EVERY FACTORIZATION SYSTEM IS ORTHOGONAL

Notation Each category $\mathcal{X}$ determines a simplicial category

$$\begin{tikzcd}
\mathcal{X}^3 \arrow[r, shift left=6, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=6, "d_2"'] & \mathcal{X}^2 \arrow[l, shift right=3, "s_0"'] \arrow[l, shift left=3, "s_1"] \arrow[r, shift left=4, "d_0"] \arrow[r, shift right=4, "d_1"'] & \mathcal{X} \arrow[l, "s_0"']
\end{tikzcd}$$

where $s_0 : \mathcal{X} \longrightarrow \mathcal{X}^2$ takes $X \in \mathcal{X}$ to $1_X : X \longrightarrow X$, where $d_0, d_1 : \mathcal{X}^2 \longrightarrow \mathcal{X}$ take $f : A \longrightarrow B$ in $\mathcal{X}^2$ to B, A, respectively, where $s_0, s_1 : \mathcal{X}^2 \longrightarrow \mathcal{X}^3$ take $f : A \longrightarrow B$ to the composable pairs $(1_A : A \longrightarrow A,\ f : A \longrightarrow B)$, $(f : A \longrightarrow B,\ 1_B : B \longrightarrow B)$, respectively, and, where $d_0, d_1, d_2 : \mathcal{X}^3 \longrightarrow \mathcal{X}^2$ take the composable pair $(f : A \longrightarrow B,\ g : B \longrightarrow C)$ in $\mathcal{X}^3$ to $g : B \longrightarrow C$, $g \circ f : A \longrightarrow B$, $f : A \longrightarrow B$, respectively. Recall that we have adjunctions $d_0 \dashv s_0 \dashv d_1 : \mathcal{X}^2 \longrightarrow \mathcal{X}$ involving a unit $\eta : 1 \Rightarrow s_0 \circ d_0$ and a counit $\varepsilon : s_0 \circ d_1 \Rightarrow 1$.

For functors $P : \mathcal{A} \longrightarrow \mathcal{C}$, $Q : \mathcal{B} \longrightarrow \mathcal{C}$, we write $\mathcal{C}(P,Q) : \mathcal{A}^{op} \times \mathcal{B} \longrightarrow \mathrm{Set}$ for the functor given by $\mathcal{C}(P,Q)(A,B) = \mathcal{C}(PA,QB)$; so $\mathcal{C}(1_{\mathcal{C}}, 1_{\mathcal{C}})$, or merely $\mathcal{C}(1,1)$, denotes the hom functor of $\mathcal{C}$.

We shall make use of the natural transformation $\theta : \mathcal{X}(d_0, d_1) \longrightarrow \mathcal{X}^2(1,1)$ whose component $\theta_{f,g} : \mathcal{X}(B,C) \longrightarrow \mathcal{X}^2(f,g)$, for objects $f : A \longrightarrow B$, $g : C \longrightarrow D$ of $\mathcal{X}^2$, is given by $\theta_{f,g}(w) = (w \circ f, g \circ w)$. More conceptually, θ is the following composite.

$$\mathcal{X}(d_0, d_1) \xrightarrow{\ s_0\ } \mathcal{X}^2(s_0 \circ d_0, s_0 \circ d_1) \xrightarrow{\ \mathcal{X}^2(\eta, \varepsilon)\ } \mathcal{X}^2(1,1)$$

We say f is *orthogonal* to g when $\theta_{f,g}$ is invertible.

