MR2730089 (2012d:18001) 18-03 00A30 01A60 03G30

Marquis, Jean-Pierre (3-MTRL-Q)

★**From a geometrical point of view.**

A study of the history and philosophy of category theory.
Logic, Epistemology, and the Unity of Science, 14.
Springer, *Dordrecht*, 2009. *x*+309 *pp.* $199.00. *ISBN* 978-1-4020-9383-8

The creation of a book such as this marks an interesting point in the history and philosophy of category theory. Based in the city of Montréal, the author is at a major categorical centre, well placed to review the history and to develop a philosophical angle. The subject of the book fits well into the publisher's series "Logic, Epistemology, and the Unity of Science", whose stated aim on page (ii) is "to provide an integrated picture of the scientific enterprise in all its diversity".

While reading the book, I used the opportunity to revisit several seminal papers and to marvel at the advances they represented. The author clearly studied each paper thoroughly, had private communications with some pioneers, and succeeded quite well in describing the mathematical community's mind set at the time of those advances.

We are reminded that a problem raised by Karol Borsuk and Samuel Eilenberg in 1937 (to determine the homotopy classes of maps from the complement of a solenoid in the 3-sphere to the 2-sphere) led to the collaboration of Eilenberg and Saunders Mac Lane, and hence to category theory. Mac Lane's work on group extensions involved a result about the group $\mathbb{Z}[\frac{1}{p}]$, which Eilenberg pointed out was dual to the topological p-adic solenoid group.

In the paper [Trans. Amer. Math. Soc. **58** (1945), 231–294; MR0013131], where categories were first defined, Eilenberg and Mac Lane finished the Introduction by saying: "This may be regarded as a continuation of the Klein Erlanger Programm, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings." The title and philosophy of the book under review are grounded in that sentence. The author argues its truth; but in an expanded sense well beyond what Eilenberg and Mac Lane could have understood before the definition of adjoint functors by Daniel Kan [Trans. Amer. Math. Soc. **87** (1958), 294–329; MR0131451]. The claim is that such a continuation was a minor aspect for Eilenberg and Mac Lane at the time; their main purpose was to define naturality in a setting appropriate for all of mathematics. They did not think categories as such warranted study.

The author carries this point further on page 84 in discussing Mac Lane's paper [Bull. Amer. Math. Soc. **56** (1950), 485–516; MR0049192], by suggesting that dual category was not mentioned therein because Mac Lane was thinking only of concrete categories. However, product categories and dual categories were defined clearly in Section 13 of the 1945 paper to show that functors of several covariant and several contravariant variables could be viewed as functors with a single domain category. The 1945 paper gave the example of direct and inverse systems as functors; again, the domain category there was quite abstract. Functor categories were also there in 1945. On the other hand, I agree that some of the notions introduced in the 1950 paper seem now to have been misdirected. Yet what were called bicategories in that paper can be seen as a step in the evolution of factorization systems. Once, when I reminded Mac Lane that Jean Bénabou had asked him for agreement to use the term bicategory in a different sense, Mac Lane had himself by then forgotten his 1950 use of the term.

Mac Lane [Proc. Nat. Acad. Sci. U. S. A. **34** (1948), 263–267; MR0025464] defined the product of two objects in a category by the universal property emphasizing that the concept dualized by formally reversing arrows, as exemplified by direct product and free product in the category of groups. It thus emerged that isomorphism was the important identity between objects in a category.

The important notion of identity between categories themselves came later. The author reminds us that the 1945 paper did not define equivalence of categories. To mimic homotopy equivalence involved thinking of a category as analogous to a space. Alexander Grothendieck defined equivalence of categories in the innocently entitled revolutionary memoir, "Sur quelques points d'algèbre homologique" [Tôhoku Math. J. (2) **9** (1957), 119–221; MR0102537]. Indeed I might expand on the footnote on page 94 of the present paper to say that Grothendieck recognized higher coherence conditions by actually defining what we now call adjoint equivalence (although his two natural transformations both went in the unit direction and there was a typographical error in not saying they were invertible; but invertibility was clearly intended since both inverses were used in the coherence conditions). To emphasize how terminology had not settled and to reinforce the author's philosophy, I point to the discussion in the paper [Proc. Cambridge Philos. Soc. **60** (1964), 721–735; MR0167509] on equivalence of categories; there G. M. Kelly used the defensible term isomorphism of categories, whereas an invertible morphism in a category he called an equivalence.

Of course, the books [S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, New Jersey, 1952; MR0050886] and [H. P. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, NJ, 1956; MR0077480] could not have existed without functors. Actually, the categories there were very concrete. The appendix, by Buchsbaum, to the latter of these books clarified the essential properties of categories of modules over rings needed for the book. The purpose of the Tôhoku paper was to develop homological algebra to include categories of module-valued sheaves, opening up new geometric thought. The Tôhoku contribution is thoroughly discussed in Section 3.2 of the present book, where the author makes a point about the paper looking at categories with extra structure. The footnote on page 99 shows that the author is aware that the Tôhoku notions of additive (including existence of finite products) and abelian are actually properties a category either has or does not; however the reader might be misled by assertions elsewhere to think a category might be additive in several different ways.

