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★**Mackey 2-functors and Mackey 2-motives.**

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As an abstract concept, Mackey functors appeared in 1971 with applications to the representation theory of finite groups [see J. A. Green, J. Pure Appl. Algebra **1** (1971), no. 1, 41–77; MR0279208; A. W. M. Dress, *Notes on the theory of representations of finite groups. Part I*, Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971; MR0360771]. For these authors a Mackey functor $\mathcal{M}$ with domain category $\mathcal{C}$ was a pair of functors $\mathcal{M}_*, \mathcal{M}^*$ equal on objects, where $\mathcal{M}_*$ (called induction on morphisms) had domain $\mathcal{C}$, and $\mathcal{M}^*$ (called restriction on morphisms) had domain $\mathcal{C}^{\rm op}$, satisfying some axioms abstracting representation theory. Given a finite group G, for Green, $\mathcal{C}$ was equivalent to the category of connected finite G-sets while for Dress, $\mathcal{C}$ was the category $[G^{\rm op}, {\rm set}]$ of finite G-sets. Dress's requirement that $\mathcal{M}^*$ should preserve finite products meant the concepts were equivalent.

Categorical insight was provided by H. Lindner [Manuscripta Math. **18** (1976), no. 3, 273–278; MR0401864], who showed that Mackey functors with domain $\mathcal{C}$ were equivalent to single finite-product-preserving functors with domain the category of spans in $\mathcal{C}$. Spans are a significant device in the book under review and the Lindner result is an instance of the authors' motivic approach, as clarified in Chapter 6.

On page 3 we learn briefly that the purpose of the book is four-fold: to lay the foundations of Mackey 2-functor theory, to catalogue justifying examples, to provide some applications, and to construct a motivic approach. On this last point, at the bottom of page 4 the authors give a correspondence between the steps in Grothendieck motive theory and what they will do with 2-motives. Algebraic varieties are replaced by finite groups, Weil cohomologies by Mackey 2-functors. The initiality of the category of pure motives becomes the initiality of the 2-category of Mackey 2-motives. Just as decomposition of a variety X into a direct sum of simpler motives means the Weil cohomology at X also decomposes, a decomposition of the 2-motive of a finite group G means every Mackey 2-functor at G decomposes.

The heroes of the book are finite groupoids: categories with a finite number of morphisms, all invertible. Finite groups are identified with one-object finite groupoids. In order to see Mackey functors as related to lower-dimensional Mackey 2-functors, as explained by Theorem B.0.12 in Appendix B at the end of the book, we must realise that a G-set X can be identified with the discrete fibration over G obtained as the category of elements of the functor $X\colon G^{\rm op} \to {\rm Set}$. The point needed is that each faithful functor $E \to G$ from a groupoid E to the group G is equivalent over G to such a discrete fibration.

Now to the definition of Mackey 2-functor. The 2-category of finite groupoids, functors, and natural transformations is denoted by gpd. The 2-category of abelian-group-enriched categories admitting finite coproducts, additive functors, and natural transformations, is denoted by ADD. For any 1-morphism u in gpd, the authors write u^* for $\mathcal{M}u$. While a Mackey 2-functor is a (strict) 2-functor $\mathcal{M}\colon {\rm gpd}^{\rm op} \to {\rm ADD}$, for the axioms, it is better to think of it as a pseudofunctor between bicategories. Finite bicategorical products are to be preserved by $\mathcal{M}$. For each faithful functor $i\colon H \to G$ between finite groupoids, there are adjunctions $i_* \dashv i^* \dashv i_*$. The two Beck-Chevalley conditions should be satisfied at each square displaying a (bicategorical) comma category in gpd of the form i/u with i faithful (they call these Mackey squares): the square commutes up to a natural transformation $\gamma : i \circ v \Rightarrow u \circ j$ (invertible since we are dealing with groupoids) and the conditions say that the mate $\gamma_l\colon j_* \circ v^* \Rightarrow u^* \circ i_*$ of γ (using adjunctions $i_* \dashv i^*$ and $j_* \dashv j^*$) and the mate $\gamma_r^{-1}\colon u^* \circ i_* \Rightarrow j_* \circ v^*$ of γ^{-1} (using adjunctions $i^* \dashv i_*$ and $j^* \dashv j_*$) should both be invertible.

