Functorial Complex Analysis

Let $\mathcal{A}$ denote the category whose objects are finite dimensional unital associative algebras over the complex number field $\mathbb{C}$ and whose arrows are algebra homomorphisms. Let

$$U : \mathcal{A} \longrightarrow \mathcal{S}$$

be the forgetful functor into the category $\mathcal{S}$ of sets. A natural transformation

$$\theta : U \longrightarrow U$$

is said to be *continuous* when, for all $A \in \mathcal{A}$, the component $\theta_A : UA \longrightarrow UA$ is continuous for the canonical topology on the set UA transported across any linear isomorphism $A \cong \mathbb{C}^{\dim A}$.

Schanuel Theorem [Sch] *There is a bijection between continuous natural transformations*

$$\theta : U \longrightarrow U$$

and analytic functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ *under which* θ *corresponds to* $f = \theta_{\mathbb{C}}$.

Proof One question which must be addressed is why, for continuous natural $\theta : U \longrightarrow U$, is $f = \theta_{\mathbb{C}}$ analytic. The two projections $p_i : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ are algebra homomorphisms, so the following squares commute for $i = 1, 2$; so $\theta_{\mathbb{C} \times \mathbb{C}} = f \times f : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{C}$.

$$\begin{tikzcd}
\mathbb{C} \times \mathbb{C} \arrow[r, "\theta_{\mathbb{C} \times \mathbb{C}}"] \arrow[d, "p_i"'] & \mathbb{C} \times \mathbb{C} \arrow[d, "p_i"] \\
\mathbb{C} \arrow[r, "\theta_{\mathbb{C}}"'] & \mathbb{C}
\end{tikzcd}$$

Consider the algebra W of upper-triangular 2×2 matrices

$$\begin{bmatrix} \lambda & \sigma \\ 0 & \mu \end{bmatrix}$$

with complex entries. For each complex number α, we have an algebra homomorphism $h_\alpha : \mathbb{C} \times \mathbb{C} \longrightarrow W$ given by

$$h_\alpha(\lambda, \mu) = \begin{bmatrix} \lambda & \alpha(\lambda - \mu) \\ 0 & \mu \end{bmatrix}.$$

Naturality implies that the following square commutes for all α.

$$\begin{tikzcd}
\mathbb{C} \times \mathbb{C} \arrow[r, "\theta_{\mathbb{C} \times \mathbb{C}}"] \arrow[d, "h_\alpha"'] & \mathbb{C} \times \mathbb{C} \arrow[d, "h_\alpha"] \\
W \arrow[r, "\theta_W"'] & W
\end{tikzcd}$$

It follows that, for all $\alpha, \lambda, \mu \in \mathbb{C}$, we have the equality

$$\theta_W \begin{bmatrix} \lambda & \alpha(\lambda - \mu) \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} f(\lambda) & \alpha(f(\lambda) - f(\mu)) \\ 0 & f(\mu) \end{bmatrix}.$$

Take $\lambda \neq \mu$ and $\alpha = 1/(\lambda - \mu)$. This gives the following identity.

$$\theta_W \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} f(\lambda) & \frac{f(\lambda)-f(\mu)}{\lambda-\mu} \\ 0 & f(\mu) \end{bmatrix}$$

Using the continuity of θ, we see that the left side of the above identity has limit

$$\theta_W \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

as μ tends to λ. It follows that the right side also has a limit as μ tends to λ; by inspection of the top right entry of the matrix, we deduce that f is differentiable. Indeed we have the formula

$$\theta_W \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} f(\lambda) & f'(\lambda) \\ 0 & f(\lambda) \end{bmatrix}.$$

It is well known that any differentiable function of a complex variable is analytic. So f is analytic.

Now suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is any analytic function. Then there is a Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

which allows us to define $\theta_A : UA \longrightarrow UA$ for any algebra $A \in \mathcal{A}$ by

$$\theta_A(a) = \sum_{n=0}^{\infty} c_n a^n .$$

Clearly this defines a continuous natural transformation $\theta : U \longrightarrow U$ with $\theta_{\mathbb{C}} = f$. This shows that the assignment of the theorem is surjective.

