Gurski, Nick

Coherence in three-dimensional category theory. (English) Zbl 1314.18002

Cambridge Tracts in Mathematics 201. Cambridge: Cambridge University Press (ISBN 978-1-107-03489-1/hbk; 978-1-139-54233-3/ebook). vii, 278 p. (2013).

For a given algebraic structure, interpreting algebraic in a very general way, we can say that coherence in the sense of this book's title concerns the equality of different expressions when evaluated in the free structures. Those equalities then hold in all models. Higher-dimensional categorical structures are algebraic in this sense. There is also a concept of equivalence between these categorical structures. For weak n-categories, such equivalences detect equality of n-cells. So a theorem stating that every weak n-categories is equivalent to one of a stricter kind (that is, one in which more equalities hold than in general) yields coherence information.

In [*R. Gordon* et al., Coherence for tricategories, Mem. Am. Math. Soc. 558, 81 p. (1995; Zbl 0836.18001)], the authors gave a definition of tricategory (= weak 3-category) and proved such an equivalence theorem. The stricter form of tricategory was a category with homs enriched in the symmetric monoidal closed category **Gray**. Here **Gray** is the category of 2-categories and 2-functors made closed with the internal hom $\mathbf{Hom}(A, B)$ taken to be 2-category with objects 2-functors $A \to B$, morphisms pseudonatural transformations, and 2-cells modifications. Bob Gordon and John Power had remarked that the reviewer's Yoneda Lemma for bicategories [*R. Street*, Cah. Topologie Géom. Différ. Catégoriques 21, 111–159 (1980; Zbl 0436.18005)] gave an easy proof that every bicategory is equivalent to a 2-category. So what became [Zbl 0836.18001] began by trying to obtain the tricategory coherence by first proving a Yoneda Lemma for tricategories.

Meanwhile, in BTC [*A. Joyal* and *R. Street*, Adv. Math. 102, No. 1, 20–78 (1993; Zbl 0817.18007)] we applied the Gordon-Power remark to one-object bicategories; that is, monoidal categories. André Joyal and the reviewer noted that less than the "presheaf" construction was needed to prove every monoidal category equivalent to a strict one. We then went on to deduce Mac Lane's coherence theorem for monoidal categories and to deduce a coherence result for strong monoidal functors.

Because of the complexity of tricategories and for the sake of easier reading, we decided that [Zbl 0836.18001] should be a geodesic path to the equivalence form of the coherence theorem for tricategories. We adapted the strategy of the work with Joyal and avoided the full Yoneda Lemma for tricategories.

Parts I and II of Nick Gurski's book successfully takes on the full program on these matters and does it in a readable way. The work begins in Section 2 by developing the several object case of the first part of [Zbl 0817.18007] to obtain coherence for bicategories and then for pseudofunctors between bicategories (called homomorphisms by Bénabou and now simply called functors by Gurski). Section 3 treats cubical functors and **Gray**. Part II culminates in using a cubical Yoneda Lemma to prove the [Zbl 0836.18001] coherence for tricategories, going on to prove the free tricategory form, and then treating coherence for the appropriate trifunctors.

Having dealt with coherence for tricategories, Gurski turns in Part III to the theory internal to these ambient structures. He generalizes from two to three dimensions the perspective of *S. Lack* [J. Pure Appl. Algebra 175, No. 1–3, 223–241 (2002; Zbl 1142.18301)] (using a private remark of John Bourke) and *A. J. Power* [J. Pure Appl. Algebra 57, No. 2, 165–173 (1989; Zbl 0668.18010)]. Indeed, Part III extends the analysis of *J. Power* [Contemp. Math. 431, 405–426 (2007; Zbl 1131.18005)].

No longer do I find that the mention of 2-category or bicategory causes a huge stir. To those researchers who find they need to go the extra step (see [*A. J. Power*, Inf. Comput. 120, No. 2, 251–262 (1995; Zbl 0829.18003)] for why you might), Gurski's book is a gift. It is so easy in this topic to flood the main thread with large complex diagrams. The author has avoided doing so, without sacrificing precision. This is a scholarly work with many subtleties, of interest to any reader, including the experts.

Reviewer: Ross H. Street (North Ryde)

MSC:

18A99	General theory of categories and functors
18B99	Special categories
18-02	Research exposition (monographs, survey articles) pertaining to category theory

Keywords:

tricategory; Gray category; coherence; braiding; bicategory

Full Text: DOI