathcal{E} \to [D^{op},Set]$ for some site (D,K). Then I_*m is also monic since I_* preserves limits. Since Im is an arrow in $[D^{op},Set]$ it follows that it is an equaliser of its kernel pair, and since I^* is left exact, it takes this equaliser to an equaliser I^*I_*m in $\mathcal{E}$. But I_* is full and faithful so that $I^*I_* \cong 1_{\mathcal{E}}$ from which it follows that m is an equaliser. □

Lemma GT12: Every jointly epic family $(\alpha_i : U_i \to U : i \in I)$ in $\mathcal{E}$ has

$$\coprod_{i,j \in I} U_{ij} \underset{(q_{ij})}{\overset{(p_{ij})}{\rightrightarrows}} \coprod_{i \in I} U_i \xrightarrow{(\alpha_i)} U$$

as a coequaliser.

Proof:

When $\mathcal{E}$ = Set we note that $\alpha_i(x) = \alpha_j(y) \Leftrightarrow (x,y) \in U_{ij}$, so that the result follows in this case. By the pointwise formation of colimits for $[D^{op},Set]$ from Set, the result follows for $\mathcal{E} = [D^{op},Set]$ for some category D.

For the general case we recall the full and faithful geometric morphism I_*. Let

$$\coprod_{i,j \in I} I_*U_{ij} \underset{(I_*q_{ij})}{\overset{(I_*p_{ij})}{\rightrightarrows}} \coprod_{i \in I} I_*U_i \xrightarrow{(\tau_i)} P$$

be a coequaliser. Then $\exists!\ m : P \to I_*U$ such that $m \circ \tau_i = I_*\alpha_i$ and this m is necessarily monic since I_* preserves limits. Then applying I^* to this situation and noting that $I^*I_* \cong 1_{\mathcal{E}}$ it follows that $Lm : LP \to U$ is monic and hence an equaliser by GT11, and epi since α_i is a jointly epic family. $\therefore$ by GT10 Lm is invertible, so that α_i form a limiting cocone. □

Proposition GT13: Consider the geometric morphism $f_* : \mathcal{E} \to [C^{op},Set]$ and $f = f^* \circ Y : C \to \mathcal{E}$ where Y denotes the Yoneda imbedding $C \to [C^{op},Set]$. Then f is cover preserving iff f_* factors through the inclusion $i : Sh(C,J) \hookrightarrow [C^{op},Set]$.

Proof:

Suppose that f is cover preserving. Let $(\alpha_i : U_i \to U : i \in I) \in J(U)$. Then

$$\coprod_{i,j \in I} fU_{ij} \underset{(fq_{ij})}{\overset{(fp_{ij})}{\rightrightarrows}} \coprod_{i \in I} fU_i \xrightarrow{(f\alpha_i)} fU$$

is a coequaliser by GT12 since by hypothesis, the $f(\alpha_i)$ form a jointly epi family. The representable $\mathcal{E}(-,E)$ takes colimits to limits, and applying this to the above diagram gives the sheaf condition for $\mathcal{E}(f-,E)$. Since $f = f^* \circ Y$ it follows that $f^* = Lan_Y(f) = colim(-,f)$ (see lines 3-7 of the proof of CK17, which apply here because $\mathcal{E}$ is cocmplete by GT7) so that by the proof of L20 $f_*(E) = \mathcal{E}(f-,E)$. That is, $f_*(E)$ is a sheaf.

For the converse, reverse each of the steps in the above argument. □

THE DUALITY BETWEEN GEOMETRY AND LOGIC

The Main Theorem

At last we are in a position to discuss the main theorem alluded to in the introduction. Again we adopt the notation $\mathcal{E}$ to denote a Grothendieck topos and (C,J) to denote a site. First we require a lemma.

Lemma GT14: Let $f : C \to \mathcal{E}$ and $g : C \to D$ be functors where C is finite complete and D is some other category. Suppose also that f is left exact. Then $Lan_g f$ is left exact.
Proof:

In any case since $\mathcal{E}$ is cocomplete, $Lan_g f$ exists by CK9 and is pointwise.
Suppose that $\mathcal{E}$ = Set. Then since f is left exact and C is finite complete it follows that el(f) is finite complete $\Rightarrow$ $el(f)^{op}$ is finite cocomplete $\Rightarrow$ $el(f)^{op}$ is filtered.
In this case we see that $Lan_g(f)(E)$ = colim(Set(g–,E),f) = colim(f,Set(g–,E)) by L7 and by the dual of L6, colim(f,Set(g–,E)) = colim($el(f)^{op} \to C \to D$) which is a filtered colimit. Thus by L18 it follows that $Lan_g(f)$ is left exact.
Next suppose that $\mathcal{E}$ = [B^{op},Set] for some category B. Then since $Lan_g f$ is expressable as a [B^{op},Set]-valued colimit, it follows that $Lan_g f$ is itself formed pointwise from Set. By the pointwise formation of limits (again) it follows from the case $\mathcal{E}$ = Set that $Lan_g f$ must be left exact.
Finally we take $\mathcal{E}$ in general. Then there is some site (B,K) and a full and faithful geometric morphism $I_* : \mathcal{E} \to [B^{op},Set]$. Now since $I_* \circ f$ is left exact and by the previous case it follows that $Lan_g(I_* \circ f)$ is left exact. But since I^* is a left adjoint it preserves left Kan extensions, and because $I^* \circ I_* \cong 1_{\mathcal{E}}$ it follows that $Lan_g(I^* \circ I_* \circ f) = Lan_g(f) = I^* \circ Lan_g(I_* \circ f)$, and since I^* is left exact this left Kan extension must be left exact also. □

