Stone and Priestley dualities via the ultrafilter and prime filter monads

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Declaration

This work has not previously been submitted for a degree or diploma at any university. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

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Abstract

The ultrafilter monad is induced by the contravariant adjunction between Boolean algebras and sets. In the late 1960s, Manes proved that its algebras are precisely compact Hausdorff spaces, thereby showing them to be algebras in the universal algebraic sense; albeit with operations of infinite arity. In this thesis, we consider the induced comparison functor from Boolean algebras to compact Hausdorff spaces, and its left adjoint. By restricting to the objects at which the unit of the comparison adjunction is an isomorphism, we extract classical Stone duality. We use the prime filter monad to obtain Priestley duality in the same way as for Stone duality.

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Introduction

Stone's representation theorem for Boolean algebras was one of the first examples of an equivalence of categories to become widely known [14]. His original formulation was not in these terms – category theory was still in a period of incubation in the mid 1930s – nonetheless, the implications for translating between the algebraic and the topological were evident [26, 27, 14]. In modern language, it expresses a dual equivalence between the category of Boolean algebras and the category of Stone spaces – totally disconnected compact Hausdorff spaces. Since then, Stone duality, as it is often referred to, has found application in areas as diverse as functional analysis (whence Stone originated), probability theory, and computer science [14]. Perhaps this is not surprising; Stone himself [27] made the remark:

> 'The theory of Boolean algebras ...bears to the theory of combinations the same relation as the theory of abstract groups to the theory of permutations.'

Stone duality has been generalised in a variety of directions. Priestley's representation theorem for bounded distributive lattices, published in the early 1970s, is one such generalisation [24, 25]. The language of category theory established by this time, Priestley showed a dual equivalence between the category of bounded distributive lattices and a category of partially ordered spaces (now) known as Priestley spaces. Analogously to Stone's totally disconnected spaces, these are often referred to as totally order disconnected spaces.

Monads arose in sheaf theory in the mid-twentieth century, however, they are now pervasive throughout category theory; including in the main category theoretic formulation of general algebra. In this context, monads are a generalisation of algebraic closure operators, allowing for the description of finitary and infinitary algebraic theories on arbitrary base categories. In the late 1960s, Manes showed that the category of algebras for the ultrafilter monad is equivalent to the category of compact Hausdorff spaces [18]. Compact Hausdorff spaces were thus proven to be algebras in the universal algebraic sense; albeit with operations of infinite arity.[1]

Partially ordered spaces appeared in Nachbin's *Topology and Order* in the mid-sixties [22]. In particular, such spaces in which the partial order is closed and the topology compact are considered the asymmetric counterpart of compact Hausdorff spaces [15, 28]. The terminology for these spaces is not consistent across the literature, however, they are now most commonly referred to

[1] The reader may wonder what precisely these operations are; we refer to Manes' description in [19].

as *compact pospaces*. In 1997, Flagg showed that compact pospaces are precisely the algebras for the prime filter monad (regarded as an endofunctor on the category of partially ordered sets) [8].

Once one has Manes' characterisation of compact Hausdorff spaces as the algebras for the ultrafilter monad, Stone duality can be derived in a canonical way – this is the first contribution of this thesis, attained in Chapter 2.

The ultrafilter monad can be induced by a contravariant adjunction between sets and Boolean algebras – the left adjoint takes a set to its (Boolean) algebra of subsets, and the right adjoint sends a Boolean algebra to its set of ultrafilters. For any monad, there is a canonical comparison functor from the domain of the right adjoint and the category of algebras for the monad. The comparison functor for the ultrafilter monad is a contravariant functor from the category of Boolean algebras to the category of compact Hausdorff spaces, sending each Boolean algebra to its associated Stone space.

To reconstruct Stone duality from this point requires two further facts: First, the comparison functor is fully faithful; this corresponds to a a canonical coequaliser diagram associated to the monad. Second, the comparison functor has a left adjoint which sends each compact Hausdorff space to its Boolean algebra of clopen (closed and open) subsets. We show the unit of the comparison adjunction is an isomorphism at a compact Hausdorff space precisely when that space is a Stone space. The result is that the comparison adjunction has a canonical restriction to the dual equivalence between Boolean algebras and Stone spaces; that is, Stone duality.

The second objective of this thesis is to give an account of Priestley duality using the same pattern – this is realised in Chapter 3. We start from the contravariant adjunction between the category of partially ordered sets and distributive lattices. The left adjoint sends a partially ordered set to its lattice of upward closed sets (up-sets), and the right adjoint sends a lattice to the set of its prime filters ordered by inclusion. Composition of these functors induces the prime filter monad, which as stated previously, has compact pospaces as its algebras. Here, the comparison functor sends a distributive lattice to its associated Priestley space. As in the case of the ultrafilter monad, the comparison adjunction restricts canonically to precisely the dual equivalence between Priestley spaces and distributive lattices – Priestley duality.

This approach is more elementary than the typical modern formulations of Stone duality and Priestley duality. Moreover, it provides a general schema for duality theorems for which we might hope have further interesting instances. We discuss this further in the final section of Chapter 4.

Chapter 1

Foundations

This chapter contains the conceptual foundations required for the bulk of this thesis. We expect that much of what is presented here will be familiar to the reader. However, by nature of the topic at hand, we draw on a variety of areas of mathematics. The reader may therefore be more familiar with some areas than others. For this reason, and for the sake of a cohesive piece of work, we provide some basic background material.

1.1 Order

We present some basic definitions and examples. The main reference for this section was [6].

Note that when we refer to an 'ordered set', we mean a set with a partial order (unless otherwise specified). If we are referring to a partially ordered set in which every pair of elements are comparable, we will generally refer to that set as a 'chain'.

Definition 1.1.1 (Filter)**.** Let P be a partially ordered set. A *filter* of P is a subset $F \subseteq P$ satisfying the following:

(i) $F \neq \varnothing$;

(ii) $x, y \in F \implies (\exists z \in F) \quad z \leqslant x$ and $z \leqslant y$;

(iii) $(\forall x \in F)(\forall y \in P) \quad x \leqslant y \implies y \in F$.

A filter F of P is a *proper filter* if F is a proper subset of P.

Notation 1.1.2. The set of filters of a partially ordered set P may themselves be ordered by inclusion; we write $\mathrm{Fil}(P)$ for the ordered set of proper filters.

Example 1.1.3 (Neighbourhood filter)**.** Let X be a topological space and let $x \in X$. The set of all neighbourhoods of a point x is a filter called the *neighbourhood filter*, or *neighbourhood system* at x.

It is obvious that the subject of our next definition will be of particular importance.

Definition 1.1.4 (Ultrafilter)**.** Let P be a partially ordered set. A proper filter F of P is called an *ultrafilter* if F is a maximal element of $\mathrm{Fil}(P)$.

Example 1.1.5. Let X be a set and let $S \subseteq X$. Then $\mathcal{F}_S \coloneqq \{T \subseteq X : S \subseteq T\}$ is a filter on X (i.e. a filter of the powerset of X ordered under inclusion). If S is a singleton, it is easy to see that that $\mathcal{F}_S$ is a maximal filter, and thus an ultrafilter.

Definition 1.1.6. A partially ordered set D is *(downward) directed* if D is non-empty and for all $x, y \in D$ there exists $z \in D$ such that $z \leqslant x$ and $z \leqslant y$. The order-theoretic dual is an *upward-directed set*[1].

Definition 1.1.7. Let P be a partially ordered set and let $Q \subseteq P$. Then Q is an *up-set* if for all $x \in Q$ and $y \in P$, if $x \leqslant y$ then $y \in Q$.

Definition 1.1.8. Let P be a partially ordered set. The *principal up-set* of an element $a \in P$ is given by

$$\uparrow a \coloneqq \{b \in P : a \leqslant b\}.$$

The *principal down-set* of a is defined dually:

$$\downarrow a \coloneqq \{b \in P : b \leqslant a\}.$$

For any $Q \subseteq P$, the up-set generated by Q is the smallest up-set which contains Q, denoted $\uparrow Q$, and $\downarrow Q$ defined dually.

Notation 1.1.9. For any function $f : X \to Y$, we denote the preimage of $U \subseteq Y$ by $f^{\leftarrow}(U)$. The image of $V \subseteq X$ is denoted $f^{\rightarrow}(V)$.

Lemma 1.1.10. *Let P be an ordered set. The set of all up-sets of P is closed under arbitrary union and arbitrary intersection* (*as is the set of down-sets, by order-theoretic duality*).

Lemma 1.1.11. *A map is order-preserving if and only if the preimage of every up-set is an up-set.*

Proof. Let P and Q be ordered sets, and let $f : P \to Q$ be a function. Assume that f is order-preserving, and let U be an up-set of Q. To show that $f^{\leftarrow}(U)$ is an up-set of P, let $x \in f^{\leftarrow}(U)$, and let $y \in P$ with $y \geqslant x$. Then $f(y) \geqslant f(x) \in U$, so $f(y) \in U$, and hence $y \in f^{\leftarrow}(U)$.

Conversely, assume that $f^{\leftarrow}(U)$ is an up-set of P whenever U is an up-set of Q. Let $x, y \in P$ with $x \leqslant y$. By assumption, $f^{\leftarrow}(\uparrow f(x))$ is an up-set which contains x, and therefore must contain y, so $f(x) \leqslant f(y)$ as required. □

We write the category of ordered sets with monotone maps as **Poset**.

[1] A warning: when we refer to a directed set it is implicit that we mean downward directed. Other authors usually reverse this convention, or indeed, refer to downward-directed ordered structures as *filtered*.

1.2 Boolean algebras and lattices: where algebra meets order

We give most of the results in this section without proof as they are routine and found in texts such as [6, 20]. Further material on universal algebra, lattices, and category-theoretic formulations of universal algebra may be found in [3].

Definition 1.2.1 (Lattice)**.** A partially ordered set L is called a *lattice* if the least upper bound and greatest lower bound of every pair of elements exist in L.

Alternatively, lattices may be defined as a class of algebras in the universal (general) algebraic sense; that is, as sets equipped with operations which are models of an equational theory. In these terms, a lattice is a set L equipped with binary operations $\vee : L^2 \to L$ and $\wedge : L^2 \to L$ – least upper bound and greatest lower bound respectively – which satisfy certain equations.

Notation 1.2.2. When considering a lattice, we use the order relation '$\leqslant$' interchangeably with the equation $x \wedge y = x$ (or equivalently $x \vee y = y$).

We briefly introduce two of the basic constructs of universal algebra; subalgebras, and homomorphisms. A *homomorphism* of algebras A to B is a function $h : A \to B$ which preserves the operations. A *subalgebra* of an algebra A is a subset $B \subseteq A$ for which the operations on B are the restriction of the operations on A. So in particular, a homomorphism of lattices preserves the binary operations $\vee$ and $\wedge$.

Definition 1.2.3 (Bounded distributive lattice)**.** A lattice L is said to be *bounded* if $\bigvee L$ and $\bigwedge L$ exist in L. We denote $\bigvee L$ as 1, and $\bigwedge L$ as 0. Algebraically, we regard these as nullary operations.

A *distributive lattice* L is a lattice in which join and meet distribute over one another; that is, for all $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad \text{and} \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{1.2.1}$$

The lattices in this thesis are always distributive and bounded. As noted previously, homomorphisms of algebras preserve the algebraic operations; thus bounded lattice homomorphisms preserve $\vee$, $\wedge$, 0, and 1. Observe that equations are preserved by homomorphisms and are thus preserved by bounded lattice homomorphisms. We denote the category of bounded distributive lattices with bounded lattice homomorphisms as **DLat**.

Boolean algebras have been of interest, both to logicians and mathematicians, long before lattices. However, it is convenient for our purposes to define them in lattice-theoretic terms.