There is a Mackey 2-Functor Rectification Theorem 3.4.3. It tells us *inter alia* that the units and counits of the two adjunctions $i_* \dashv i^* \dashv i_*$ can be re-chosen whereby γ_r^{-1} becomes the inverse of γ_l. The proof involves some rather tricky analysis of the Mackey square for i/i and the diagonal functor $d\colon H \to i/i$. To indicate the kind of information in that square let me point out that, if i is also full then d is an equivalence and we have a Mackey square with d and γ identities, in which case, γ_l is the unit for $i_* \dashv i^*$. For any faithful $i\colon H \to G$ the authors prove that their choice of adjunctions is such that the counit for $i^* \dashv i_*$ is left inverse to the unit for $i_* \dashv i^*$. Consequently, the functor $i_*\colon \mathcal{M}H \to \mathcal{M}G$ is separably monadic provided $\mathcal{M}$ takes values in additive categories in which idempotents split.

Before moving to Chapter 4 on Examples, I should admit that the authors do define Mackey 2-functors where gpd is replaced by a more general groupoid-enriched category $\mathbb{G}$ and faithful functors by some class $\mathbb{J}$ of morphisms of $\mathbb{G}$. At first $\mathbb{G}$ is a locally full sub-2-category of gpd (closed under what you would expect) but for later purposes it is even more general.

All additive derivators [see M. Groth, Algebr. Geom. Topol. **13** (2013), no. 1, 313–374; MR3031644] give examples. Derivators are 2-functors $\mathcal{D}\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{ADD}$ satisfying various conditions. The restriction of an additive $\mathcal{D}$ to $\mathrm{gpd}^{\mathrm{op}}$ is a Mackey 2-functor (Theorem 4.1.1). The main work in showing this is to prove that the left and right adjoints of $i^* = \mathcal{D}i$ are isomorphic for i a faithful functor between finite groupoids. Some examples of additive derivators are explained: model categories with additive homotopy category, linear representations, derived categories, and spectra; but there are others. The other sections in Chapter 4 discuss quotients of Mackey 2-functors by Mackey sub-2-functors, extending group examples to groupoids, and Mackey 2-functors of equivariant objects.

Chapter 5 is the longest (50 pages) and deals with bicategories of spans in groupoid-enriched categories. String diagrams as well as pasting diagrams are employed to discuss the desired universal properties of span constructions.