It remains to show the assignment is injective. Suppose $\theta, \phi : U \longrightarrow U$ are continuous and natural with $\theta_{\mathbb{C}} = f = \phi_{\mathbb{C}}$. Every algebra A is isomorphic to a subalgebra of a matrix algebra $M_n(\mathbb{C})$, so, by naturality, it suffices to show that θ, ϕ have equal components at each algebra $M_n(\mathbb{C})$. Every matrix is a limit of a sequence of matrices with distinct eigenvalues. By continuity of θ, ϕ it suffices to show their components at $M_n(\mathbb{C})$ equal on matrices with distinct eigenvalues. Yet every matrix with distinct eigenvalues is similar diagonal matrix and conjugation by an invertible matrix is an algebra homomorphism. By naturality, it suffices to show that θ, ϕ agree on diagonal matrices. By naturality using the algebra homomorphism $\mathbb{C}^n \longrightarrow M_n(\mathbb{C})$ which identifies n-vectors with diagonal matrices, it suffices to see that θ, ϕ have the same components at $\mathbb{C}^n$. By naturality using the projections (as before in the case $n = 2$), we see that the components of θ, ϕ at $\mathbb{C}^n$ are both equal to $f \times f \times \ldots \times f$. **Q. E. D.**

This motivates a definition of derivative for an arbitrary natural transformation $\theta : U \longrightarrow U$. We introduce the functor $T : \mathcal{A} \longrightarrow \mathcal{A}$ which assigns to each algebra $A \in \mathcal{A}$ the algebra $T(A)$ of upper-triangular 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

with entries $a, b \in A$. There are continuous natural transformations

$$\eta : U \longrightarrow UT \quad \text{and} \quad \tau : UT \longrightarrow U$$

whose components at the algebra A are given by

$$\eta_A(a) = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \quad \text{and} \quad \tau_A \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = b .$$

Definition The *derivative* of a natural transformation $\theta : U \longrightarrow U$ is the composite

$$\theta' \ : \ U \xrightarrow{\eta} UT \xrightarrow{\theta_T} UT \xrightarrow{\tau} U .$$

Proposition 1 *If* $\theta : U \longrightarrow U$ *is a continuous natural transformation with* $\theta_{\mathbb{C}} = f : \mathbb{C} \longrightarrow \mathbb{C}$ *then the derivative of* θ *is the unique continuous natural transformation* $\theta' : U \longrightarrow U$ *satisfying the equation* $(\theta')_{\mathbb{C}} = f'$.

Proof In the proof of Schanuel's Theorem we saw that

$$\theta_W \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} f(\lambda) & f'(\lambda) \\ 0 & f(\lambda) \end{bmatrix}$$

where W is the algebra of upper-triangular 2×2 matrices. Clearly $T(\mathbb{C})$ is a subalgebra of W, so, by naturality, $\theta_{T(\mathbb{C})}$ is just the restriction of θ_W to $T(\mathbb{C})$. So the above displayed matrix equation holds with W replaced by $T(\mathbb{C})$. Using the definition of θ', we obtain the equation $(\theta')_{\mathbb{C}} = \tau_{\mathbb{C}} \circ \theta_{T(\mathbb{C})} \circ \eta_{\mathbb{C}}$ which can be evaluated at $\lambda \in \mathbb{C}$ to yield $(\theta')_{\mathbb{C}}(\lambda) = f'(\lambda)$. Since the derivative f' of an analytic function f is analytic, indeed θ' corresponds to f' under the bijection of Schanuel's Theorem. **Q. E. D.**

We would like to contrast this approach to derivatives with that of synthetic algebraic geometry **[DG]** which would begin with the category $\mathcal{C}$ of *finitely presented commutative* $\mathbb{C}$-algebras and the category

$$\mathcal{E} = [\mathcal{C}, \mathcal{S}]$$

of set-valued functors on $\mathcal{C}$. The forgetful functor $R : \mathcal{C} \longrightarrow \mathcal{S}$ is a ring object of $\mathcal{E}$. Of course R is represented by the polynomial algebra $\mathbb{C}[x]$ in a single indeterminate x; that is, there is a natural bijection

$$R(A) \cong \mathcal{C}(\mathbb{C}[x], A).$$

Note that $\mathbb{C}[x]$ is finitely presented but not finite dimensional and so is not an object of the category $\mathcal{A}$. The analogue of the Schanuel Theorem here is a consequence of the Yoneda Lemma: *natural transformations* $\theta : R \longrightarrow R$ *are in natural bijection with polynomial functions* $p : \mathbb{C} \longrightarrow \mathbb{C}$ *via the equation* $\theta_{\mathbb{C}} = p$.