Definition 1.2.4 (Boolean algebra)**.** A *Boolean algebra* is a bounded distributive lattice B in which every element has a complement; that is for all $a \in B$, there exists $b \in B$ such that

$$a \vee b = 1 \quad \text{and} \quad a \wedge b = 0 \tag{1.2.2}$$

We generally denote the complement of an element a as $a^{\perp}$. The category of Boolean algebras with Boolean algebra homomorphisms is written as **BA**.

Example 1.2.5. Any powerset under the inclusion order forms a distributive lattice, and in particular, a Boolean algebra. Complement is given by set complement and, conveniently, $\cap$ corresponds to $\wedge$ and $\cup$ corresponds to $\vee$. This holds for any *field of sets*; that is, a subalgebra of the powerset algebra of some set.

Although we defined filters previously, they admit a nicer description in lattices.

Definition 1.2.6. Let L be a lattice. Then $F \subseteq L$ is a filter on L if;

(i) F is a non-empty up-set;

(ii) $(x, y \in F) \implies x \wedge y \in F$.

We call F a *prime filter* if:

(iii) F is a proper subset of L, and;

(iv) $(\forall x, y \in L) \quad x, y \notin F \implies x \vee y \notin F$.

Remark 1.2.7 (FIP)**.** Let F be a non-empty family of subsets of some set X. If for every pair $A, B \in F$, we have $A \cap B \in F$, we say that F has the *finite intersection property*. Recall from Example 1.2.5 that intersection corresponds to $\wedge$ in a field of sets. Thus by Item (ii) of Definition 1.2.6, any proper filter on a set X has the finite intersection property. We are often interesting in families of sets with the finite intersection property as they can be used to generate prime filters – this will be shown later in this section.

Definition 1.2.8 (Ideal and prime ideal)**.** The dual of the definition of a filter gives that of an *ideal*. As would be expected, a *prime ideal* is defined as the order-theoretic dual of a prime filter.

Lemma 1.2.9. *Let L be a lattice and let $F \subseteq L$. Then F is a prime filter if and only if $L \setminus F$ is a prime ideal.*

Proof. Let I be a prime ideal. Then I is a non-empty down-set, so its complement is a non-empty up-set. Prime implies proper (*i.e.*, I is a proper subset of L) therefore $L \setminus I$ is also a proper subset of L.

Now let $x, y \in L \setminus I$, that is to say $x, y \notin I$. Then by primeness of I, we have $x \wedge y \notin I$ and so $x \wedge y \in L \setminus I$. Whence $L \setminus I$ is a proper filter.

Also, if $x, y \notin L \setminus I$ then $x, y \in I$, which by definition of an ideal means $x \wedge y \in I$. It is evident that $x \wedge y \notin L \setminus I$, thus $L \setminus I$ is a prime filter.

The converse follows by order-theoretic duality, so we conclude this proof. □

Lemma 1.2.10. *Maximal filters on distributive lattices are prime.*

Proof. Let F be an ultrafilter (and thus a maximal filter) of a distributive lattice L. Assume $a \vee b \in F$ and $b \notin F$. We show $a \in F$. First, define the following:

$$F_b := \{x \in L : (\exists c \in F)\ x \vee (b \wedge c) = x\}$$

Certainly F_b is a filter that strictly contains F, so we conclude $F_b = L$. Now $a \in F_b$, so there exists $c \in F$ such that $a \vee (b \wedge c) = a$. But $a \vee c \in F$, and by assumption $a \vee b \in F$. Using the distributivity of L, we have

$$(a \vee b) \wedge (a \vee c) = a \vee (b \wedge c) = a \in F.$$ □

Corollary 1.2.11. *A filter on a Boolean algebra is prime if and only if it is an ultrafilter.*

1.2.1 Prime ideal theorems

We now prove the distributive prime ideal theorem (DPI) using Zorn's lemma.

Theorem 1.2.12 (DPI)**.** *Let G be a filter of a distributive lattice L and let J be an ideal of L with $G \cap J = \varnothing$. Then there exists a prime filter F of L and a prime ideal I of L such that $G \subseteq F$, $J \subseteq I$, and $I \cap F = \varnothing$.*

Proof. Define $\mathcal{E} := \{H \in \mathrm{Fil}(L) : G \subseteq H \text{ and } H \cap J = \varnothing\}$ ordered by inclusion. Since $G \in \mathcal{E}$, we have that $\mathcal{E}$ is non-empty. Now consider a non-empty chain in $(\mathcal{E}; \subseteq)$, say, $\mathcal{C}$, with $G \subseteq C \subseteq \bigcup \mathcal{C}$ for some $C \in \mathcal{C}$. Now every element of $\mathcal{E}$ has empty intersection with J, and thus

$$\bigcup \mathcal{C} \cap J = \bigcup \{C \cap J : C \in \mathcal{C}\} = \varnothing.$$

It is immediate that $\bigcup \mathcal{C}$ is an up-set. To see $\bigcup \mathcal{C}$ is closed under $\wedge$; let $a, b \in \bigcup \mathcal{C}$. Then $a \in C$ and $b \in D$ for some $C, D \in \mathcal{C}$. Without loss of generality, we may assume that $C \subseteq D$, so $a, b \in D$. We then have $a \wedge b \in D$, and so $a \wedge b \in \bigcup \mathcal{C}$. Therefore $\bigcup \mathcal{C}$ is a filter, and so is an element of $\mathcal{E}$, so $\mathcal{E}$ is chain-complete. By Zorn's Lemma there is a maximal element F of $(\mathcal{E}; \subseteq)$ above G, and so F is a prime filter containing G and disjoint from J. Moreover, by Lemma 1.2.9, $L \setminus F$ is a prime ideal containing J. □

Corollary 1.2.13. *Given a lattice L and a pair of elements $a, b \in L$ with $a \nleq b$, there is a prime filter containing a but not b.*

We used Zorn's lemma in our proof, however, the distributive prime ideal theorem (DPI) is strictly weaker than the axiom of choice; that is, there are models of Zermelo–Fraenkel set theory in which DPI (or equivalent) hold, but the Axiom of Choice does not. Here are two variations on the same theme:

Lemma 1.2.14 (The Ultrafilter Lemma)**.** *Any proper filter $\mathcal{F}$ on a set X can be extended to an ultrafilter on X.*

Theorem 1.2.15 (The Boolean Prime Ideal Theorem)**.** *Given a proper ideal I on a Boolean algebra B, there exists a prime ideal $\mathcal{I}$ on B such that $I \subseteq \mathcal{I}$.*

Corollary 1.2.16. *Given a Boolean algebra B and a pair of elements $a, b \in B$ with $a \neq b$, there is an ultrafilter $\mathcal{F}$ containing precisely one of a and b.*

Although the ultrafilter lemma and the Boolean prime ideal theorem appear be weaker than the distributive prime ideal theorem, all three are equivalent. Proofs appear in a number of contexts in the literature, see for example [7, 6]. In later chapters we will use whichever is best suited to the situation.

1.3 Topology and ultrafilters

In this section we provide a view of topology in terms of ultrafilters. Unlike the previous sections, the approach presented here is not emphasised in the usual introductory texts. We therefore assume less familiarity on the part of the reader and provide proofs of key results.

Notation 1.3.1. Let X be a topological space. Recall that Example 1.1.3 gave the definition of the neighbourhood filter at a point $x \in X$ as the filter on the powerset consisting of all neighbourhoods of x. We write this as $\mathrm{Nbhd}(x)$.

If F and G are filters with $F \subseteq G$, the filter G is said to *refine* the filter F.

Definition 1.3.2. Let X be a topological space and let $x \in X$. An ultrafilter $\mathcal{F}$ is said to *converge* to x, denoted by $\mathcal{F} \rightsquigarrow x$, when every neighbourhood of x is in $\mathcal{F}$. In other words, $\mathcal{F}$ refines the neighbourhood filter at x.

Proposition 1.3.3. *For any topological space X, the interior and closure of $A \subseteq X$ can be formulated in terms of ultrafilter convergence:*

$$\textit{Interior:} \qquad \mathring{A} = \{a \in A : (\forall \mathcal{F} \in \mathrm{Ult}\, X)\ \mathcal{F} \rightsquigarrow a \implies A \in \mathcal{F}\}; \tag{1.3.1a}$$

$$\textit{Closure:} \qquad \overline{A} = \{x \in X : (\exists \mathcal{F} \in \mathrm{Ult}\, X)\ \mathcal{F} \rightsquigarrow x \text{ and } A \in \mathcal{F}\} \tag{1.3.1b}$$

where $\mathrm{Ult}\, X$ *is the set of ultrafilters on* X.

Proof. Let A be a subset of a topological space X. To see that the forward inclusion in (1.3.1a) holds, note that an open set is a neighbourhood of all its points, so $a \in \mathring{A}$ means there exists an open set U with $a \in U \subseteq A$. An ultrafilter $\mathcal{F} \rightsquigarrow a$ must refine the neighbourhood filter $\mathrm{Nbhd}(a)$, and since filters are upward-closed (under the usual inclusion ordering in the powerset), A (as well as U) is in $\mathrm{Nbhd}(a) \subseteq \mathcal{F}$.

For the converse: we must show that any $a \in A$ as on the right side of (1.3.1a) is in the interior of A, *i.e.*, that $A \in \mathrm{Nbhd}(a)$. Suppose by way of contradiction $A \notin \mathrm{Nbhd}(a)$. Then $X \setminus A \cap N \neq \varnothing$

for all $N \in \mathrm{Nbhd}(a)$, so $X \setminus A \cup \mathrm{Nbhd}(a)$ generates a proper filter on X. Let $\mathcal{F}$ be any ultrafilter extending it. Then $\mathcal{F}$ contains $\mathrm{Nbhd}(a)$, so $\mathcal{F} \rightsquigarrow a$. But then A must be in $\mathcal{F}$ by assumption, which is the contradiction we sought.

The second equation (1.3.1b) is the dual of (1.3.1a) via the de Morgan laws. □

Proposition 1.3.4. *For any ultrafilter $\mathcal{F}$ on a space X, define*

$$\lim \mathcal{F} := \bigcap_{B \in \mathcal{F}} \bar{B} \tag{1.3.2}$$

Then the set of convergence points of $\mathcal{F}$ is given by $\lim \mathcal{F}$.

Proof. An ultrafilter $\mathcal{F}$ on a space X converges to a point x if and only if $x \in \mathring{B}$ implies $B \in \mathcal{F}$. Equivalently, $B \notin \mathcal{F}$ implies $x \notin \mathring{B}$. Ultrafilters are prime and satisfy the FIP by Lemma 1.2.10 and Remark 1.2.7 respectively, so $B \notin \mathcal{F}$ just when $X \setminus B \in \mathcal{F}$. Moreover, we have $x \notin \mathring{B}$ just when $x \in X \setminus \mathring{B}$, which is evidently the case just when $x \in \overline{X \setminus B}$. We thus have that x is a point of convergence of $\mathcal{F}$ precisely when $B \in \mathcal{F}$ implies $x \in \bar{B}$. □

Definition 1.3.5. Let $f : X \to Y$ be a function, and let $\mathcal{U}$ be an ultrafilter on X. The the *pushforward* of $\mathcal{U}$ along f is given by

$$f_!(\mathcal{U}) := \{V \subseteq Y : f^{\leftarrow}(V) \in \mathcal{U}\} \tag{1.3.3}$$

Since preimages preserve union and intersection, it is easily observed that $f_!(\mathcal{U})$ is an ultrafilter on Y.

