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Homological algebra. The interplay of homology with distributive lattices and orthodox semigroups. (English) Zbl 1280.18001

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Homological algebra concerns algebraic aspects of the homology of geometric structures. It deals with chain complexes and constructions on them. In particular, the construction of a spectral sequence from a filtered chain complex lies in this area. Traditional books on homological algebra develop these concepts and constructions in an abelian category and apply the work to specific categories of modules over a ring or categories of sheaves of modules. Ultimately the subject is aimed at understanding traditional mathematical structures such as topological spaces, groups, rings, algebras, and Lie algebras.

The book under review is at a different level. Perhaps the subject might be called meta-homological-algebra. It analyses in detail the homological theories behind the structures arising in homological algebra. Yet this meta-mathematics, in the spirit of [*H. Rasiowa* and *R. Sikorski*, The mathematics of metamathematics. Warszawa: Panstwowe Wydawnictwo Naukowe (1963; Zbl 0122.24311)], can be studied using mathematics. The more relevant structures for this study are modular lattices, distributive lattices, and orthodox semigroups.

To arrive at the notion of homological theory it is fundamental to understand where the models should be taken. The author's tenet is that the practice of homological algebra belongs in p-exact categories, where the p is for *D. Puppe* [Math. Ann. 148, 1–30 (1962; Zbl 0109.25201)]; they are called exact categories in [*B. Mitchell*, Theory of categories. New York and London: Academic Press (1965; Zbl 0136.00604)]. Abelian categories are p-exact categories with binary products. It is necessary to take the more general context since the universal models of homological theories constructed here are seldom abelian categories.

In providing one presentation of what the book is about, the Introduction discusses coherence. In a distributive p-exact category, we have coherence: diagrams of canonical morphisms between subquotients all commute. Only the trivial abelian category is distributive. So, to gain coherence, it is necessary to move to a distributive expansion p-category Dst$\mathbf{E}$ for a general p-exact category $\mathbf{E}$. The author says that Chapter 1 also provides a more elementary introduction to the book. It leads to the Birkhoff theorem on modular lattices generated by two chains, since this is the basis of many of the universal models constructed later. The category Mlc of modular lattices and connections (these morphisms are particular kinds of adjoint pairs between the lattices) plays an important role: it is a p-exact category which receives the transfer (or subobject) functor $\mathrm{Sub} : \mathbf{E} \longrightarrow \mathrm{Mlc}$.

Properties of p-exact categories $\mathbf{E}$ are presented in Chapter 2. While not necessarily having all pullbacks and pushouts, they do have pullbacks along arbitrary morphisms of monomorphisms and pushouts along arbitrary morphisms of epimorphisms. A relation from A to B in $\mathbf{E}$ is an isomorphism class of diagrams $A \leftarrowtail X \twoheadrightarrow Y \twoheadleftarrow Z \rightarrowtail B$ with the first and last morphisms monomorphisms and the middle two epimorphisms. The locally ordered 2-category Rel$\mathbf{E}$ has the same objects as $\mathbf{E}$, the morphisms are relations, and the 2-cells are induced by morphisms of diagrams with identities at A and at B. The trick is in the composition of relations which goes back to *M. S. Tsalenko* [Sov. Math., Dokl. 5, 416–418 (1964); translation from Dokl. Akad. Nauk SSSR 155, 292–294 (1964; Zbl 0192.10801); Mat. Sb., N. Ser. 73(115), 564–584 (1967; Zbl 0164.01401)]. There is a functor $\mathbf{E} \longrightarrow \mathrm{Rel}\mathbf{E}$ taking $f : A \to B$ to the class of $A@ > 1_A >> \leftarrowtail A@ > e >> \twoheadrightarrow Y@ > 1_Y >> \twoheadleftarrow Y@ > m >> \rightarrowtail B$, where $f = me$ is the image factorization. There is a reversal operation on relations allowing the definition of exact squares in $\mathbf{E}$ as the commutative squares which remain commutative when taken into Rel$\mathbf{E}$ and having an opposite pair of sides reversed.

Chapters 3 and 4 focus on 2-categories of the form Rel$\mathbf{E}$ for $\mathbf{E}$ p-exact; these 2-categories are so-called RE-categories. The 2-category of RE-categories is complete, and this is used in Chapter 5 to construct desired universal models. The appropriate theories (RE-theories) T can be defined in a semantical way: they consist of a small graph Δ with some distinguished graph morphisms $t : \Delta \to \mathbf{A}$ into RE-categories $\mathbf{A}$, subject to three axioms. A universal model of T is a graph morphism $t_0 : \Delta \to \mathbf{A}_0$ through which all the distinguished graph morphisms t factor.

Chapter 6 provides explicit universal models of homological theories. This is done for bifiltered objects, sequences of relations, bifiltered chain complexes (in which lie spectral sequences, amongst other things), real filtered chain complexes, Eilenberg exact systems (as arise from homologies of quotients F_pC/F_qC, $q \leq p$ in a filtration F_*C of a chain complex C), Massey exact couples, and so on.

There is an appendix on the required category theory and another including the proof of the Eilenberg exact system example. The book is a fine culmination to many papers of the author going back to 1967.

Reviewer: Ross H. Street (North Ryde)

MSC:

18-02	Research exposition (monographs, survey articles) pertaining to category theory
18Gxx	Homological algebra in category theory, derived categories and functors
18D05	Double categories, 2-categories, bicategories and generalizations (MSC2010)
18E10	Abelian categories, Grothendieck categories
18G50	Nonabelian homological algebra (category-theoretic aspects)

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homological theory; modular lattices; Puppe exact categories; spectral sequence; Zeeman diagram; orthodox semigroups; relation in a category; chain complexes; distributive lattices; exact categories; coherence

Full Text: DOI