The present book, correctly in my view, attributes great significance to the 1958 definition by Kan of adjoint functors. Working rather independently on his combinatorial approach to homotopy theory, Kan recognized the omnipresence of adjoints (with examples provided by tensor and hom, the singular functor and geometric realization, loop space and suspension) and was encouraged to publish by Eilenberg, who suggested the name motivated by adjoint operators in functional analysis. Incidentally, this leads me to agree with the present author's comments on Stone's remarks (see page 111) about Eilenberg and Mac Lane lacking knowledge of contemporary Hilbert space theory.

Kan's paper includes the definition of the colimit and limit (called direct and inverse limit) of an arbitrary functor, showing they give the values of adjoints for diagonal functors. What are now called Kan extensions are also there with the formula in terms of limits. The formula added the subdivision category to our list of constructions on categories themselves.

As to be expected the present book discusses foundations for mathematics that are appropriate for, and may even involve, category theory. Many pages are dedicated to the views of logicians such as Kreisel about the special needs, or not, of category theory. The author argues that there was possibly some justification for Kreisel's view in the 1960s, but no longer. He concludes by the end of the book (page 289): "Moreover, unlike the use of formal systems in foundations, the use of category theory has actually shaped the development of mathematics itself, thus extending the notion of foundations to include the structure of modern mathematics."

The influence of Lawvere's work in the early 1960s is examined: the proposal for the category of categories as foundations, algebraic theories as categories, and axioms for the category of sets.

The importance that Eilenberg and Mac Lane attached to functors is by this time confirmed. Kan saw his spaces as functors: simplicial sets were functors defined on the dual of the simplicial category. Lawvere saw algebras as functors: models of the algebraic theory.

The author returns often to develop his view on how category theory generalizes and continues the Klein Programm. In the first chapter the Klein concept of geometry using the transformation groups is nicely explained using a philosophical fable involving three geometers A, B and C doing apparently different things with apparently different spaces until geometer K turns up to show them they are doing the same geometry via isomorphisms of transformation groups. The author claims (page 102) that "from 1957 onwards, it became possible to think that abstract categories are to mathematical structures what transformation groups are to geometric structures". Then (page 113): "My main claim is that adjoint functors are to categories what automorphisms are to geometric spaces."

Topos theory is discussed in the last chapter before the Conclusion. The Grothendieck view of toposes as generalized spaces and maps as adjoint pairs fits perfectly with the author's generalization of Klein. Some insight into elementary toposes and geometric theories is included.

A work such as this cannot include everything and the author notes that he does not say much about higher-dimensional categories, monoidal categories, quantum logic, or linear logic. There are places where I thought enriched category theory might naturally have been mentioned: at the bottom of page 24, the definition of a geometry involves a Lie group so that categories enriched in smooth manifolds are appropriate; on page 47 we have homs forming abelian groups. Enriched categories are mentioned in a footnote on page 140. The footnote on page 210 mentions the Rosebrugh-Wood characterization of the category of sets in terms of adjoints involving the Yoneda-Grothendieck embedding: this accords with the philosophy of the book and might have been discussed more. The same remark applies to the lack of mention of the lex-total properties of Grothendieck abelian categories and toposes. To me the Definition 6.5 on page 227 of the finite limit Ehresmann sketch is unduly involved: to say simply that it is a small category with chosen finite cones seems more in the spirit of the book. Finally, I am surprised Myles Tierney's paper [in *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, NS, 1971)*, 13–42, Lecture Notes in Math., 274, Springer, Berlin, 1972; MR0373888] on forcing is not in the References.

I have a few minor criticisms. Too often the author says something like "we will see more clearly why later" without saying where; so it is hard to be on the lookout. Sometimes paraphrasing leaves a bit to be desired: I am happy with the 1945 paper's definition using aggregates but not with the paraphrasing on page 45; also, subfunctor in the generality of page 64 requires inclusions which the 1945 paper had because they restricted to categories of groups; the explanation around the square diagram on page 130 misses something which is in the Kan paper.

There are some typographical errors in mathematical explanations that could cause confusion: on page 25, line 5, X' should be X; there is a G missing in the diagram at the top of page 149; page 153, line 6, "diagrams" should be "triangles"; page 157, lines 6 and 8, delete "distributive"; page 275, line 17, and page 276, three lines before Section 7.5, one of the categories in the equivalence is missing.

May I conclude by recommending the book. I am glad it was written and am glad it was assigned to me to review. *R. H. Street*