Proposition 1.3.6. *A function $f : X \to Y$ between topological spaces X and Y is continuous if and only if it preserves ultrafilter convergence; that is, for any ultrafilter $\mathcal{U}$ on X,*

$$\mathcal{U} \rightsquigarrow x \implies f_!(\mathcal{U}) \rightsquigarrow f(x) \tag{1.3.4}$$

Proof. First assume $f : X \to Y$ is a continuous map between topological spaces X and Y. Let $x \in X$ and assume $\mathcal{U}$ is an ultrafilter on X with $\mathcal{U} \rightsquigarrow x$. Then for any $V \ni f(x)$ open in Y, we have $f^{\leftarrow}(V)$ is open (as f is continuous) and contains x. Thus $f^{\leftarrow}(V) \in \mathcal{U}$, which implies $V \in f_!(\mathcal{U})$, so $f_!(\mathcal{U})$ converges to $f(x)$.

Conversely, assume (1.3.4) holds and let V be open in Y. Then for any ultrafilter $\mathcal{U}$ which converges to x in $f^{\leftarrow}(V)$, we have $f_!(\mathcal{U}) \rightsquigarrow f(x)$ by hypothesis (1.3.4). Now V is open and contains $f(x)$, which by Eq. (1.3.1a) of Proposition 1.3.3 implies $V \in f_!(\mathcal{U})$ and so $f^{\leftarrow}(V) \in \mathcal{U}$. Thus for any ultrafilter $\mathcal{U} \rightsquigarrow x$ in $f^{\leftarrow}(V)$, we have $f^{\leftarrow}(V) \in \mathcal{U}$, hence $f^{\leftarrow}(V)$ is open by Eq. (1.3.1a), Proposition 1.3.3. □

Recall that a topological space X is called *compact* if every open cover of X has a finite subcover, and *Hausdorff* if distinct points are separated by disjoint neighbourhoods.

Lemma 1.3.7. *A topological space X is compact if and only if every ultrafilter on X has at least one point of convergence.*

Proof. We first assume X is compact. Let $\mathcal{U}$ be an ultrafilter on X. From Proposition 1.3.4, we have that the set of points to which $\mathcal{U}$ converges is given by $\bigcap_{B\in\mathcal{U}} \bar{B}$. Suppose this set is empty, *i.e.*, $\mathcal{U}$ does not converge to any point in X. Now, X is compact means that for any family of closed sets with empty intersection, there is a finite subfamily of sets whose intersection is also empty (to see this apply de Morgan's law to the definition of compactness); therefore there must be some finite collection of closed sets in $\mathcal{U}$, say $\mathcal{K}$, such that $\bigcap \mathcal{K} = \varnothing$. But by the finite intersection property of ultrafilters, this is a contradiction. Hence $\lim \mathcal{U} \neq \varnothing$ as required.

Conversely, let X be an arbitrary topological space and assume every ultrafilter on X converges to some point in X. Suppose that X is *not* compact. Let $\mathcal{C}$ be some open cover of X with no finite subcover. Define $\mathcal{K}$ as the family of sets consisting of $X \setminus A$ for all $A \in \mathcal{C}$. We have

$$\bigcup \mathcal{C} = X \implies \bigcap \mathcal{K} = \varnothing. \tag{1.3.5}$$

Since $\mathcal{C}$ has no finite subcover, $\mathcal{K}$ has the finite intersection property, and thus by Lemma 1.2.14, extends to an ultrafilter $\mathcal{U}$ on X. But each $B \in \mathcal{K}$ is closed, and by assumption, the set of points to which $\mathcal{U}$ converges – namely $\bigcap_{B\in\mathcal{U}} \bar{B}$ – is non-empty. This is the contradiction we sought, so we conclude this proof. □

Lemma 1.3.8. *A topological space X is Hausdorff if and only if every ultrafilter on X converges to at most one point in X.*

Proof. Every ultrafilter converges to at most one point in X is precisely to say that every ultrafilter on X refines at most one neighbourhood filter. It is a consequence of the finite intersection property of ultrafilters that this statement is equivalent to the following: for all $x, y \in X$, there exists $U \in \mathrm{Nbhd}(x)$ and $V \in \mathrm{Nbhd}(y)$ with $U \cap V = \varnothing$ – the Hausdorff condition. □

There are many ways of defining Stone spaces. This definition has the advantage of making apparent a certain resemblance to the Boolean prime ideal theorem (Theorem 1.2.15).

Definition 1.3.9. A compact space X is called a *Stone space* or *Boolean space* if it satisfies the following: for all $x, y \in X$ with $x \neq y$, there is a clopen subset of X containing x but not y.

In the remainder of this section we prove some results specific to compact Hausdorff spaces which will be useful later.

Definition 1.3.10. Let X and Y be topological spaces, and let $f : X \to Y$ be a surjective continuous map. Then f is called a *quotient map* if U is open in Y whenever $f^{\leftarrow}(U)$ is open in X, for every $U \subseteq Y$.

Proposition 1.3.11. *Let X and Y be compact Hausdorff spaces, and let $f : X \to Y$ be a surjective continuous map. Then f is a quotient map.*

Proof. Let $U \subseteq Y$ with $f^{\leftarrow}(U)$ open, and let $U' := Y \setminus U$. Then $f^{\leftarrow}(U')$ is closed, so $f^{\rightarrow}(f^{\leftarrow}(U'))$ is closed as f preserves convergence of ultrafilters to a unique point (by Lemmas 1.3.7 and 1.3.8). But $f^{\rightarrow}(f^{\leftarrow}(U')) = U'$ as f is surjective. So U' is closed, and thus U is open as required. □

Corollary 1.3.12. *An injective and surjective map between compact Hausdorff spaces is a homeomorphism.*

1.3.1 Ordered topological spaces

Definition 1.3.13. A partially ordered set equipped with a topology is called a *partially ordered (topological) space*. Perhaps confusingly, what is often referred to as a *pospace* is a partially ordered space X in which the partial order is closed; that is, the set $R^{\leqslant} := \{(x, y) \in X \times X : x \leqslant y\}$ is closed in the product space $X \times X$.

Notation 1.3.14. For any partially ordered space, we write $\mathrm{Nhd}^{\uparrow}(x)$ for the neighbourhoods of x that are up-sets and $\mathrm{Nhd}^{\downarrow}(x)$ for the neighbourhoods of x that are down-sets; that is,

$$\mathrm{Nhd}^{\uparrow}(x) = \{U \subseteq X : U \in \mathrm{Nbhd}(x) \text{ and } U \text{ is an up-set}\} \tag{1.3.1a}$$

and

$$\mathrm{Nhd}^{\downarrow}(x) = \{V \subseteq X : V \in \mathrm{Nbhd}(x) \text{ and } V \text{ is a down-set}\} \tag{1.3.1b}$$

Proposition 1.3.15. *Let X be a partially ordered space. Then the following are equivalent:*

(i) X is a pospace, i.e. the partial order $R^{\leqslant} \subseteq X \times X$ is closed;

(ii) For all $x, y \in X$ such that $x \nleqslant y$, there are neighbourhoods $U \ni x$ and $V \ni y$ such that U is an up-set and $U \cap V = \varnothing$.

(iii) For all $x, y \in X$ such that $x \nleqslant y$, there are neighbourhoods $U \ni x$ and $V \ni y$ such that V is a down-set and $U \cap V = \varnothing$.

(iv) For all $x, y \in X$ such that $x \nleqslant y$, there are neighbourhoods $U \ni x$ and $V \ni y$ such that U is an up-set, V is a down-set and $U \cap V = \varnothing$.

(v) If, for all $x, y \in X$, and for each open $U \ni x, V \ni y$, there exist $u \in U$ and $v \in V$ with $u \leqslant v$; then $x \leqslant y$.

Proof. Item (i) $\iff$ Item (v) and Item (i) $\implies$ Items (iii) to (v). Item (i) holds iff $X^2 \setminus R^{\leqslant} = \{(x, y) \in X \times X : x \nleqslant y\}$ is open; that is, just when for all $x \nleqslant y$, there are neighbourhoods $U' \ni x$ and $V' \ni y$ such that $U' \times V' \subseteq X^2 \setminus R^{\leqslant}$. This is equivalent to the contrapositive of Item (v); and the existence of such U', V' implies the existence of neighbourhoods $U \ni x$ and $V \ni y$ such that

U is an up-set, V is a down-set – by putting $U := {\uparrow} U'$, $V := {\downarrow} V'$ – and $U \cap V' = U' \cap V = U' \cap V' = \varnothing$. Else there would exist $z \in {\uparrow} U' \cap V'$, that is $x' \leq z$ for some $x' \in U'$, but then $(x', z) \in R^{\leqslant} \cap (U' \times V') = \varnothing$ (and similarly for $U' \cap V \neq \varnothing$ and $U' \cap V' \neq \varnothing$).

Item (ii) $\Longrightarrow$ Item (i) Assume Item (ii) holds, and by contradiction suppose $R^{\leqslant}$ is not closed. Then there exists a point of X^2 such that

$$(x, y) \in \overline{R^{\leqslant}} \setminus R^{\leqslant} \tag{1.3.2}$$

Then, by Item (ii), there exist a neighbourhood up-set U of x and a neighbourhood V of y such that $U \cap V = \varnothing$. Since $U \times V$ is a neighbourhood of (x, y), we have $(x, y) \in \overline{R^{\leqslant}}$ implies that $(U \times V) \cap R^{\leqslant} \neq \varnothing$. That is, there exist $u \in U$ and $v \in V$ such that $u \leqslant v$. Since U is an up-set, $v \in V$ so $U \cap V \neq \varnothing$: contradiction.

Item (iii) $\Longrightarrow$ Item (i) and Item (iv) $\Longrightarrow$ Item (i) are shown analogously. □

From Item (ii), we obtain:

Corollary 1.3.16. *Every pospace is a Hausdorff space.*

Definition 1.3.17. A *compact pospace* is just what it sounds like; a pospace whose topology is compact.

It should be noted that, unfortunately, the terminology here is not standardised across the literature. For example, in [10], compact pospaces are referred to simply as compact ordered spaces.

Analogously with Stone spaces and BPI, we give a definition of Priestley spaces which resembles the distributive prime ideal theorem (Theorem 1.2.12).

Definition 1.3.18. A *Priestley space* X is a compact pospace such that for all $x, y \in X$ with $x \nleq y$, there exists a clopen up-set U such that $x \in U$ and $y \notin U$.

Although it is beyond the scope of this thesis to further discuss the correspondence between Priestley and Stone separation condions, and BPI and DPI, we refer the curious reader to Erné's paper *Prime ideal theory for general algebras* [7]. This precise topic is covered, along with several other correspondences between separation conditions for topological spaces and prime ideal theorems for algebras.

1.4 Monads and algebras

This section collates some of the requisite category theory for this thesis. Many of the proofs in this section are standard material in texts on category theory such as [16] and [2] and so are omitted. For more context and advanced content, the reader may consult [17, 4, 1].

1.4.1 Adjunctions and Monads

Let $\mathcal{A}$ and $\mathcal{B}$ be categories with functors $\mathcal{A} \underset{\mathrm{U}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{B}$.