The bicategory $\widehat{\mathrm{Span}}(=\widehat{\mathrm{Span}}(\mathbb{G};\mathbb{J}))$ of Mackey 2-motives is defined in Chapter 6. For the purposes of this review, let me denote by $\widehat{\mathcal{S}}$ the tricategory defined as follows: objects are those of $\mathbb{G}$; morphisms $G \xrightarrow{(u,P,i)} H$ are spans $G \xleftarrow{u} P \xrightarrow{i} H$ in $\mathbb{G}$ with $i \in \mathbb{J}$; 2-morphisms $(u,P,i) \overset{(w,R,k,\alpha,\beta)}{\Longrightarrow} (v,Q,j)$ are spans $P \xleftarrow{w} R \xrightarrow{k} Q$ with $uw \overset{\alpha}{\Rightarrow} vk$ and $iw \overset{\beta}{\Rightarrow} jk$ in $\mathbb{G}$; and 3-morphisms $(w,R,k,\alpha,\beta) \xrightarrow{(\xi,f,\zeta)} (w',R',k',\alpha',\beta')$ consist of $R \xrightarrow{f} R'$ and $w \overset{\xi}{\Rightarrow} w'f$ and $\zeta\colon k \overset{\zeta}{\Rightarrow} k'f$ in $\mathbb{G}$ such that $v\zeta \cdot \alpha = \alpha' f \cdot u\xi$ and $j\zeta \cdot \beta = \beta' f \cdot i\xi$. Then $\widehat{\mathrm{Span}}$ is the bicategory obtained from $\widehat{\mathcal{S}}$ by taking isomorphism classes of 2-morphisms. There is an identity-on-objects pseudofunctor $(-)^*\colon \mathbb{G}^{\mathrm{op}} \to \widehat{\mathrm{Span}}$ taking $H \xrightarrow{g} G$ to $G \xleftarrow{g} H \xrightarrow{1_H} H$. For $(H \xrightarrow{i} G) \in \mathbb{J}$, we also have the morphism $i_* = (1_H, H, i)$ in $\widehat{\mathrm{Span}}$. The great thing is that $i_* \dashv i^* \dashv i_*$ and, for the 2-morphism γ in a Mackey square, γ_r^{-1} is indeed the inverse of γ_l. Theorem 6.1.13 tells us that the pseudofunctor $(-)^*\colon \mathbb{G}^{\mathrm{op}} \to \widehat{\mathrm{Span}}$ has the universal property with respect to these properties. (A rare typographical error: there is an important widehat missing in the first line of page 138.) As a consequence (Theorem 6.3.6), the bicategory of Mackey 2-functors is biequivalent to the bicategory of finite-product-preserving pseudofunctors $\widehat{\mathrm{Span}} \to \mathrm{ADD}$.

Chapter 7 is directed at the decomposition results for the original $\mathbb{G} = \mathrm{gpd}$ case. For these decompositions to involve direct sums, certain additive structures need to be freely adjoined to previous constructions. The enabling result is Theorem 7.4.5, which identifies the linearised monoid $\widehat{\mathrm{Span}}(G,G)(1_G,1_G)$ as isomorphic to the crossed Burnside ring $\mathrm{B}^c(G)$ of the finite group G. If G^c denotes the G-set G with conjugation action then $\mathrm{B}^c(G)$ is the Grothendieck ring of the slice category $[G,\mathrm{set}]/G^c$. Now $[G,\mathrm{set}]/G^c$ is the monoidal centre of the cartesian monoidal category $[G,\mathrm{set}]$ [for example, see B. J. Day, E. Panchadcharam and R. H. Street, in *Hopf algebras and generalizations*, 1–17, Contemp. Math., 441, Amer. Math. Soc., Providence, RI, 2007; MR2381533]. The monoidal centre of a monoidal category $\mathcal{V}$ is obtained by regarding $\mathcal{V}$ as a one-object bicategory and forming $\mathrm{Bicat}(\mathcal{V},\mathcal{V})(1_{\mathcal{V}},1_{\mathcal{V}})$ where Bicat is the tricategory of bicategories, pseudofunctors and pseudonatural transformations. This makes me wonder whether there is something more to be made of the fact that the two sides of the isomorphism of Theorem 7.4.5 come from instances of a construction which yields a braided monoidal category $\mathcal{X}(X,X)(1_X,1_X)$ from any chosen object X of a tricategory $\mathcal{X}$; the tricategories in mind are Bicat and $\widehat{\mathcal{S}}$.

Using the identification in Theorem 7.4.5, the authors obtain Corollary 7.5.4: for any finite group G and any Mackey functor $\mathcal{M}$ taking values in Cauchy complete additive categories, each finite-product decomposition of $\mathrm{B}^c(G)$ into a product of rings gives a related finite direct sum decomposition of the additive category $\mathcal{M}(G)$.

Appendix A provides background category and bicategory material while, as mentioned, Appendix B is about ordinary Mackey functors. *R. H. Street*