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Seminar on Triples and Categorical Homology Theory.

Lectures from the seminar held at Forschungsinstitut für Mathematik, ETH, Zürich, 1966/1967.
Edited by Beno Eckmann and Myles Tierney.
Reprint of the 1969 original.
With a preface to the reprint by Michael Barr.
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This is a reprinting of the Springer Lecture Notes in Mathematics Volume 80 [*Seminar on triples and categorical homology theory (ETH 1966/1967)*, Springer, Berlin, 1969; MR0240157]—what I have known as "The Zurich Triples Book". This newly typeset re-publication provides a real service: one of my graduate students says he finds it painful to read papers not in LaTeX. And these papers are important original sources for researchers learning category theory and how to apply it.

A triple T on a category $\mathcal{A}$ is a monoid in the monoidal endofunctor category $[\mathcal{A},\mathcal{A}]$ with composition as tensor product. The term was used by Eilenberg-Moore in lieu of the term standard construction as used by Godement. However, I believe Eilenberg himself soon after suggested the term monad, which is now commonly used; it is mentioned in the volume, but some people still prefer to cling to triple.

The eleven papers in the volume are listed above and in the review of the original [MR0240157]. As a supplement to the reviews of the individual articles, with the benefit of hindsight, I shall discuss the remarkable ideas contained in the articles.

A monad on **Set** which preserves filtered colimits is equivalent to a theory of F. W. Lawvere [Repr. Theory Appl. Categ. No. 5 (2004), 1–121 (electronic); MR2118935]. Linton, who wrote the first three papers in the volume under review, dedicates the first paper to generalizing Lawvere theories to accommodate all monads. Linton also lifts Lawvere's structure-semantics adjunction to this context. The right Kan extension of a functor along itself, when it exists, is a monad on the codomain category, called the codensity monad of the functor: this had been studied by H. Appelgate, A. Kock and M. Tierney. Linton shows how this concept relates to his general notion of algebraic theory based on a functor.

J. M. Beck's thesis [Repr. Theory Appl. Categ. No. 2 (2003), 1–59 (electronic); MR1987896] provided monadicity theorems characterizing when a functor $U\colon\mathcal{A}\to\mathcal{X}$ is equivalent over $\mathcal{X}$ to the underlying functor from some category $\mathcal{X}^T$ of Eilenberg-Moore T-algebras (also sometimes called T-modules). Linton's second paper improves considerably on Beck's theorem, when $\mathcal{X}$ is sufficiently like **Set**. Later, improving on Linton's work, J. Duskin [in *Reports of the Midwest Category Seminar, III*, 74–129, Springer, Berlin, 1969; MR0252471] obtained a monadicity theorem, using coequalizers of equivalence relations (rather than split, absolute or reflexive coequalizers) in general categories.

It is trivial that any limits existing in $\mathcal{X}$ lift to $\mathcal{X}^T$; not so, colimits. In particular, it is of great interest when coequalizers exist in $\mathcal{X}^T$. For example, the way a monoidal structure on $\mathcal{X}$ can lead to one on $\mathcal{X}^T$ for a monoidal monad T is via a coequalizer. Linton's third paper begins by reducing the construction of general coequalizers in $\mathcal{X}^T$(given that $\mathcal{X}$ has them) to the case of reflexive coequalizers. He then gives some sufficient conditions for these to exist.

The paper of Manes presents some results of his thesis showing that, under the compact Hausdorff assumption, topological algebras are algebraic. In particular, the category of compact Hausdorff spaces and continuous functions is monadic over **Set**. The monad involved assigns, to each set X, the set βX of ultrafilters on X. While Moore-Smith characterized general topological spaces in terms of convergence of nets, M. Barr [in *Reports of the Midwest Category Seminar, IV*, 39–55, Lecture Notes in Mathematics, Vol. 137. Springer, Berlin, 1970; MR0262140] obtained a neat ultrafilter-convergence version: topological spaces are relational (or lax) β-modules. There have been recent developments in this direction, for example, by M. Mahmoudi, C. Schubert and W. Tholen [Appl. Categ. Structures **14** (2006), no. 3, 243–249; MR2248546]. By the way, Duskin [op. cit.] used his result mentioned above to show that the opposite of the category of compact Hausdorff spaces is monadic over **Set**.

Rings are not merely simultaneous abelian groups and monoids; there is an extra axiom: the distributive law. The fifth paper of the collection under review brilliantly abstracts the content of this situation to pairs of monads on a given category. A distributive law of a monad S over a monad T on the same category $\mathcal{X}$ is a natural transformation $\lambda\colon ST \Rightarrow TS$ satisfying five conditions. This allows the lifting of the monad T to a monad $\tilde{T}$ on $\mathcal{X}^S$ and also determines a monad structure on the composite functor TS; what is more, the category $(\mathcal{X}^S)^{\tilde{T}}$ is isomorphic to $\mathcal{X}^{TS}$. Distributive laws have been used quite a bit in the literature. Many people realized that the notion of monad could make sense on any object in any 2-category. The reviewer pointed out in [J. Pure Appl. Algebra **2** (1972), no. 2, 149–168; MR0299653] that distributive laws themselves are precisely the monads in an appropriate 2-category of monads; dualities allow one to consider distributive laws involving two comonads and mixed distributive laws involving a monad and comonad. More recently, S. Lack and the reviewer [J. Pure Appl. Algebra **175** (2002), no. 1-3, 243–265; MR1935981] defined a concept called wreath, generalizing distributive laws, so that TS is called the wreath product. Distributive laws are implicit in works on quantum groups involving Yang-Baxter operators and twisted products; as an example, see [D. Hobst and B. Pareigis, J. Algebra **242** (2001), no. 2, 460–494; MR1848955].