Definition 1.4.1. Recall that F is called the *left adjoint* of U (and U the *right adjoint* of F), denoted $\mathrm{F} \dashv \mathrm{G}$, if we have natural transformations

$$1_{\mathcal{A}} \xrightarrow{\eta} \mathrm{UF} \qquad\qquad \mathrm{FU} \xrightarrow{\varepsilon} 1_{\mathcal{B}} \tag{1.4.1}$$

such that the *triangle identities* are satisfied; that is, for all $X \in \mathcal{A}$ and all $B \in \mathcal{B}$, the following diagrams commute:

$$\begin{array}{ccc} \mathrm{F}X & \xrightarrow{\mathrm{F}\eta_X} & \mathrm{FUF}X \\ & \underset{1_{\mathrm{F}X}}{\searrow} & \downarrow \varepsilon_{\mathrm{F}X} \\ & & \mathrm{F}X \end{array} \qquad \begin{array}{ccc} \mathrm{U}B & \xrightarrow{\eta_{\mathrm{U}B}} & \mathrm{UFU}B \\ & \underset{1_{\mathrm{U}B}}{\searrow} & \downarrow \mathrm{U}\varepsilon_B \\ & & \mathrm{U}B \end{array} \tag{1.4.2a, 1.4.2b}$$

Recall also that an adjunction $\mathrm{F} \dashv \mathrm{U}$ is an *adjoint equivalence of categories* when the components of the unit η and counit ε are isomorphisms, and is a *dual equivalence*, or *duality*, when the adjunction is contravariant.

Theorem 1.4.2. *Right adjoints preserve limits and left adjoints preserve colimits.*

Definition 1.4.3 (Monad)**.** A *monad*, sometimes called a *triple*, (T, η, μ) on a category $\mathcal{A}$ is an endofunctor $\mathrm{T} : \mathcal{A} \to \mathcal{A}$ equipped with two natural transformations

$$\eta : 1_{\mathcal{A}} \to \mathrm{T} \qquad \text{and} \qquad \mu : \mathrm{T}^2 \to \mathrm{T}$$

such that the following diagrams commute:

$$\begin{array}{ccc} \mathrm{T}^3 & \xrightarrow{\mathrm{T}\mu} & \mathrm{T}^2 \\ {\scriptstyle \mu_{\mathrm{T}}}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathrm{T}^2 & \xrightarrow[\mu]{} & \mathrm{T} \end{array} \qquad \begin{array}{ccc} \mathrm{T} & \xrightarrow{\eta_{\mathrm{T}}} & \mathrm{T}^2 \\ {\scriptstyle \mathrm{T}\eta}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathrm{T}^2 & \xrightarrow[\mu]{} & \mathrm{T} \end{array} \tag{1.4.3a, 1.4.3b}$$

The natural transformation μ is often referred to as the 'multiplication' for the monad. A monad on $\mathcal{A}$ is a monoid object in the category of endofunctors on $\mathcal{A}$. In this sense, η corresponds to the monoid unit (identity) and μ corresponds to the multiplication.

Definition 1.4.4. A T-*algebra* for a monad (T, η, μ) on a category $\mathcal{A}$ is given by a pair (X, ξ), where X is an object in $\mathcal{A}$ and $\xi : \mathrm{T}X \to X$ is a morphism in $\mathcal{A}$ called the *structure map*, making the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathrm{T}X \\ & {}_{1_X}\searrow & \downarrow{\scriptstyle \xi} \\ & & X \end{array} \qquad \begin{array}{ccc} \mathrm{T}^2X & \xrightarrow{\mathrm{T}\xi} & \mathrm{T}X \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \xi} \\ \mathrm{T}X & \xrightarrow[\xi]{} & X \end{array} \tag{1.4.4a, 1.4.4b}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be categories with adjunction

$$\mathcal{A} \underset{\mathrm{U}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{B} \ (\mathrm{F} \dashv \mathrm{U}) \qquad 1_{\mathcal{A}} \xrightarrow{\eta} \mathrm{UF} \qquad \mathrm{FU} \xrightarrow{\varepsilon} 1_{\mathcal{B}}$$

and monad (T, η, μ) given by $\mathrm{T} = \mathrm{UF}$.

There is a canonical category of algebras,[2] denoted $\mathcal{A}^{\mathrm{T}}$, which has T-algebras as objects, while a morphism $h\colon (X, \xi) \to (Y, \delta)$ is given by a morphism in $\mathcal{A}$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{T}X & \xrightarrow{\mathrm{T}h} & \mathrm{T}Y \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \delta} \\ X & \xrightarrow[h]{} & Y \end{array} \tag{1.4.5}$$

Definition 1.4.5. The *free* and *forgetful* functors (respectively) for monad T, with category of algebras $\mathcal{A}^{\mathrm{T}}$, are given by:

$$\begin{array}{ccc} \mathrm{F}^{\mathrm{T}}\colon \mathcal{A} & \longrightarrow & \mathcal{A}^{\mathrm{T}} \\ Y & & (\mathrm{T}Y, \mu_Y) \\ {\scriptstyle f}\uparrow & \longmapsto & \uparrow{\scriptstyle \mathrm{T}f} \\ X & & (\mathrm{T}X, \mu_X) \end{array} \qquad \begin{array}{ccc} \mathrm{U}^{\mathrm{T}}\colon \mathcal{A}^{\mathrm{T}} & \longrightarrow & \mathcal{A} \\ (Y, \delta) & & Y \\ {\scriptstyle h}\uparrow & \longmapsto & \uparrow{\scriptstyle h} \\ (X, \xi) & & X \end{array} \tag{1.4.6a, 1.4.6b}$$

Proposition 1.4.6. *Every monad arises from an adjunction. In particular, given a monad* (T, η, μ) *on a category* $\mathcal{A}$*, there exists a category* $\mathcal{B}$ *with adjunction* $\mathcal{A} \underset{\mathrm{U}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{B}$ $(\mathrm{F} \dashv \mathrm{U})$*, with unit* $\eta\colon 1_{\mathcal{A}} \to \mathrm{UF}$ *and counit* $\varepsilon\colon \mathrm{FU} \to 1_{\mathcal{B}}$ *such that* $\mathrm{UF} = \mathrm{T}$ *and* $\mathrm{U}\epsilon_{\mathrm{F}} = \mu$ *and unit remaining the same.*

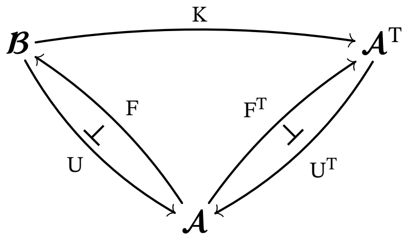
Proposition 1.4.7. *Given an adjunction* $\mathcal{A} \underset{\mathrm{U}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{B}$ $(\mathrm{F} \dashv \mathrm{U})$ *with unit* $\eta\colon 1_{\mathcal{A}} \to \mathrm{UF}$ *and counit* $\varepsilon\colon \mathrm{FU} \to 1_{\mathcal{B}}$*, gives rise to a monad* $\mathrm{T} = \mathrm{UF}$ *with multiplication* $\mu = \mathrm{U}\epsilon_{\mathrm{F}}\colon \mathrm{UFUF} \to \mathrm{UF}$ *and unit* η*.*

[2] This category is sometimes referred to as the *Eilenberg-Moore category* of algebras.

1.5 The adjoint triangle theorem

Throughout this section, assume we have an adjunction $\mathcal{A} \underset{\mathrm{U}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{B}$ ($\mathrm{F} \dashv \mathrm{U}$) with unit $\eta : 1_{\mathcal{A}} \to \mathrm{UF}$ and counit $\varepsilon : \mathrm{FU} \to 1_{\mathcal{B}}$, monad (T, η, μ) where $\mathrm{T} = \mathrm{UF}$ and multiplication $\mu = \mathrm{U}\varepsilon_{\mathrm{F}} : \mathrm{UFUF} \to \mathrm{UF}$.

Definition 1.5.1. The *comparison functor* $\mathrm{K} : \mathcal{B} \to \mathcal{A}^{\mathrm{T}}$ is the canonical functor making the following diagram commute:



(1.5.1)

Moreover, the action on objects and morphisms is given by:

$$\mathrm{K} : \mathcal{B} \longrightarrow \mathcal{A}^{\mathrm{T}}$$

$$\left(B \xrightarrow{g} C \right) \longmapsto \left((\mathrm{U}B, \mathrm{U}\varepsilon_B) \xrightarrow{\mathrm{U}g} (\mathrm{U}B, \mathrm{U}\varepsilon_C) \right) \tag{1.5.2}$$

The following result is proved in Lemma 4.3.3, [4].

Proposition 1.5.2. *For each* T*-algebra* (X, ξ)*, the structure map* ξ *is the coequaliser of the maps* $(\mu_X, \mathrm{T}\xi)$ *in* $\mathcal{A}^{\mathrm{T}}$*; the forgetful functor* U^{T} *sends the coequaliser diagram in* $\mathcal{A}^{\mathrm{T}}$ *to the coequaliser diagram*

$$\mathrm{UFUF}X \underset{\mathrm{U}\varepsilon_{\mathrm{F}X}}{\overset{\mathrm{UF}\xi}{\rightrightarrows}} \mathrm{UF}X \xrightarrow{\ \xi\ } X \tag{1.5.3}$$

in $\mathcal{A}$*. Furthermore, this diagram is an* absolute *coequaliser, meaning it is preserved by all functors.*

Definition 1.5.3. The *canonical presentation* of an object $B \in \mathcal{B}$ is given by the following diagram:

$$\mathrm{FUFU}B \underset{\varepsilon_{\mathrm{FU}B}}{\overset{\mathrm{FU}\varepsilon_B}{\rightrightarrows}} \mathrm{FU}B \xrightarrow{\ \varepsilon_B\ } B \tag{1.5.4}$$

The next result is proved in [23].

Proposition 1.5.4. *The comparison functor* K *is full and faithful if and only if the canonical presentation* (1.5.4) *is a coequaliser.*

There are several variations on the adjoint triangle theorem; the form we present here is essentially the same as proved in [23].[3] The proof provided therein does not use any particularly sophisticated methods, though it will be apparent in later chapters that the result is quite powerful.

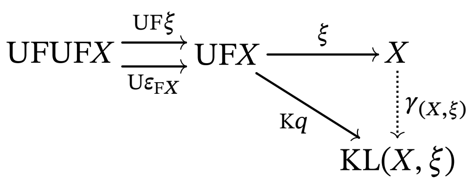
Theorem 1.5.5 (The adjoint triangle theorem (Dubuc))**.** *The comparison functor* $\mathrm{K}\colon \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}}^{\mathrm{T}}$ *has a left adjoint if and only if, for each* T*-algebra* (X, ξ)*, the pair* $(\varepsilon_{\mathrm{F}X}, \mathrm{F}\xi)$ *has a coequaliser*

$$\mathrm{FUF}X \underset{\varepsilon_{\mathrm{F}X}}{\overset{\mathrm{F}\xi}{\rightrightarrows}} \mathrm{F}X \xrightarrow{\ q\ } Q \tag{1.5.5}$$

in $\boldsymbol{\mathcal{B}}$.

(i) The left adjoint $\mathrm{L}\colon \boldsymbol{\mathcal{A}}^{\mathrm{T}} \to \boldsymbol{\mathcal{B}}$ *sends* $(X, \xi) \mapsto Q$*;*

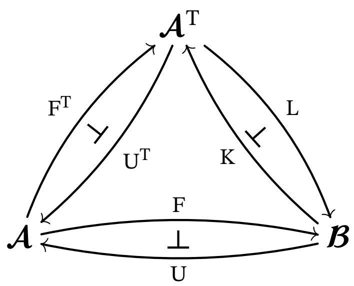
(ii) The unit of the adjunction: $\gamma_{(X,\xi)}$ *is the unique morphism of* $\boldsymbol{\mathcal{A}}^{\mathrm{T}}$ *such that* $\gamma_{(X,\xi)} \circ \xi = \mathrm{K}q$*, that is;*



(1.5.6a)

(iii) The counit of the adjunction: δ_B *is the unique morphism such that* $\delta_B \circ c' = \varepsilon_B$*, where* $c' = \mathrm{coeq}\,(\varepsilon_{\mathrm{FU}B}, \mathrm{FU}\varepsilon_B)$ *for each* $B \in \boldsymbol{\mathcal{B}}$.