Let LexCovPres(C,$\mathcal{E}$) be the full subcategory of [C,$\mathcal{E}$] consisting of all of the left exact cover preserving functors. Similarly, Lex(C,$\mathcal{E}$) consists of all of the left exact functors. Let Geom($\mathcal{E}$,Sh(C,J)) be the full subcategory of [$\mathcal{E}$,Sh(C,J)] consisting of all of the geometric morphisms. Then we obtain the following result:

Theorem GT15: For (C,J) a finite complete site, LexCovPres(C,$\mathcal{E}$) $\approx$ Geom($\mathcal{E}$,Sh(C,J)).
Proof:

Since $\mathcal{E}$ is cocomplete it follows by CK17 that [C,$\mathcal{E}$] $\approx$ CoCts[[C^{op},Set],$\mathcal{E}$] and that this equivalence is given by taking left Kan extensions along $Y : C \to [C^{op},Set]$. Under these circumstances it follows also that each $Lan_Y(f) = f^* : [C^{op},Set] \to \mathcal{E}$ has a right adjoint which we will denote by f_*. Hence this equivalence of categories restricts by GT14 to an equivalence Lex(C,$\mathcal{E}$) $\approx$ Geom($\mathcal{E}$,[C^{op},Set]), and this restricts yet further by GT13 to the desired result. □

An Illustrative Example

In order to provide some insight into the meaning of this last result we present an illustrative example. Here we study the theory of local rings, which is typically viewed from a logical perspective. However we will use the above result to demonstrate the way in which the theory of local rings can be viewed within something analogous to a spatial perspective.

We will focus our attention on the category of finitely presented commutative rings with identity and denote this category by FPCR. Such a ring R is a local ring when $\forall r \in R$, r is a unit or (1–r) is a unit. This is very much the logical perspective.

By definition every A $\in$ FPCR, is expressable as a quotient $\mathbf{Z}[x_1, \ldots, x_n] / I$. This is just a coequaliser diagram involving rings of polynomials over **Z** and A as the coequaliser. Furthermore, the $\mathbf{Z}[x_1, \ldots, x_n]$ themselves are just coproducts $\mathbf{Z}[x_1] \otimes \ldots \otimes \mathbf{Z}[x_n]$. Thus every A $\in$ FPCR is expressable as a finite colimit of $\mathbf{Z}[x]$'s. Furthermore, since all finite colimits are obtainable precisely from finite coproducts and coequalisers, it follows that FPCR is in fact finitely cocomplete.

Consider a left exact functor M : $\text{FPCR}^{op} \rightarrow$ Set. Now, in the previous paragraph we demonstrated that FPCR^{op} is finitely complete and that every element of this category is obtainable as some finite limit of $\mathbf{Z}[x]$'s. Thus, since M preserves all finite limits, it is determined by its value on $\mathbf{Z}[x]$.

Usually the axioms for a commutative ring with identity are defined by the use of logical notation within the framework of set theory. However it is possible to define these axioms by the use of commutative diagrams. For example the associative axiom for the ring R within the category Set could be expressed as:

$$\begin{array}{ccc} R\times R\times R & \xrightarrow{\bullet\times 1_R} & R\times R \\ {\scriptstyle 1_R\times\bullet}\downarrow & & \downarrow{\scriptstyle\bullet} \\ R\times R & \xrightarrow[\bullet]{} & R \end{array}$$

where $\bullet : R\times R \rightarrow R$ denotes the multplication map. Specifically for a <u>ring object</u> in Set we need a set R and the following collection of maps:

$$1 \;\underset{0}{\overset{1}{\rightrightarrows}}\; R \;\underset{\bullet}{\overset{}{\leftleftarrows}}\; R\times R$$

(where the object 1 is the ring $\{0,1\}$, the arrow 1 takes $1 \mapsto 1 \in R$ and the arrow 0 takes $1 \mapsto 0 \in R$) along with a series of commutative diagrams that give the axioms for a commutative ring in the way demonstrated above. Notice of course that this definition of ring object is in fact good for any category C with finite products. A <u>coring</u> object for a category C with finite coproducts is defined in the same way as a ring object except that all of the arrows are reversed and R×R is replaced by $R\coprod R$ (ie $R\otimes R$).