Proposition 1 *For all objects* $(f : A \longrightarrow B, g : B \longrightarrow C)$, $(f' : A' \longrightarrow B', g' : B' \longrightarrow C')$ *of* $\mathcal{X}^3$, *the following square is a pullback.*

$$\begin{tikzcd}
\mathcal{X}^3((f,g),(f',g')) \arrow[r, "d_1"] \arrow[d, "d_1 \circ d_0"'] & \mathcal{X}^2(g \circ f, g' \circ f') \arrow[d, "\mathcal{X}^2((1,g),(f',1))"] \\
\mathcal{X}(B,B') \arrow[r, "\theta_{f,g'}"'] & \mathcal{X}^2(f,g')
\end{tikzcd}$$

Definitions A *decomposition* on a category $\mathcal{X}$ is a normalized splitting of $d_1 : \mathcal{X}^3 \longrightarrow \mathcal{X}^2$; that is, a functor $N : \mathcal{X}^2 \longrightarrow \mathcal{X}^3$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "s_0"] \arrow[dr, "s_0 \circ s_0"'] & \mathcal{X}^2 \arrow[d, "N"] \arrow[dr, "1_{\mathcal{X}^2}"] & \\ & \mathcal{X}^3 \arrow[r, "d_1"'] & \mathcal{X}^2 \end{tikzcd}$$

Put $\mathcal{E} = \{ f : A \longrightarrow B \mid d_0\, N(f) \text{ invertible} \}$, $\mathcal{M} = \{ f : A \longrightarrow B \mid d_2\, N(f) \text{ invertible} \}$[1]. A *factorization system* is a decomposition with $d_2\, N(f) \in \mathcal{E}$ and $d_0\, N(f) \in \mathcal{M}$ for all arrows $f : A \longrightarrow B$ in $\mathcal{X}$.

Suppose $N : \mathcal{X}^2 \longrightarrow \mathcal{X}^3$ is a decomposition. We shall make use of the natural transformation $\phi : \mathcal{X}(d_0, d_1) \longrightarrow \mathcal{X}^3(N,N)$ obtained by composing θ with the effect of N on homs; explicitly, the component

$$\phi_{f,g} : \mathcal{X}(B,C) \longrightarrow \mathcal{X}^3(N(f), N(g))$$

is given by $\phi_{f,g}(w) = (w \circ f,\ s \circ w \circ m,\ g \circ w)$ where we have put

$$N(f) = (e : A \longrightarrow I,\ m : I \longrightarrow B) \quad \text{and} \quad N(g) = (s : C \longrightarrow J,\ i : J \longrightarrow D).$$

Proposition 2 *If* $f \in \mathcal{E}$ *and* $g \in \mathcal{M}$ *then* $\phi_{f,g}$ *is invertible.*

Proof The hypotheses mean m and s are both invertible. Any $(u, x, v) : N(f) \longrightarrow N(g)$ has $u = s^{-1} \circ x \circ e$, $v = i \circ x \circ m^{-1}$. The inverse of $\phi_{f,g}$ takes (u, x, v) to $s^{-1} \circ x \circ m^{-1}$. **Q.E.D.**

Proposition 3 *Every factorization system* $N : \mathcal{X}^2 \longrightarrow \mathcal{X}^3$ *is a fully faithful functor.*

Proof Since $N : \mathcal{X}^2(f,g) \longrightarrow \mathcal{X}^3(N(f), N(g))$ has d_1 as a left inverse, it suffices to prove d_1 is injective. The pullback of an injective function is injective so, by applying Proposition 1 to the objects $N(f)$, $N(g)$ of $\mathcal{X}^3$, we see that it suffices to prove $\theta_{e,i}$ is injective. So it suffices to prove $\phi_{e,i} = N \circ \theta_{e,i}$ is injective. But N is a factorization system, so $e \in \mathcal{E}$ and $i \in \mathcal{M}$. The result now follows from Proposition 2. **Q.E.D.**

Corollary *For a factorization system, each* $f \in \mathcal{E}$ *is orthogonal to each* $g \in \mathcal{M}$.

Proof Propositions 2 and 3 and the equality $\phi_{f,g} = N \circ \theta_{f,g}$ imply $\theta_{f,g}$ is invertible. **Q.E.D.**

[1] Clearly both these classes include all identity arrows.