Thus when the comparison functor has a left adjoint, we have the following diagram:



(1.5.7)

Notation 1.5.6. Note that when context allows the reader to infer our meaning, we will often omit subscripts for the sake of readability. Similarly, we will generally omit explicit reference to the application of forgetful functors as it is easily deduced by the reader.

[3] Though there is a small typographical error in the original, what is presented here is correct.

Chapter 2

The ultrafilter monad and Stone duality

We begin this chapter with a description of the adjunction which gives rise to the ultrafilter monad. We then prove the result due to Manes [18] that the category of algebras for the ultrafilter monad is equivalent to the category of compact Hausdorff spaces. In the last section we show that by restricting the comparison adjunction in a canonical way, we obtain Stone duality.

2.1 The ultrafilter monad

In this section we describe the adjunction that gives rise to the ultrafilter monad. We omit the routine proofs that the mappings are functorial and that the functors are adjoint.

2.1.1 The powerset and ultrafilter functors

Much of the material in this subsection appears in some form in [2].

Given a set X and subset $S \subseteq X$, consider the characteristic function $\chi : X \to \{0, 1\}$, defined by

$$\chi_S(x) = \begin{cases} 1 & x \in S, \\ 0 & x \notin S \end{cases} \tag{2.1.1}$$

So for any $S \subseteq X$, we have a characteristic function from which S can be recovered via its preimage, that is $\chi_S^{\leftarrow}(\{1\}) = S$. Furthermore, every mapping $X \to \{0, 1\}$ is of this form.

Let 2 denote the two-element set. Recall that the observation was made in Example 1.2.5 that every powerset may be regarded a Boolean algebra with operations given by the algebra of (sub)sets. Thus the representable functor **Set**$(-, 2)$ lifts to the powerset functor as defined below.

Definition 2.1.1. The *powerset functor* sends a set X to its powerset algebra $\mathcal{P}X$, and a function $f : X \to Y$ is sent to its preimage mapping $f^{\leftarrow} : \mathcal{P}Y \to \mathcal{P}X$, as in the diagram:

$$\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{BA}^{\mathrm{op}}$$

$$\begin{array}{ccc} Y & & \mathcal{P}X \\ \uparrow f & \longmapsto & \downarrow f^{\leftarrow} \\ X & & \mathcal{P}X \end{array} \tag{2.1.2}$$

Consider the two element Boolean algebra $\{1, 0\}$, which we write as $\mathbf{2}$. For any Boolean algebra B, a Boolean algebra homomorphism $u : B \to \mathbf{2}$ preserves $\vee$, $\wedge$, complement, and 0 and 1. It is easily observed that the axioms for prime filters listed in Definition 1.2.6 are satisfied for $u^{\leftarrow}(\{1\})$, and every prime filter in B can be given as $u^{\leftarrow}(\{1\})$ for some $u : B \to \mathbf{2}$ (prime filters in Boolean algebras are the same as ultrafilters by Corollary 1.2.11). It turns out that the representable functor $\mathbf{BA}(-, \mathbf{2})$ is isomorphic to the ultrafilter functor.

Definition 2.1.2. The *ultrafilter functor* sends Boolean algebras A, B to their sets of ultrafilters and sends a Boolean algebra homomorphism h to its preimage mapping, as in the following diagram:

$$\mathcal{U} : \mathbf{BA}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

$$\begin{array}{ccc} B & & \mathrm{Ult}\, B \\ \uparrow h & \longmapsto & h^{\leftarrow} \downarrow \\ A & & \mathrm{Ult}\, A \end{array} \tag{2.1.3}$$

where $\mathrm{Ult}\, B$ denotes set of ultrafilters in B (recalling $\mathrm{Ult}\, X$ was the ultrafilters on a set X in Proposition 1.3.3).

2.1.2 The ultrafilter adjunction

The ultrafilter and powerset functors form an adjunction

$$\mathbf{Set} \underset{\mathcal{U}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathbf{BA}^{\mathrm{op}} \ (\bot) \qquad 1_{\mathbf{Set}} \xrightarrow{\ \eta\ } \mathcal{U}\mathcal{P} \qquad \mathcal{P}\mathcal{U} \xrightarrow{\ \varepsilon\ } 1_{\mathbf{BA}} \,.$$

For all $X \in \mathbf{Set}$ and for all $B \in \mathbf{BA}$, the unit and counit have components given by:

$$\begin{array}{rl} \eta_X : X \longrightarrow \mathcal{U}\mathcal{P}X & \qquad \varepsilon_B : \mathcal{P}\mathcal{U}B \longrightarrow B \\ x \mapsto \{S \in \mathcal{P}X : x \in S\} & \qquad \{\mathcal{U} \in \mathcal{U}B : b \in \mathcal{U}\} \mapsto b \end{array}$$

Notation 2.1.3. Let X be a set and let B be a Boolean algebra. Put

Principal ultrafilter at $x \in X$:

$$\mathcal{U}_x := \{S \in \mathcal{P}X : S \ni x\} \tag{2.1.1a}$$

Ultrafilters containing an element $b \in B$:

$$[b] := \{\mathcal{U} \in \mathcal{U}B : b \in \mathcal{U}\} \tag{2.1.1b}$$

Note that we gave a set-based definition of the pushforward of an ultrafilter in Definition 1.3.5. We now give a functorial definition that will be useful in the context of the ultrafilter monad. Let X and Y be sets. If $u : \mathcal{P}X \to \mathbf{2}$ is a Boolean algebra homomorphism and $f : X \to Y$ a function of sets, then $u \circ f^{\leftarrow} : \mathcal{P}Y \to \mathcal{P}X \to \mathbf{2}$ is called the *pushforward* of u along f. Identifying u with the corresponding ultrafilter $\mathcal{U} \subseteq \mathcal{P}X$, its pushforward along f is given by:

$$f_!(\mathcal{U}) = \{V \subseteq Y : f^{\leftarrow}(V) \in \mathcal{U}\} \tag{2.1.2}$$

which is what was given in Definition 1.3.5.

Lemma 2.1.4. *The counit is an injective homomorphism $B \to \mathcal{P}\mathcal{U}B$ for each Boolean algebra B.*

Proof. This is a consequence of Theorem 1.2.15 and Corollary 1.2.16. □

Definition 2.1.5. The ultrafilter monad is the composite functor $\beta = \mathcal{U}\mathcal{P} :$ **Set** $\to$ **Set** sending $X \mapsto \mathcal{U}\mathcal{P}X$ and a function $f : X \to Y$ to its pushforward $f_! : \mathcal{U}\mathcal{P}X \to \mathcal{U}\mathcal{P}Y$. The unit and multiplication for the ultrafilter monad are given by the following:

$$\begin{aligned} \eta_X : X &\to \beta X \\ x &\mapsto \{A \subseteq X : A \ni x\} \end{aligned} \qquad\qquad \begin{aligned} \mu_X : \beta\beta X &\to \beta X \\ \mathfrak{U} &\mapsto \{A \subseteq X : [A] \in \mathfrak{U}\} \end{aligned}$$

2.2 Equivalence of β-algebras and KHaus

The main result of this section originally appeared in [18]. It was also proved in [14].

For each compact Hausdorff space X, furnish X with a function $\xi : \beta X \to X$ defined by

$$\mathcal{F} \rightsquigarrow x \implies \xi(\mathcal{F}) = x. \tag{2.2.1}$$

for all $\mathcal{F} \in \beta X$. By Lemmas 1.3.7 and 1.3.8 every ultrafilter on a compact Hausdorff space converges to a unique point, so ξ is well-defined.

Lemma 2.2.1. *The map $\xi : \beta X \to X$ endows a compact Hausdorff space X with β-algebra structure.*

Proof. Let X be a compact Hausdorff space. We want ξ to satisfy the conditions for the structure map of a β-algebra, given in Definition 1.4.4. The unit η_X sends every $x \in X$ to its principal ultrafilter $\mathcal{F}_x$. Any principal ultrafilter must converge to its principal point, so (1.4.4a) commutes.

Let $\mathfrak{U}$ be an ultrafilter on βX. To satisfy (1.4.4b), we require that the ultrafilters given by

$$\xi_!(\mathfrak{U}) = \{U \subseteq X : \xi^{\leftarrow}(U) \in \mathfrak{U}\} \quad \text{and} \quad \mu(\mathfrak{U}) = \{V \subseteq X : [V] \in \mathfrak{U}\}$$

converge to the same point. Let V be a closed subset of X, then from Proposition 1.3.3, we have $V = \{x \in X : (\exists \mathcal{F} \in \mathcal{U}\mathcal{P}X)\ \mathcal{F} \twoheadrightarrow x \text{ and } V \in \mathcal{F}\} = \xi^{\rightarrow}[V]$. Every $\mathcal{F} \in [V]$ is in $\xi^{\leftarrow}(V)$ which implies any ultrafilter $\mathfrak{U} \in \beta\beta X$ which contains $[V]$ must also contain $\xi^{\leftarrow}(V)$, as ultrafilters are upward closed. We therefore have

$$\bigcap_{V \in \mu(\mathfrak{U})} \overline{V} \subseteq \bigcap_{U \in \xi_!(\mathfrak{U})} \overline{U} \tag{2.2.2}$$

One inclusion is sufficient as $\mu(\mathfrak{U})$ and $\xi_!(\mathfrak{U})$ each converges to a unique point. □

Define a functor **KHaus** $\rightarrow$ $\mathbf{Set}^{\beta}$ on objects by sending a compact Hausdorff space X to the β-algebra just defined. By Proposition 1.3.6, this assignation on objects extends to a fully faithful functor from the category of compact Hausdorff spaces to the category of β-algebras. We now show it is essentially surjective on objects.

Proposition 2.2.2. *Let (X, ξ) be a β-algebra. The function $\xi^{\rightarrow}[-] : \mathcal{P}X \rightarrow \mathcal{P}X$ defined by*

$$\xi^{\rightarrow}[A] = \{\xi(\mathcal{F}) : \mathcal{F} \in [A]\} := \overline{A} \tag{2.2.3}$$

is a topological closure operator.

Proof. Note that $x \in \xi^{\rightarrow}[A] \iff [A] \cap \xi^{\leftarrow}\{x\} \neq \varnothing$ and $\mathcal{F} \in [A] \iff A \in \mathcal{F}$. Then for all $x \in A$, we have $A \in \eta(x)$ and $\xi\eta(x) = x$, so $x \in \xi[A]$, which means $A \subseteq \xi^{\rightarrow}[A]$.

Finite union: Since $\mathcal{F}$ is upward closed, the union of any $A \in \mathcal{F}$ with $B \in \mathcal{P}X$ is in $\mathcal{F}$. Conversely, $A \cup B \in \mathcal{F}$ implies $A \in \mathcal{F}$ or $B \in \mathcal{F}$ by definition of a prime filter (ultrafilters in Boolean algebras are prime – Lemma 1.2.10). So $\mathcal{F} \in [A \cup B]$ precisely when $\mathcal{F} \in [A]$ or $\mathcal{F} \in [B]$, thus $\xi(\mathcal{F}) \in \xi^{\rightarrow}[A \cup B] \iff \xi(\mathcal{F}) \in \xi^{\rightarrow}[A]$ or $\xi(\mathcal{F}) \in \xi^{\rightarrow}[B]$ so $\xi^{\rightarrow}[A] \cup \xi^{\rightarrow}[B] = \xi^{\rightarrow}[A \cup B]$. No ultrafilter contains $\varnothing$, so $\xi^{\rightarrow}[\varnothing] = \varnothing$.