Notice that the functor $T : \mathcal{A} \longrightarrow \mathcal{A}$ can be defined on all $\mathbb{C}$-algebras, finite dimensional or not, and as such restricts to take commutative algebras to commutative algebras. In particular, we can regard T as a functor $T : \mathcal{C} \longrightarrow \mathcal{C}$. can mimic the definition of derivative to endomorphisms of R in $\mathcal{E}$: the *derivative* of $\theta : R \longrightarrow R$ is the composite

$$\theta' \ : \ R \xrightarrow{\eta} RT \xrightarrow{\theta_T} RT \xrightarrow{\tau} R .$$

¿How does this relate to synthetic algebraic geometry? There is a specific object $D \in \mathcal{E}$ which is the subobject of R given by

$$D(A) = \{ a \in A : a^2 = 0 \} .$$

Clearly D is represented by the quotient algebra $\mathbb{C}[x]/\langle x^2 \rangle$. The *tangent bundle* of an object $S \in \mathcal{E}$ is the cartesian internal hom $[D, S]$ given by

$$\begin{aligned} [D,S](A) &= \mathcal{E}(D(-) \times \mathcal{C}(A,-), S(-)) \\ &\cong \mathcal{E}(\mathcal{C}(\mathbb{C}[x]/\langle x^2 \rangle, -) \times \mathcal{C}(A,-), S(-)) \\ &\cong \mathcal{E}(\mathcal{C}(\mathbb{C}[x]/\langle x^2 \rangle \otimes A, -), S(-)) \\ &\cong \mathcal{E}(\mathcal{C}(A[x]/\langle x^2 \rangle, -), S(-)) \\ &\cong S(A[x]/\langle x^2 \rangle) . \end{aligned}$$

Notice that $A[x]/\langle x^2\rangle \cong T(A)$ where $a + b\,x$ corresponds to the matrix

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

So the tangent bundle of S can be identified with the composite functor ST. It is convenient to write the elements of $T(A)$ in the form $a + b\,\delta$ where $a, b \in A$, where a is identified with its product with the identity matrix, and where δ is the matrix with 1 as the top right entry and other entries all 0.

Actually, the tangent bundle involves a *canonical projection* $ev_0 : [D,S] \longrightarrow S$ which is induced by the arrow $0 : 1 \longrightarrow D$ in $\mathcal{E}$ whose component at A picks out the element 0 of $D(A)$. There is a natural transformation $\pi : T \longrightarrow 1_{\mathcal{C}}$ whose component at A is the algebra homomorphism $\pi_A : T(A) \longrightarrow A,\ a + b\,\delta \longmapsto a$. This induces an arrow $S\pi : ST \longrightarrow S$ which corresponds to the canonical projection under the identification of ST with $[D,S]$. For any arrow $s : 1 \longrightarrow S$, the *tangent space to* S *at* s is the pullback ST_s of s and $S\pi$.

Each arrow $\theta : S \longrightarrow X$ in $\mathcal{E}$ induces the following commutative square.

$$\begin{tikzcd}
ST \arrow[r, "\theta_T"] \arrow[d, "S\pi"'] & XT \arrow[d, "X\pi"] \\
S \arrow[r, "\theta"'] & X
\end{tikzcd}$$

Therefore, each global point $s : 1 \longrightarrow S$ induces an arrow $\theta'(s) :\ ST_s \longrightarrow XT_{\theta\circ s}$, called the *derivative of* θ *at* s, such that the following diagram commutes.

$$\begin{tikzcd}
ST_s \arrow[rr] \arrow[dr, "\theta'(s)"] \arrow[ddr] & & ST \arrow[dr, "\theta_T"] & \\
& XT_{\theta\circ s} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{pullback}"] & & XT \arrow[d, "X\pi"] \\
& 1 \arrow[r, "s"'] & S \arrow[r, "\theta"'] & X
\end{tikzcd}$$

So there is a sense in which every arrow in $\mathcal{E}$ is differentiable.