Ems compose

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(1) First recall the 2-cell aspect of the universal property of X^{2}.

$$\begin{tikzcd} K \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X^{2} \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] \arrow[r, phantom, "\Downarrow\lambda"] & X \end{tikzcd}$$

If ρ, σ are 2-cells such that

$$\begin{tikzcd} d_1 u \arrow[r, Rightarrow, "\rho"] \arrow[d, Rightarrow, "\lambda u"'] & d_1 v \arrow[d, Rightarrow, "\lambda v"] \\ d_0 u \arrow[r, Rightarrow, "\pi"'] & d_0 v \end{tikzcd}$$ commutes

then there exists a unique 2-cell $\omega : u \Rightarrow v$ such that

$$d_1 \omega = \rho \quad \& \quad d_0 \omega = \pi .$$

(2) A <u>decomposition</u> on X consists of j, ε, μ such that

$$\begin{tikzcd} X^{2} \arrow[r, bend left=60, "d_1"] \arrow[r, "j" description] \arrow[r, bend right=60, "d_0"'] \arrow[r, phantom, bend left=30, "\Downarrow\varepsilon"] \arrow[r, phantom, bend right=30, "\Downarrow\mu"] & X \end{tikzcd} = \begin{tikzcd} X^{2} \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] \arrow[r, phantom, "\Downarrow\lambda"] & X \end{tikzcd} .$$

Define $e : X^2 \longrightarrow X^2$ by

$$\begin{tikzcd} X^2 \arrow[r, "e"] & X^2 \arrow[r, bend left, "d_1"] \arrow[r, bend right, "d_0"'] \arrow[r, phantom, "\Downarrow\lambda"] & X \end{tikzcd} = \begin{tikzcd} X^2 \arrow[r, bend left, "d_1"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & X \end{tikzcd}.$$

Axiom 1 $\quad \mu e : je \Rightarrow j$ is invertible.

Axiom 2 If

$$\begin{tikzcd} & jg \arrow[dr, Rightarrow, "\mu g"] \arrow[dd, Rightarrow, "\theta"] & \\ d_1 g \arrow[ur, Rightarrow, "\varepsilon g"] \arrow[dr, Rightarrow, "\varepsilon g"'] & & d_0 g \\ & jg \arrow[ur, Rightarrow, "\mu g"'] & \end{tikzcd}$$

commutes then θ is invertible.

③ Theorem Suppose (j, ε, μ) is a decomposition on X satisfying Axioms 1 & 2. For $f : A \longrightarrow X^3$, put

$$\varphi_i = \left(\begin{tikzcd} A \arrow[r, "f"] & X^3 \arrow[r, "d_i"] & X^2 \arrow[r, bend left, "d_1"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & X \end{tikzcd} \right)$$

for $i = 0, 1, 2$. Then

φ_0, φ_2 invertible implies φ_1 invertible.

<u>Proof</u> By ①, commutativity of

$$\begin{tikzcd}
d_1 d_1 = d_1 d_2 \arrow[r, "\lambda d_2"] \arrow[d, "\lambda d_1"'] \arrow[dr, phantom, "\mathrm{I}"] & d_0 d_2 = d_1 d_0 \arrow[d, "\lambda d_0"] \\
d_0 d_1 \arrow[r, equal] & d_0 d_0
\end{tikzcd}$$

implies $\exists!\ \tau : d_1 \Rightarrow d_0 : X^3 \to X^2$ such that

$$d_1 \tau = \lambda d_2 : d_1 d_1 \Rightarrow d_1 d_0$$

$$\& \ d_0 \tau = 1 \quad : d_0 d_1 = d_0 d_0 .$$

Thus we have commutativity in

$$\begin{tikzcd}
d_1 d_1 f \arrow[r, "d_1 \tau f = \lambda d_2 f"] \arrow[d, "\varphi_1 = \varepsilon d_1 f"'] \arrow[dr, phantom, "\mathrm{II}"] & d_1 d_0 f \arrow[d, "\varepsilon d_0 f = \varphi_0"] \\
j d_1 f \arrow[r, "j \tau f"] \arrow[d, "\mu d_1 f"'] \arrow[dr, phantom, "\mathrm{III}"] & j d_0 f \arrow[d, "\mu d_0 f"] \\
d_0 d_1 f \arrow[r, "d_0 \tau f = 1"] & d_0 d_0 f ;
\end{tikzcd}$$