It remains to be shown that $\xi^{\rightarrow}[-]$ is idempotent. We require that for $\mathcal{F} \ni \xi^{\rightarrow}[A]$ we have $\xi(\mathcal{F}) \in \xi^{\rightarrow}[A]$. Now $\xi^{\rightarrow}[A] \cap B \in \mathcal{F}$ and is therefore non-empty for each $B \in \mathcal{F}$, so there exists some $x \in \xi^{\rightarrow}[A] \cap B \in \mathcal{F}$ and so by definition of $\xi^{\rightarrow}[A]$, we have $\xi^{\leftarrow}\{x\} \cap [A]$ is non-empty. This implies $\xi^{\leftarrow}(B) \cap [A]$ is non-empty for all $B \in \mathcal{F}$, and as ultrafilters have the finite intersection property, $\xi^{\leftarrow}(B) \cap \xi^{\leftarrow}(B') \neq \varnothing$ for all $B, B' \in \mathcal{F}$. Whence there is some ultrafilter $\mathfrak{U}$ on βX such that $[A], \xi^{\leftarrow}(B) \in \mathfrak{U}$ for each $B \in \mathcal{F}$.

We therefore have $B \in \xi_!(\mathfrak{U})$ for all $B \in \mathcal{F}$, thus $\xi_!(\mathfrak{U}) = \mathcal{F}$. We know $A \in \mu(\mathfrak{U})$ and so $\xi(\mathcal{F}) = \xi(\xi_!(\mathfrak{U})) = \xi(\mu(\mathfrak{U})) \in \xi^{\rightarrow}[A]$. Thence $\xi^{\rightarrow}[-]$ is idempotent, which completes showing that $\xi^{\rightarrow}[-]$ is a topological closure operator. □

Proposition 2.2.3. *In the topology given by closure operator $\xi^{\rightarrow}[-] : \mathcal{P}X \rightarrow \mathcal{P}X$, each ultrafilter $\mathcal{F}$ has a unique point of convergence $\xi(\mathcal{F})$.*

Proof. For any $K \in \mathcal{F}$ closed, $\xi(\mathcal{F}) \in K$, so $\xi(\mathcal{F})$ is a point of convergence of $\mathcal{F}$. We now show that $\xi(\mathcal{F})$ is the only point of convergence for $\mathcal{F}$. Say $\mathcal{F} \rightsquigarrow x$ for some $x \in X$, then x is in the closure of each $A \in \mathcal{F}$. Thus for all $A \in \mathcal{F}$, we have $[A] \cap \xi^{\leftarrow}\{x\} \neq \varnothing$, and $A \cap B \neq \varnothing$ for all $B \in \mathcal{F}$, since $\mathcal{F}$ satisfies the finite intersection property. Let $\mathfrak{U}$ be any ultrafilter on βX that refines the filter generated by $\{[A] \subseteq \beta X : A \in \mathcal{F}\} \cup \{\xi^{\leftarrow}\{x\}\}$. By construction, $A \in \mu(\mathfrak{U})$ for all $A \in \mathcal{F}$, and so $\mu(\mathfrak{U}) = \mathcal{F}$. It follows from the monad axioms that $\xi(\mathcal{F}) = \xi(\mu(\mathfrak{U})) = \xi(\xi_!(\mathfrak{U}))$. But $\xi^{\leftarrow}\{x\} \in \mathfrak{U}$ implies $\{x\} \in \xi_!(\mathfrak{U})$, i.e., $\xi_!(\mathfrak{U}) = \eta(x)$. So $\xi(\mathcal{F}) = \xi(\xi_!(\mathfrak{U})) = \xi(\eta(x)) = x$ as desired. □

It follows that the functor **KHaus** → $\mathbf{Set}^\beta$ is surjective on objects, and so what we have shown in this section thus far gives us (originally shown in [18]) the following:

Theorem 2.2.4 (Manes)**.** *The category of β-algebras for the ultrafilter monad is equivalent to the category of compact Hausdorff spaces.*

2.3 Stone Duality via the comparison functor

In this section, we prove the comparison functor is full and faithful by showing the canonical presentation is a coequaliser. We then show the left adjoint of the comparison functor sends a compact Hausdorff space to the Boolean algebra of its clopen subsets. Lastly, we show the restriction of the comparison adjunction to the objects at which the unit is an isomorphism yields classical Stone duality.

The comparison functor for the ultrafilter monad is given by

$$\begin{array}{ccc} \mathcal{K}: \mathbf{BA}^{\mathrm{op}} & \longrightarrow & \mathbf{Set}^\beta \\ B & & (\mathcal{U}B, \mathcal{U}\varepsilon_B) \\ h\uparrow & \longmapsto & \downarrow \mathcal{U}h \\ A & & (\mathcal{U}A, \mathcal{U}\varepsilon_A) \end{array} \tag{2.3.1}$$

We now have a complete description of the adjoint triangle (1.5.1) for the ultrafilter monad:

- The free functor $\mathcal{P}^\beta : \mathbf{Set} \to \mathbf{Set}^\beta$ acts on a set X by sending it to the free algebra $(\beta X, \mu_X)$ – this is the special case of the Stone-Cěch compactification of X regarded as a discrete space.
- The forgetful functor $\mathcal{U}^\beta : \mathbf{Set}^\beta \to \mathbf{Set}$ simply maps a β-algebra to its underlying set.

2.3.1 The canonical presentation of a Boolean algebra

To show the comparison functor $\mathcal{K} : \mathbf{BA}^{\mathrm{op}} \to \mathbf{Set}^\beta$ is full and faithful, we require that the canonical presentation (given in (1.5.4)) of a Boolean algebra B exhibits ε_B as the coequaliser of $\varepsilon_{\mathcal{P}\mathcal{U}B}$ and $\mathcal{P}\mathcal{U}\varepsilon_B$ in $\mathbf{BA}^{\mathrm{op}}$.

Recall it was observed in Lemma 2.1.4 that ε_B is an injective homomorphism of Boolean algebras (and is thus an epimorphism in $\mathbf{BA}^{\mathrm{op}}$). For each $\mathfrak{U} \in \mathcal{UPU}B$, put

$$\lim \mathfrak{U} := \mathcal{U}\varepsilon_B(\mathfrak{U}) = \{b \in B : \varepsilon_B(b) \in \mathfrak{U}\} \tag{2.3.1}$$

The definitions given for the ultrafilter adjunction yield

$$\mathcal{PU}\varepsilon_B(\mathcal{S}) = \{\mathfrak{U} \in \mathcal{UPU}B : \lim \mathfrak{U} \in \mathcal{S}\} \tag{2.3.2a}$$

and

$$\varepsilon_{\mathcal{PU}B}(\mathcal{S}) = \{\mathfrak{U} \in \mathcal{UPU}B : \mathcal{S} \in \mathfrak{U}\} = [\mathcal{S}] \tag{2.3.2b}$$

for all $\mathcal{S} \in \mathcal{PU}B$. We claim that $\mathcal{S} \in \mathcal{PU}B$ satisfies

$$(\forall \mathfrak{U} \in \mathcal{UPU}B) \qquad \lim \mathfrak{U} \in \mathcal{S} \iff \mathcal{S} \in \mathfrak{U} \tag{2.3.3}$$

precisely when $\mathcal{S} = [b]$ for some $b \in B$.

Since the diagram of a canonical presentation always commutes, it is certainly true that (2.3.3) holds when there exists $b \in B$ such that $\mathcal{S} = [b]$.

Lemma 2.3.1. *If $\mathcal{S} \in \mathcal{PU}B$ satisfies (2.3.3), then for all $\mathcal{F} \in \mathcal{S}$ there exists some $b \in \mathcal{F}$ such that $[b] \subseteq \mathcal{S}$.*

Proof. Suppose, for a contradiction, that there is some ultrafilter $\mathcal{F} \in \mathcal{S}$ such that $[b] \nsubseteq \mathcal{S}$ for all $b \in \mathcal{F}$. This implies that for each $b \in \mathcal{F}$, the set $[b] \cap \mathcal{U}B \setminus \mathcal{S}$ is non-empty and so

$$\{[b] \in \mathcal{PU}B : b \in \mathcal{F}\} \cup \{\mathcal{U}B \setminus \mathcal{S}\} \tag{2.3.4}$$

extends to an ultrafilter $\mathfrak{U}$ on $\mathcal{PU}B$. Evidently $\lim(\mathfrak{U}) = \mathcal{F}$ so by (2.3.6) we have $\mathcal{S} \in \mathfrak{U}$. But $\mathcal{U}B \setminus \mathcal{S} \in \mathfrak{U}$ by construction. □

Proposition 2.3.2. *If $\mathcal{S} \in \mathcal{PU}B$ satisfies (2.3.3), then $\mathcal{S} = [b]$ for some $b \in B$.*

Proof. Suppose otherwise; then for all $[b] \subseteq \mathcal{S}$ it must be so that $\mathcal{S} \setminus [b] \neq \varnothing$. We then have that

$$\{\mathcal{S} \setminus [b] : [b] \subseteq \mathcal{S}\} \tag{2.3.5}$$

is a collection of non-empty sets closed under binary $\cap$, and thus extends to an ultrafilter on $\mathcal{PU}B$, say $\mathfrak{U}$. Now for all $[b] \subseteq \mathcal{S}$, we have $\mathcal{S} \setminus [b] \subseteq \mathcal{S}$, so $\mathcal{S} \in \mathfrak{U}$ (as any ultrafilter is an up-set). Also, $\lim \mathfrak{U} \in \mathcal{S}$ by assumption (2.3.3). It follows from Lemma 2.3.1 that there exists $a \in \lim \mathfrak{U}$ such that $[a] \subseteq \mathcal{S}$. Since $a \in \lim \mathfrak{U}$, we also have $[a] \in \mathfrak{U}$ by definition of lim (2.3.1). But $\mathfrak{U}$ contains $\mathcal{S} \setminus [a]$ as $\mathfrak{U}$ extends (2.3.5), and since $[a] \cap \mathcal{S} \setminus [a] = \varnothing$, it follows that $\varnothing \in \mathfrak{U}$, which is a contradiction. Whence $\mathcal{S} = [b]$ for some $b \in B$ as required. □

From the results of this subsection, we can state the the following:

Lemma 2.3.3. *The diagram*

$$B \xrightarrow{\varepsilon_B} \mathcal{F}\mathcal{U}B \underset{\varepsilon_{\mathcal{F}\mathcal{U}B}}{\overset{\mathcal{F}\mathcal{U}\varepsilon_B}{\rightrightarrows}} \mathcal{F}\mathcal{U}\mathcal{F}\mathcal{U}B \tag{2.3.6}$$

is an equaliser of Boolean algebras (and so a coequaliser in $\mathbf{BA}^{\mathrm{op}}$*).*

By Proposition 1.5.4 we obtain:

Corollary 2.3.4. *The comparison functor* $\mathcal{K}\colon \mathbf{BA}^{\mathrm{op}} \to \mathbf{Set}^{\beta}$ *is full and faithful.*

2.3.2 Left adjoint for the comparison functor

We use Theorem 1.5.5 to construct the left adjoint of $\mathcal{K}$.

$$\begin{array}{ccc} \mathcal{L}\colon \mathbf{Set}^{\beta} & \longrightarrow & \mathbf{BA}^{\mathrm{op}} \\ (X,\xi) & & \mathcal{P}\mathcal{U}\mathcal{P}X \underset{\varepsilon_{\mathcal{P}X}}{\overset{\mathcal{P}\xi}{\rightrightarrows}} \mathcal{P}X \xrightarrow{q} \mathcal{L}(X,\xi) \\ {\scriptstyle g}\Big\downarrow & \longmapsto & \Big\uparrow{\scriptstyle \mathcal{L}g} \\ (Y,\sigma) & & \mathcal{P}\mathcal{U}\mathcal{P}Y \underset{\varepsilon_{\mathcal{P}Y}}{\overset{\mathcal{P}\sigma}{\rightrightarrows}} \mathcal{P}Y \xrightarrow{c} \mathcal{L}(Y,\sigma) \end{array} \tag{2.3.1}$$

Proposition 2.3.5. *The counit* $\delta\colon \mathcal{L}\mathcal{K} \to 1_{\mathbf{BA}}$ *is invertible at each* $B \in \mathbf{BA}$.