There is a canonical stucture of commutative ring on the object R in the category $\mathcal{E}$ (since products in $\mathcal{E}$ are formed valuewise and each value $R(A)$ of R is naturally a ring by virtue of the algebra structure on A). It follows that, for each object S of $\mathcal{E}$, we have a commutative ring $p_2 : R \times S \longrightarrow S$ in the category $\mathcal{E}/S$. We shall show that, if $S : \mathcal{C} \longrightarrow \mathcal{S}$ preserves pullbacks, then the object $S\pi : ST \longrightarrow S$ of $\mathcal{E}/S$ has a canonical structure of module over the ring $p_2 : R \times S \longrightarrow S$. Then we have a right to call $S\pi : ST \longrightarrow S$ a *vector bundle*. This is stronger than just saying that each tangent space ST_s is an R-module in $\mathcal{E}$.

Observe that $\pi : T \longrightarrow 1_{\mathcal{C}}$ is an abelian group in $[\mathcal{C},\mathcal{C}]/1_{\mathcal{C}}$ with the components of the addition given by

$$T(A) \times_A T(A) \longrightarrow T(A), \quad (a + b\,\delta,\ a + c\,\delta) \longmapsto a + (b + c)\,\delta\,.$$

It follows that any functor $S : \mathcal{C} \longrightarrow \mathcal{S}$ which preserves pullbacks determines an abelian group $S\pi : ST \longrightarrow S$ in the category $\mathcal{E}/S$. The multiplication of R restricts to give an action

$$R \times D \longrightarrow D, \quad (\alpha, a) \longmapsto \alpha\, a$$

of R on D and so corresponds to an arrow $R \longrightarrow [D,D]$. But $[D,D]$ acts on $[D,S]$ by internal composition

$$[D,D] \times [D,S] \longrightarrow [D,S],$$

so R acts on $[D,S]$ by restriction of scalars. This transports to an action of R on ST. Note that the pullback of $p_2 : R \times S \longrightarrow S$ and $S\pi : ST \longrightarrow S$ is just $R \times ST$, so we have an action of the ring $p_2 : R \times S \longrightarrow S$ on the object $S\pi : ST \longrightarrow S$ in $\mathcal{E}/S$. This abelian group and this action give the module structure on $S\pi : ST \longrightarrow S$.

We now wish to consider general linear groups in $\mathcal{E}$. Let V, W be any R-modules in $\mathcal{E}$. Then there is an R-module $\mathrm{Lin}(V,W)$ in $\mathcal{E}$ which is the intersection of the equalizers of the following two pairs of arrows.

$$\begin{tikzcd}
 & {[V, W \times W]} \arrow[dr, "{[1,+]}"] & \\
{[V,W]} \arrow[ur, "{[1,\mathrm{diag}]}"] \arrow[rr, "{[+,1]}"'] & & {[V \times V, W]}
\end{tikzcd}$$

$$\begin{tikzcd}
 & {[R \times V, R \times W]} \arrow[dr, "{[.\,,1]}"] & \\
{[V,W]} \arrow[ur, "R \times -"] \arrow[rr, "{[.\,,1]}"'] & & {[R \times V, W]}
\end{tikzcd}$$

The universal property of this construction is that arrows $Z \longrightarrow \mathrm{Lin}(V,W)$ in $\mathcal{E}$ are in natural bijection with $(p_2 : R \times Z \longrightarrow Z)$-module homomorphisms

$$\begin{tikzcd}
V \times Z \arrow[rr] \arrow[dr, "\mathrm{proj}_2"'] & & W \times Z \arrow[dl, "\mathrm{proj}_2"] \\
 & Z &
\end{tikzcd}$$

in $\mathcal{E}/Z$. We obtain an R-algebra $\mathrm{Lin}(V) = \mathrm{Lin}(V,W)$ by taking internal composition as multiplication.