the top square of this gives commutativity in

$$\begin{tikzcd}
d_1 e d_1 f = d_1 d_1 f \arrow[rr, equal] \arrow[d, shift right=2, "\lambda e d_1 f"'] \arrow[d, shift left=2, "\varepsilon d_1 f"] \arrow[drr, phantom, "\mathrm{IV}"] & & d_1 d_2 f \arrow[d, "\lambda d_2 f"] \\
d_0 e d_1 f = j d_1 f \arrow[r, "j \tau f"'] & j d_0 f \arrow[r, "\varphi_0^{-1}"'] & d_0 d_2 f .
\end{tikzcd}$$

So, by ①, there exists a unique

$$\sigma : ed_1 f \longrightarrow d_2 f$$

such that $d_1\sigma = 1$ and $d_0\sigma = \varphi_0^{-1}\cdot(j\tau f)$.

Thus we have commutativity in

$$\begin{tikzcd}
d_1ed_1f \arrow[rr, equal, "d_1\sigma = 1"] \arrow[d, "\lambda d_1 f = \varepsilon e d_1 f"'] \arrow[drr, phantom, "\underline{\mathrm{V}}"] & & d_1d_2f \arrow[d, "\varepsilon d_2 f = \varphi_2"] \\
jed_1f \arrow[rr, "j\sigma"] \arrow[d, "\mu e d_1 f"'] \arrow[drr, phantom, "\underline{\mathrm{VI}}"] & & jd_2f \arrow[d, "\mu d_2 f"] \\
d_0ed_1f \arrow[rr, "d_0\sigma"] \arrow[dr, "j\tau f"'] & & d_0d_2f \arrow[dl, "\varphi_0"] \\
& jd_0f &
\end{tikzcd}$$

and so in

$$\begin{tikzcd}
& jd_1f \arrow[d, "(\mu e d_1 f)^{-1}"] \arrow[ddr, bend left, "\underline{\mathrm{III}}\ \ \mu d_1 f = (\mu d_0 f)(j\tau f)"] & \\
& jed_1f \arrow[d, "j\sigma"] \arrow[dr, "\underline{\mathrm{VI}}"] & \\
d_1d_1f \arrow[uur, bend left, "\varepsilon d_1 f = \lambda e d_1 f \ \text{decomp.}"] \arrow[ur, "\varepsilon e d_1 f"] \arrow[r, "\varphi_2"'] \arrow[dr, "1"] \arrow[ddr, bend right, "\varepsilon d_1 f"'] \arrow[r, phantom, shift left=4, "\underline{\mathrm{V}}"] & jd_2f \arrow[r, "(\lambda d_0 f)(\mu d_2 f)"] \arrow[d, "\varphi_2^{-1}"] & d_0d_1f \\
& d_1d_2f \arrow[ur, "(\lambda d_0 f)(\lambda d_2 f) \overset{\mathrm{I}}{=} \lambda d_1 f \ \text{decomp.}"'] \arrow[d, "\varphi_1"] & \\
& jd_1f \arrow[uur, bend right, "\mu d_1 f"'] &
\end{tikzcd}$$

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(Notice that we used less than Axiom 1: merely that $\mu ed, f$ is invertible.)

By Axiom 2,

$$\theta = \varphi_1 \cdot \varphi_2^{-1} \cdot j\sigma \cdot (\mu ed, f)^{-1}$$

is invertible. So $j\sigma$ is a split monic. From $\overline{\mathrm{V}}$ and the invertibility of φ_2, we see that $j\sigma$ is split epic. So $j\sigma$ is invertible. So

$$\varphi_1 = \theta \cdot (\mu ed, f) \cdot (j\sigma)^{-1} \cdot \varphi_2$$

is invertible as required. QED