Proof. This follows from Lemma 2.3.3 and the universal property of the coequaliser. □

Proposition 2.3.6. *The functor* $\mathcal{L}\colon \mathbf{Set}^{\beta} \to \mathbf{BA}^{\mathrm{op}}$ *sends each* β*-algebra* (X,ξ) *to the Boolean algebra of its clopen subsets.*

Proof. Let (X,ξ) be a β-algebra. By Theorem 1.5.5, the functor $\mathcal{L}$ sends (X,ξ) to the coequaliser diagram

$$\mathcal{L}(X,\xi) \xleftarrow{q} \mathcal{P}X \underset{\mathcal{P}\xi}{\overset{\varepsilon_{\mathcal{P}X}}{\leftleftarrows}} \mathcal{P}\mathcal{U}\mathcal{P}X \tag{2.3.2}$$

in $\mathbf{BA}^{\mathrm{op}}$. Hence $q\colon \mathcal{L}(X,\xi) \to \mathcal{P}X$ is an equaliser in $\mathbf{BA}$, and therefore its underlying function exhibits $\mathcal{L}(X,\xi)$ as the elements of $\mathcal{P}X$ on which $\varepsilon_{\mathcal{P}X}$ and $\mathcal{P}\xi$ coincide. We thus claim that $A \subseteq X$ is clopen in (X,ξ) just when

$$\varepsilon_{\mathcal{P}X}(A) = [A] \quad \text{and} \quad \mathcal{P}\xi(A) = \xi^{\leftarrow}(A) \tag{2.3.3}$$

are equal.

Recall that an ultrafilter $\mathcal{F} \rightsquigarrow \xi(\mathcal{F})$ in X (2.2.1). Then by Proposition 1.3.3, Eq. (1.3.1a) we have $A \subseteq X$ open if and only if $\xi^{\leftarrow}(A) \subseteq [A]$ and by Eq. (1.3.1b) (the de Morgan dual of (1.3.1a)), A is closed if and only if $[A] \subseteq \xi^{\leftarrow}(A)$. □

Proposition 2.3.7. *For each β-algebra (X, ξ), the unit $\gamma_{(X,\xi)} : (X, \xi) \longrightarrow \mathcal{K}\mathcal{L}(X, \xi)$ is surjective.*

Proof. By definition of the comparison functor $\mathcal{K} \cdot \mathcal{U}^{\beta} \cong \mathcal{U}$, so from Theorem 1.5.5 (Dubuc), we have that $\gamma_{(X,\xi)}$ is the unique function making the diagram

$$\begin{array}{ccccc} \mathcal{U}\mathcal{P}\mathcal{U}\mathcal{P}X & \overset{\mathcal{U}\mathcal{P}\xi}{\underset{\mathcal{U}\varepsilon_{\mathcal{P}X}}{\rightrightarrows}} & \mathcal{U}\mathcal{P}X & \xrightarrow{\ \xi\ } & X \\ & & & \searrow^{\mathcal{U}q} & \vdots\ \gamma \\ & & & & \mathcal{U}\mathcal{L}(X,\xi) \end{array} \tag{2.3.4}$$

commute (where $q = \mathrm{coeq}\,(\varepsilon_{\mathcal{P}X}, \mathcal{P}\xi)$ in $\mathbf{BA}^{\mathrm{op}}$). The structure map for any β-algebra is an absolute coequaliser by Proposition 1.5.2, so ξ is a surjective function. For $\gamma_{(X,\xi)}$ to be shown surjective, it suffices to show that $\mathcal{U}q$ is surjective. Let $\mathcal{G} \in \mathcal{U}\mathcal{L}(X, \xi)$. We want to show that there exists $\mathcal{F} \in \mathcal{U}\mathcal{P}X$ such that $\mathcal{G} = \mathcal{U}q(\mathcal{F})$. The action of the ultrafilter functor on morphisms means this is equally to say that

$$\mathcal{G} = \{a \in \mathcal{L}(X, \xi) : q(a) \in \mathcal{F}\} \tag{2.3.5}$$

It was shown in Proposition 2.3.6 that $\mathcal{L}(X, \xi)$ is (isomorphic) to the algebra of clopen subsets of the space (X, ξ), and q is a Boolean algebra embedding $\mathcal{L}(X, \xi)$ into the powerset algebra $\mathcal{P}X$. Thus $\mathcal{G}$ is an ultrafilter on the clopen subsets of X. It follows that $\mathcal{G}$ has the finite intersection property in $\mathcal{P}X$. So by Lemma 1.2.14 (BPI), the image of $\mathcal{G}$ under q extends to some ultrafilter $\mathcal{F}$ on $\mathcal{P}X$.

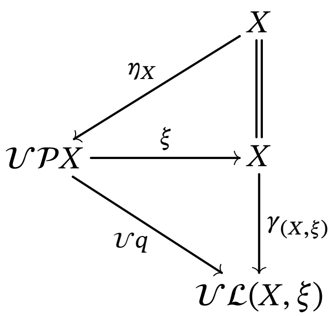
It remains to be shown that $\mathcal{U}q$ sends $\mathcal{F}$ to $\mathcal{G}$. By definition of the action of $\mathcal{U}$ on morphisms, we have

$$\mathcal{U}q(\mathcal{F}) = \{a \in \mathcal{L}(X, \xi) : q(a) \in \mathcal{F}\} \tag{2.3.6}$$

For each $a \in \mathcal{G}$, we have $q(a) \in \mathcal{F}$ by construction. For the converse; observe that for any $a \in \mathcal{U}q(\mathcal{F})$ that is not in $\mathcal{G}$, its complement $a^{\perp}$ must be. But q preserves complementarity, so this would imply $q(a) \in \mathcal{F}$ and $q(a)^{\perp} \in \mathcal{F}$, which is absurd. □

Lemma 2.3.8. *The unit is an injective mapping of β-algebras $\gamma : (X, \xi) \to \mathcal{K}\mathcal{L}(X, \xi)$ if and only if (X, ξ) is a Stone space.*

Proof. Let (X, ξ) be a β-algebra. We first note that X endowed with β-algebra structure by ξ is (isomorphic to) a compact Hausdorff space by Proposition 2.2.3. Using the unit condition for a monad and Theorem 1.5.5 (Dubuc), we have diagram of sets



(2.3.7)

It is easy to see from the diagram that the following holds:

$$\mathcal{U}q \circ \eta_X = \gamma \circ \xi \circ \eta_X = \gamma \tag{2.3.8}$$

Then (denoting the principal ultrafilter at x as $\mathcal{F}_x$ as usual) for all $x \in X$:

$$\begin{aligned} \gamma(x) &= \mathcal{U}q \circ \eta(x) \\ &= \mathcal{U}q(\mathcal{F}_x) \\ &= \{a \in \mathcal{L}(X,\xi) : q(a) \in \mathcal{F}_x\} \end{aligned}$$

from Eq. (2.3.6).

Now for any $x \in X$, put $A_x := \{a \in \mathcal{L}(X,\xi) : q(a) \in \mathcal{F}_x\}$. Evidently γ is injective precisely when $A_x = \gamma(x) \neq \gamma(y) = A_y$ for all distinct $x, y \in X$. That is to say – without loss of generality – that there exists some $a \in A_x \backslash A_y$. But this is equivalent to the existence of a clopen $q(a) \in \mathcal{F}_x \backslash \mathcal{F}_y$, and conversely, every clopen containing x but not y is of this form (Proposition 2.3.6). Therefore to say there exists $a \in A_x$ which is not in A_y is to say there is a clopen in X containing x but not y. We conclude $\gamma_{(X,\xi)}$ is injective if and only if (X, ξ) is a Stone space as required. □

By Corollary 1.3.12 and Proposition 2.3.7, the unit γ is surjective and therefore a quotient map of compact Hausdorff spaces. Therefore γ is injective just in case it is an isomorphism, and thus we obtain:

Corollary 2.3.9. *By restricting the $\mathcal{K} \vdash \mathcal{L}$ adjunction to the objects at which γ is an isomorphism, we obtain an equivalence of categories* $\mathbf{BA}^{\mathrm{op}} \simeq \mathbf{Stone}$.

Chapter 3

The prime filter monad and Priestley duality

In this chapter, we start by describing the prime filter and up-set functors, thereby detailing the adjunction they give between partially ordered sets and bounded distributive lattices. This adjunction gives rise to the prime filter monad. We then show the category of algebras of the prime filter monad is equivalent to the category of compact pospaces, proved originally by Flagg [8]. In the final section we prove that the comparison adjunction restricts canonically to Priestley duality.

3.1 The prime filter monad

This section is based on the description of the prime filter monad given in [8].

3.1.1 The up-set and prime filter functors

Recall it was proved in Lemma 1.1.11 that preimages of up-sets are up-sets under monotone maps.

Definition 3.1.1. The *up-set functor* sends an ordered set $(X, \leqslant)$ to its algebra of up-sets $\mathsf{Up}(X, \leqslant)$, and a monotone map $f : (X, \leqslant) \to (Y, \leqslant)$ is sent to its preimage mapping $f^{\leftarrow} : \mathsf{Up}(Y, \leqslant) \to \mathsf{Up}(X, \leqslant)$, as in the diagram:

$$\mathsf{Up} : \ \mathbf{Poset} \longrightarrow \mathbf{DLat}^{\mathrm{op}}$$

$$\begin{array}{ccc} (Y, \leqslant) & & \mathsf{Up}(Y, \leqslant) \\ \uparrow f & \longmapsto & \downarrow f^{\leftarrow} \\ (X, \leqslant) & & \mathsf{Up}(X, \leqslant) \end{array} \tag{3.1.1}$$

Definition 3.1.2. The *prime filter functor* sends distributive lattices C, D to the set of their prime filters, ordered by inclusion, and sends a lattice homomorphism h to its preimage mapping, as in the following diagram:

$$\mathsf{P} : \ \mathbf{DLat}^{\mathrm{op}} \longrightarrow \mathbf{Poset}$$

$$\begin{array}{ccc} D & & (\mathsf{P}D, \subseteq) \\ \uparrow h & \longmapsto & h^{\leftarrow} \downarrow \\ C & & (\mathsf{P}C, \subseteq) \end{array} \tag{3.1.2}$$

Notation 3.1.3. Let $(X, \leqslant)$ be an ordered set and let L be a distributive lattice. We introduce (and recall) some notation:

Principal up-set at $x \in (X, \leqslant)$:

$$\uparrow x := \{y \in X \,:\, x \leqslant y\} \tag{3.1.3a}$$

Up-set generated by $S \subseteq X$:

$$\uparrow S := \{y \in X \,:\, (\exists x \in S)\; x \leqslant y\} \tag{3.1.3b}$$

For emphasis (we may neglect it), if $A \subseteq (X, \leqslant)$ is an up-set, we write:

$$A^{\uparrow} \in \mathsf{Up}(X, \leqslant) \tag{3.1.3c}$$

Principal (prime) filter at $a \in L$:

$$\mathcal{F}^{a} := \{b \in L \,:\, a \leqslant b\} \tag{3.1.3d}$$

Prime filters containing $a \in L$:

$$[\![a]\!] = \{\mathcal{F} \in \mathsf{P}L \,:\, a \in \mathcal{F}\} \tag{3.1.3e}$$