For any objects X, Y of $\mathcal{E}$, there is an object $\mathrm{Inv}(X,Y)$ which is the intersection of the equalizers of the following pairs of arrows.

$$\begin{tikzcd}
 & 1 \arrow[dr, "\text{"}1_X\text{"}"] & \\
{[X,Y] \times [Y,X]} \arrow[ur] \arrow[rr, "_\circ_"'] & & {[X,X]}
\end{tikzcd}$$

$$\begin{tikzcd}
 & 1 \arrow[dr, "\text{"}1_Y\text{"}"] & \\
{[X,Y] \times [Y,X]} \arrow[ur] \arrow[dr, "\mathrm{switch}"'] & & {[Y,Y]} \\
 & {[Y,X] \times [X,Y]} \arrow[ur, "_\circ_"'] &
\end{tikzcd}$$

The universal property of this construction is that arrows $Z \longrightarrow \mathrm{Inv}(X,Y)$ in $\mathcal{E}$ are in natural bijection with invertible arrows

$$\begin{tikzcd}
X \times Z \arrow[rr] \arrow[dr, "\mathrm{proj}_2"'] & & Y \times Z \arrow[dl, "\mathrm{proj}_2"] \\
 & Z &
\end{tikzcd}$$

in $\mathcal{E}/Z$. It is easily seen that the composite

$$\mathrm{Inv}(X,Y) \longrightarrow [X,Y] \times [Y,X] \xrightarrow{\mathrm{proj}_1} [X,Y]$$

is a monomorphism, so we regard $\mathrm{Inv}(X,Y)$ as a subobject of $[X,Y]$.

The *general linear group* of an R-module V in $\mathcal{E}$ is $GL(V) = \mathrm{Lin}(V) \cap \mathrm{Inv}(V,V)$ which is a group under internal composition.

As an example, let us calculate $\mathrm{Lin}(n) = \mathrm{Lin}(R^n)$ and $GL(n) = GL(R^n)$. The elements of $\mathrm{Lin}(n)(A)$ are in bijection with natural transformations $\mathcal{C}(A,-) \longrightarrow \mathrm{Lin}(R^n)$, and so, in bijection with module homomorphisms as below.

$$\begin{tikzcd}
R^n \times \mathcal{C}(A,-) \arrow[rr] \arrow[dr, "\mathrm{proj}_2"'] & & R^n \times \mathcal{C}(A,-) \arrow[dl, "\mathrm{proj}_2"] \\
& \mathcal{C}(A,-) &
\end{tikzcd}$$

However, the functor $R^n \times \mathcal{C}(A,-)$ is representable with representing object the polynomial algebra $A[x_1, \ldots, x_n]$; so mere commutative triangles as above in $\mathcal{E}$ are in bijection with commutativ triangles as below in $\mathcal{C}$.

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
A[x_1, \ldots, x_n] \arrow[rr] & & A[x_1, \ldots, x_n]
\end{tikzcd}$$

But such triangles are in bijection with lists of n elements of $A[x_1, \ldots, x_n]$; that is, with lists of r polynomials over A. In order that the corresponding triangle in $\mathcal{E}$ should represent a module homomorphism, these polynomials should be homogeneous of degree 1; but such a list can be identified with an $n \times n$ matrix in A. The module homomorphism is invertible if and only if the corresponding matrix is invertible. So we have the natural isomorphisms

$$\mathrm{Lin}(n)(A) \cong \mathrm{Mat}(n,A), \qquad GL(n)(A) \cong GL(n,A).$$

Proposition 2 *For any* R*-module* V *in* $\mathcal{E}$*, the tangent space of* $GL(V)$ *at the identity element is isomorphic to the* R*-module* $\mathrm{Lin}(V)$.

Proof We just indicate the case $V = R^n$. The tangent bundle of $GL(n)$ is $GL(n)T(A) \cong GL(n, T(A))$, and this consists of invertible $n \times n$ matrices of the form $a + t\,\delta$ where a, t are $n \times n$ matrices over A. The fibre over the identity element consists of the matrices of the form $1 + t\,\delta$. (Each such is invertible with inverse $1 - t\,\delta$ since $\delta\delta = 0$.) These are in bijection with elements t of $\mathrm{Lin}(n)(A)$. **Q. E. D.**