Lawvere's contribution to the volume under review deals with ordinal sums and equational doctrines. The suggestion is to study categories with equational structure using monads on the category Cat of categories (these monads are the doctrines). The connection between monads and simplicial sets goes back to Godement, but Lawvere gives deep insights into the interplay between the simplicial category $\mathbf{\Delta}$ and monads, showing for example that the Eilenberg-Moore construction is adjoint to the Kleisli construction. He obtains a doctrine of adjoint monads and of Frobenius monads which are relevant to recent papers on topological quantum field theory; he also reveals his interest in examples involving toposes.

In their paper on categories with models, Appelgate and Tierney consider a functor $I\colon \mathbf{M} \to \mathbf{A}$ and look at when the Eilenberg-Moore category of G-coalgebras for the density comonad $G = \mathrm{Ran}_I(I)$ is equivalent to the presheaf category on $\mathbf{M}$. They give lots of examples where this illustrates the passage from local object to global objects: simplicial spaces, simplicial modules, manifolds, G-bundles, and G-spaces. Appelgate and Tierney's later paper [in *Reports of the Midwest Category Seminar. IV*, 56–99, Lecture Notes in Math., 137, Springer, Berlin, 1970; MR0265429] would create a tower of comonads built on this.

Barr and Beck define their general homology $H_n(X, E)_G$, where X is an object of a category $\mathbf{C}$, G is a comonad on $\mathbf{C}$, and $E: \mathbf{C} \to \mathcal{A}$ is a functor into an abelian category. It is the homology of the associated chain complex in $\mathcal{A}$ of the simplicial object in $\mathcal{A}$ obtained from applying E to the usual simplicial object resolving X using G. Many existing homology constructions are shown to fit this mold. For non-additive examples Barr and Beck note that a comonad G on $\mathbf{C}$ moves up to a comonad G/X on the slice category $\mathbf{C}/X$ and E to $E_X: \mathbf{C}/X \xrightarrow{\text{domain}} \mathbf{C} \xrightarrow{E} \mathcal{A}$, yielding homology groups $H_n(p, E_X)_{G/X}$ for any object $p: C \to X$ of $\mathbf{C}/X$. In his 1967 thesis, Beck [op. cit.] observed that for an object Π of the category $\mathbf{Gp}$ of groups, the abelian group objects in $\mathbf{Gp}/\Pi$ are Π-modules. This observation is used to obtain the usual homology of groups from the comonad on $\mathbf{Gp}$ generated by the underlying functor $\mathbf{Gp} \to \mathbf{Set}$ and its left adjoint. General properties (like exactness) and axioms characterizing $H_n(X, E)_G$ are proved. In additive situations, these homologies are shown to be calculable using the Eilenberg-Moore theory of relative projectives [S. Eilenberg and J. C. Moore, Mem. Amer. Math. Soc. No. **55** (1965), 39 pp.; MR0178036]. Barr recently refreshed ideas on this subject in his book [*Acyclic models*, Amer. Math. Soc., Providence, RI, 2002; MR1909353].

Paper nine, also by Barr, defines the derived functors $H^n(G; -, E): \mathcal{C} \to \mathcal{A}$ of a functor E (as above) with respect to a comonad G on the domain category $\mathcal{C}$ of E: take the homology of the composite simplicial object $\mathcal{C} \xrightarrow{G^\bullet} [\Delta^{\text{op}}, \mathcal{C}] \xrightarrow{[1,E]} [\Delta^{\text{op}}, \mathcal{A}]$ in the functor category $[\mathcal{C}, \mathcal{A}]$. For a distributive law $\lambda: G_1 G_2 \to G_2 G_1$ between comonads, so that $G_2 G_1$ is also a comonad, the author is interested in $H^n(G_2 G_1; -, E)$ and calculates it in some examples. There is an Appendix with many computational proofs, which the author calls generally unenlightening; however, I once entertained a vacation scholar by having him prove these results with string diagrams, and that was fun.

For the penultimate paper of the volume, by Barr as well, I cannot do better than recommend the excellent review of the original publication [M. Barr, in *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, 357–375, Springer, Berlin, 1969; MR0271192], written by Duskin, who would later find an interpretation for general monad cohomology [Mem. Amer. Math. Soc. **3** (1975), issue 2, no. 163, v+135 pp.; MR0393196].

M. André's book [*Méthode simpliciale en algèbre homologique et algèbre commutative*, Lecture Notes in Math., 32, Springer, Berlin, 1967; MR0214644] appeared when I was a graduate student, and it interested me greatly. I noticed the connection with Kan extensions. As a fresh postdoctoral fellow at the University of Illinois, I was happy to see the last paper of SLNM 80, by F. Ulmer, wherein he examined these matters in depth. I recommend Kelly's combined review of two slightly later papers by Ulmer [in *Category Theory, Homology Theory and their Applications, I (Battelle Institute Conference, Seattle, Wash., 1968, Vol. One)*, 181–204, Springer, Berlin, 1969; MR0257186; in *Category Theory, Homology Theory and their Applications, II (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Two)*, 278–308, Springer, Berlin, 1969; MR0257187].

In the Preface to the reprint, Barr remarks that "the papers in this volume have become more an end than a beginning". I cannot agree; I still see monads and concepts from this volume in lots of ongoing research. May I also add my thanks to the body of volunteers who did the retyping. *R. H. Street*