It is not difficult to check that $\mathbf{Z}[x]$ is coring object. Our collection of coring maps is:

$$\mathbf{Z} \;\underset{0}{\overset{1}{\leftleftarrows}}\; \mathbf{Z}[x] \;\underset{x\cdot y}{\overset{x+y}{\rightrightarrows}}\; \mathbf{Z}[x,y]$$

where the labels on the arrows indicate where $x \in \mathbf{Z}[x]$ gets sent. Thus $\mathbf{Z}[x]$ is a ring object in FPCR^{op} and since M takes commuting diagrams to commuting diagrams it follows that $R = M(\mathbf{Z}[x])$ is a ring object in Set. So we can in fact define a ring object in Set as a left exact functor M : $\text{FPCR}^{op} \rightarrow$ Set, the advantage being that this definition is good for any Grothendieck topos $\mathcal{E}$.

It is not possible to set up the axioms of a local ring S just using commutative diagrams as we did above because we need to say something about arbitrary elements in S. This is where covering families and Grothendieck topologies become useful.

The following coequalisers:

$$\mathbf{Z}[x] \underset{1}{\overset{xy}{\rightrightarrows}} \mathbf{Z}[x,y] \longrightarrow \mathbf{Z}[x,y]/(xy-1)$$

$$\mathbf{Z}[x] \underset{1}{\overset{(1-x)y}{\rightrightarrows}} \mathbf{Z}[x,y] \longrightarrow \mathbf{Z}[x,y]/((1-x)y-1)$$

give a pair of maps out of $\mathbf{Z}[x]$ in FPCR and thus a pair of maps into $\mathbf{Z}[x]$ in FPCR^{op}. We take the sieve generated by this covering family and impose the minimal topology J on FPCR^{op} that contains this sieve as a covering sieve.

We can now insist that M be cover-preserving. Firstly since M preserves limits, applying M to the above equaliser diagrams for FPCR^{op} we obtain equaliser diagrams. From this it follows that under these diagrams $M(\mathbf{Z}[x,y]/(xy-1))$ is identified with the subset $\{(r,s) \in R\times R : rs = 1\} \subseteq R\times R$ which in turn is identified with the subset $\{r \in R : r \text{ is a unit}\} \subseteq R$. Similarly the element $\mathbf{Z}[x,y]/((1-x)y-1) \to \mathbf{Z}[x]$ of the covering family for $\mathbf{Z}[x]$ in FPCR^{op} given above is sent to the inclusion $\{r \in R : (1-r) \text{ is a unit}\} \subseteq R$. Thus the condition that M is cover-preserving is precisely that these inclusions are jointly epi, that is, as subsets they cover all of R. Hence R is a local ring object in Set. Thus we can define a local ring object in Set as a cover preserving functor $M : \mathrm{FPCR}^{op} \to \mathrm{Set}$, and we can define the theory of local rings in Set as the collection of all of these, that is, LexCovPres(FPCR^{op},Set). Of course, this notion of a theory of local rings is good for any Grothendieck topos $\mathcal{E}$.

Now we can exploit our theorem to give an alternative perspective of the theory of local rings. Specifically we have the equivalence LexCovPres(FPCR^{op},Set) $\approx$ Geom(Set,Sh(FPCR^{op},J)).

We can use analogy with topological spaces to provide another interpretation for $f_* : \mathrm{Set} \to \mathrm{Sh}(C,J)$ a geometric morphism where (C,J) is a site. Firstly, Set $\cong$ Sh(1) where 1 is the one point topological space. In topology a geometric morphism $f_* : \mathrm{Sh}(1) \to \mathrm{Sh}(X)$ is induced from a continuous function $f : 1 \to X$. But, such a continuous function is nothing more than a point of the topological space X.

Therefore we can interpret the theory of local rings in Set as being analogous a topological space whose points are local rings. This provides a spatial alternative to the logical theory of local rings. Notice how effective the category theory was in objectifying the usually vague notion of "theory", so that this spatial analogy could be given in precise mathematical language.

BIBLIOGRAPHY

M.Barr, **C.Wells**, *Toposes Triples and Theories*, Grundlehren der math. Wiss. 278, Springer Verlag, Berlin, 1985.

P.T.Johnstone, *Topos Theory*, Academic Press, New York, 1977.

A.Kock, *Synthetic Differential Geometry*, LMS Lecture Notes 51, Cambridge University Press, Cambridge, 1981.

S.MacLane, *Categories for the Working Mathematician*, Springer-Verlag, New York, 1971.

S.MacLane, **I.Moerdijk**, *Sheaves in Geometry and Logic – A First Introduction to Topos Theory*, Springer-Verlag, New York, 1992.

R.Street, *A Survey of Topos Theory*, unpublished notes, Macquarie University, April 1978.

B.R.Tennison, *Sheaf Theory*, London Math. Soc. Lecture Notes 20, Cambridge University Press, Cambridge, 1975.

I would also wish to acknowledge a significant debt to Professor Ross Street for the many informal lectures/discussions given by him throughout the course of this year, and the generosity with which he shared his time.