Let G be any functor from $\mathcal{C}$ to the category of groups which preserves pullbacks. Then G is a group in $\mathcal{E}$ and the tangent space GT_η at the unit global element $\eta : 1 \longrightarrow G$ is an R-module which we denote by $\mathrm{Lie}(G)$. We have a short exact sequence

$$1 \longrightarrow \mathrm{Lie}(G) \xrightarrow{\mathrm{incl}} GT \xrightarrow{G\pi} G \longrightarrow 1$$

which is split by $G\iota$ where $\iota : 1 \longrightarrow T$ is the natural transformation whose component at A takes $a \in A$ to $a \in T(A)$. We can define an action of G on $\mathrm{Lie}(G)$ by taking the component at A to be the function

$$G(A) \times \mathrm{Lie}(G)(A) \longrightarrow \mathrm{Lie}(G)(A), \quad (g,x) \longmapsto (G\iota)(x)\ g\ (G\iota)(x)^{-1}.$$

Corresponding to this action there is the *adjoint representation*

$$\mathrm{Ad} : G \longrightarrow [\mathrm{Lie}(G), \mathrm{Lie}(G)]$$

which actually lands in GL(Lie(G)). This induces a homomorphism

$$\mathrm{Lie(Ad)}: \mathrm{Lie(G)} \longrightarrow \mathrm{Lie(GL(Lie(G)))}$$

on the tangent spaces at the identities. By Proposition 2, $\mathrm{Lie(GL(Lie(G)))} \cong \mathrm{Lin(Lie(G))}$ whose composite with Lie(Ad) is denoted by

$$\mathrm{ad}: \mathrm{Lie(G)} \longrightarrow \mathrm{Lin(Lie(G))},$$

and this corresponds to a "bilinear" arrow

$$\mathrm{Lie(G)} \times \mathrm{Lie(G)} \longrightarrow \mathrm{Lie(G)}$$

called the *Lie bracket.* With this, Lie(G) becomes a Lie algebra in $\mathcal{E}$.

As an example of course we could take $G = GL(n)$. Then $\mathrm{Lie(G)} = \mathrm{Mat(n)}$ and the bracket has component at A given by the commutator

$$\mathrm{Mat(n, A)} \times \mathrm{Mat(n, A)} \longrightarrow \mathrm{Mat(n, A)}, \quad (s, t) \longmapsto [s, t] = s\,t - t\,s.$$

If we take, say, the orthogonal subgroup O(n) of GL(n) (where $O(n)(A) = O(n, A)$ is the group of orthogonal matrices with entries in A), it is easy to see that Lie(O(n))(A) is the Lie subalgebra of Mat(n, A) consisting of the skew symmetric matrices.

The problem with both the toposes $\mathcal{E}$ and $[\mathcal{A}, \mathcal{S}]$ is that they do not contain the complex analytic manifolds in a suitable way. The category $\mathcal{E}$ is suitable for complex algebraic varieties, indeed, complex schemes; we need to replace C by a suitable category of analytic algebras. Since we are really interested in the continuous arrows in the category $[\mathcal{A}, \mathcal{S}]$, we might consider replacing it by the topos $[\mathcal{A}, \mathfrak{T}]$ where $\mathfrak{T}$ is a suitable topos of topological spaces (such as **[Jns**; page 21]); but we also seem to need to extend $\mathcal{A}$. We make the following suggestions about such an approach.

Take $\mathbb{A}$ to be the category of complex Banach algebras whose dimensions as vector spaces are countable; the arrows are continuous algebra homomorphisms. Take $\mathfrak{T}$ to be Johnstone's topos containing sequential spaces. Since each object of $\mathbb{A}$ is a sequential space, we have an underlying functor $U : \mathbb{A} \longrightarrow \mathfrak{T}$. Put $\mathfrak{E} = [\mathbb{A}, \mathfrak{T}]$ which, of course, is again a topos. Notice that Shanuel's theorem modifies easily to imply a bijection between endomorphisms on U in $\mathfrak{E}$ and analytic endofunctions on $\mathbb{C}$. Thus the topos $\mathfrak{E}$ contains a model of the "line" from the viewpoint of complex manifolds.

References

[DG] M. DeMeyer and P. Gabriel, *Introduction to Algebraic Geometry and Algebraic Groups* (North-Holland Mathematical Studies **39**, 1980)

[Sch] S. Schanuel, Continuous extrapolation to triangular matrices characterizes smooth functions, *J. Pure Appl. Algebra* **24** (1982) 59-71.

[Jns] P.T. Johnstone, *Topos Theory* (Academic Press, London Math Society Monographs **10** (1977)).