The unit and counit for the prime filter adjunction are given by the following:

$$\begin{aligned} \hat{\eta}_X : X &\to \mathsf{P}\,\mathsf{Up}\,X \\ x &\mapsto \{A^{\uparrow} \in \mathsf{Up}(X, \leqslant) \,:\, A^{\uparrow} \ni x\} \end{aligned} \qquad \begin{aligned} \hat{\varepsilon}_D : \ \mathsf{Up}\,\mathsf{P}D &\to D \\ [\![a]\!] &\mapsto a \end{aligned}$$

Lemma 3.1.4. *The counit is an injective homomorphism $D \to \mathsf{Up}\,\mathsf{P}D$ for each distributive lattice D.*

Proof. This follows from Theorem 1.2.12 (DPI), Corollary 1.2.13. □

Definition 3.1.5. The prime filter monad is the composite functor $\hat{\beta} = \mathsf{P}\,\mathsf{Up} : \ \mathbf{Poset} \to \mathbf{Poset}$ The multiplication for the prime filter monad is given by the following:

$$\begin{aligned} \hat{\mu}_X : \hat{\beta}\hat{\beta}X &\to \hat{\beta}X \\ \mathfrak{F} &\mapsto \{A \in \mathsf{Up}(X, \leqslant) \,:\, [\![A]\!] \in \mathfrak{F}\} \end{aligned}$$

3.2 Equivalence of $\hat{\beta}$-algebras and KPos

For any ordered set $(X, \leqslant)$ and for any prime filter $\mathcal{F}$ on $\mathsf{Up}(X, \leqslant)$, define $\mathcal{F}^\sharp$ as the filter on $\mathsf{Up}(X, \leqslant)$ generated by

$$\mathcal{F} \cup \{X \setminus A : A \in \mathsf{Up}(X, \leqslant) \setminus \mathcal{F}\} = \mathcal{F} \cup \{X \setminus A : (A \in \mathsf{Up}\, X)\, A \notin \mathcal{F}\}. \tag{3.2.1}$$

We define a function $\rho_X : \beta X \to \hat{\beta}(X, \leqslant)$ by

$$\rho_X(\mathcal{U}) = \mathcal{U} \cap \mathsf{Up}(X, \leqslant). \tag{3.2.2}$$

Lemma 3.2.1. *For each ordered set* $(X, \leqslant)$*,*

(i) given an ultrafilter $\mathcal{U} \in \beta X$*, we have* $\rho_X(\mathcal{U}) = \mathcal{F} \in \hat{\beta}(X, \leqslant)$ *if and only if* $\mathcal{F}^\sharp \subseteq \mathcal{U}$*; and*

(ii) ρ_X *is surjective.*

Proof. The intersection of an ultrafilter $\mathcal{U}$ on X with $\mathsf{Up}(X, \leqslant)$ yields the restriction of $\mathcal{U}$ to its elements which are up-sets, *i.e.* this restriction is a directed, upward-closed subset of $\mathsf{Up}(X, \leqslant)$ with the finite intersection property. This is a prime filter on $\mathsf{Up}(X, \leqslant)$, so ρ is well-defined.

Item (i): Observe that $A^\uparrow \in \mathcal{F}$ on $\mathsf{Up}(X, \leqslant)$ if and only if $A^\uparrow \notin \mathsf{Up}\, X \setminus \mathcal{F}$, which is to say we have

$$X \setminus A^\uparrow \notin \mathcal{U} \setminus \mathcal{F} \iff X \setminus A^\uparrow \notin \mathcal{U} \tag{3.2.3}$$

by the finite intersection property of ultrafilters. Thus $\rho_X(\mathcal{U}) = \mathcal{F}$ just when

$$A^\uparrow \in \mathcal{F} \implies A^\uparrow \in \mathcal{U} \quad \text{and} \quad A^\uparrow \in \mathsf{Up}(X, \leqslant) \setminus \mathcal{F} \implies X \setminus A^\uparrow \in \mathcal{U} \tag{3.2.4}$$

which is precisely when $\mathcal{F}^\sharp \subseteq \mathcal{U}$.

Item (ii): For any prime filter $\mathcal{F}$ on $\mathsf{Up}(X, \leqslant)$, we have $\mathsf{Up}(X, \leqslant) \setminus \mathcal{F} := \mathcal{I}$ is a prime ideal. We thus have $B, B' \in \mathcal{I}$ implies $B \cup B' \in \mathcal{I}$ since $\mathcal{I}$ is an ideal. Consequently, $(X \setminus B) \cap (X \setminus B') = X \setminus (B \cup B') \neq \varnothing$. If $A \in \mathcal{F}$ and $B \in \mathcal{I}$, then $A \nsubseteq B$, since $\mathcal{F}$ is upward-closed, so $A \cap (X \setminus B) \neq \varnothing$. Whence for each prime filter $\mathcal{F}$ on $\mathsf{Up}(X, \leqslant)$, we have $\mathcal{F}^\sharp$ is a proper filter on X and so extends to an ultrafilter. It follows by Item (ii) that ρ is surjective. □

Lemma 3.2.2. *Given an ordered set* $(X, \leqslant)$*, the following diagrams of sets commute:*

$$\begin{array}{ccc} & X & \\ {\scriptstyle \eta_X}\swarrow & & \searrow{\scriptstyle \hat{\eta}_X} \\ \beta X & \xrightarrow[\rho_X]{} & \hat{\beta}(X, \leqslant) \end{array} \qquad \begin{array}{ccccc} \beta\beta X & \xrightarrow{\beta(\rho_X)} & \beta\hat{\beta}(X, \leqslant) & \xrightarrow{\rho_{\beta X}} & \hat{\beta}\hat{\beta}(X, \leqslant) \\ {\scriptstyle \mu_X}\downarrow & & & & \downarrow{\scriptstyle \hat{\mu}_X} \\ \beta X & & \xrightarrow[\rho_X]{} & & \hat{\beta}(X, \leqslant). \end{array} \tag{3.2.5a,3.2.5b}$$

Proof. It is immediate from the definitions that triangle (3.2.5a) commutes. For the diagram (3.2.5b), we trace $\mathfrak{U} \in \beta\beta X$ around the two sides:

$$\begin{array}{c} \mathfrak{U} \\ \big\downarrow \\ \{A \subseteq X \,:\, [A] \in \mathfrak{U}\} \longmapsto \{A^{\uparrow} \in \mathsf{Up}(X) \,:\, [A^{\uparrow}] \in \mathfrak{U}\} \end{array}$$

and

$$\begin{array}{r} \mathfrak{U} \longmapsto \{\mathcal{H} \subseteq \hat{\beta}(X, \leqslant) \,:\, \rho_X^{\leftarrow}(\mathcal{H}) \in \mathfrak{U}\} \longmapsto \{\mathcal{H} \in \mathsf{Up}(\hat{\beta}(X, \leqslant)) \,:\, \rho_X^{\leftarrow}(\mathcal{H}) \in \mathfrak{U}\} \\ \big\downarrow \\ \{A \in \mathsf{Up}(X) \,:\, \rho_X^{\leftarrow}([\![A]\!]) \in \mathfrak{U}\}\,. \end{array}$$

Observing that for any up-set A in X, we have

$$[A] = \{\mathcal{U} \in \beta X \,:\, A \in \mathcal{U}\} = \{\mathcal{U} \in \beta X \,:\, A \in \mathcal{U} \cap \mathsf{Up}(X)\} = \rho^{\leftarrow}([\![A]\!])\,. \tag{3.2.6}$$

We therefore obtain the equality desired. □

For any ordered set $(X, \leqslant)$, define a preorder $\lesssim$ on βX using ρ (3.2.2) as follows:

$$\mathcal{U} \lesssim \mathcal{V} \iff \rho(\mathcal{U}) \leqslant \rho(\mathcal{V}) \iff \mathcal{U} \cap \mathsf{Up}(X, \leqslant) \subseteq \mathcal{V} \cap \mathsf{Up}(X, \leqslant) \tag{3.2.7}$$

We write $\beta(X, \lesssim)$ to denote βX under this preordering. On doing so, $\rho_X : \beta(X, \lesssim) \to \hat{\beta}(X, \leqslant)$ preserves and reflects order; as it is also surjective, we obtain the following:

Lemma 3.2.3. *The map $\rho_X : \beta(X, \lesssim) \to \hat{\beta}(X, \leqslant)$ exhibits $\hat{\beta}(X, \leqslant)$ as the partial order collapse of the preorder on $\beta(X, \lesssim)$; i.e., if Y is any ordered set, and $\varphi : \beta(X, \lesssim) \to (Y, \leqslant)$ is a monotone map, then there is a unique monotone map $\hat{\varphi} : \hat{\beta}(X, \leqslant) \to (Y, \leqslant)$ such that $\hat{\varphi} \circ \rho = \varphi$ as depicted below.*

$$\begin{array}{ccc} \beta(X, \lesssim) & \xrightarrow{\ \rho\ } & \hat{\beta}(X, \leqslant) \\ {\scriptstyle \varphi}\big\downarrow & \swarrow{\scriptstyle \hat{\varphi}} & \\ (Y, \leqslant) & & \end{array} \tag{3.2.8}$$

Lemma 3.2.4. *For a partially ordered space $(X, \leqslant)$, the following are equivalent:*

(i) $(X, \leqslant)$ is a compact pospace.

(ii) X is compact Hausdorff, and taking limits of ultrafilters yields a monotone map $\xi : \beta(X, \lesssim) \to (X, \leqslant)$.

Proof. Assume Item (i) holds. As $(X, \leqslant)$ is Hausdorff by Corollary 1.3.16, X is compact and Hausdorff. We thus have by definition that ξ sends each ultrafilter of a compact Hausdorff space to its unique convergent point. To show $\xi : \beta X \to X$ is monotone, it must be shown that if $\mathcal{F} \lesssim \mathcal{G}$ in $\beta(X, \lesssim)$ then $\xi(\mathcal{F}) \leqslant \xi(\mathcal{G})$ in $(X, \leqslant)$. So given ultrafilters $\mathcal{F} \lesssim \mathcal{G}$ with $\xi(\mathcal{F}) = x$ and $\xi(\mathcal{G}) = y$, by definition of convergence $\mathrm{Nbhd}(x) \subseteq \mathcal{F}$ and $\mathrm{Nbhd}(y) \subseteq \mathcal{G}$. From Item (ii) Proposition 1.3.15, to show $x \leqslant y$ it suffices to show that every $U \in \mathrm{Nhd}^{\uparrow}(x)$ and $V \in \mathrm{Nbhd}(y)$ have non-empty intersection. Let $A \in \mathrm{Nbhd}(x)$ and $B \in \mathrm{Nbhd}(y)$, then $A \in \mathcal{F}$ and $B \in \mathcal{G}$. We have assumed $\mathcal{F} \lesssim \mathcal{G}$, so $\uparrow A \in \mathcal{G}$ (3.2.7), which means $\varnothing \neq \uparrow A \cap B$ by the finite intersection property of ultrafilters.

We show Item (ii) implies Item (i). We use Item (iii), Proposition 1.3.15. Assume that every $U \in \mathrm{Nbhd}(x)$ and $V \in \mathrm{Nhd}^{\downarrow}(y)$ have non-empty intersection. To show that X is a pospace, it is sufficient to show that $x \leq y$. By assumption, $\mathrm{Nbhd}(x) \cup \mathrm{Nhd}^{\downarrow}(y)$ has the finite intersection property, and is hence extended by an ultrafilter $\mathcal{F}$ which clearly converges to x, as does $\rho(\mathcal{F}) \cup \mathrm{Nbhd}(y)$. To see this, suppose for the contrary that for all $U \in \rho(\mathcal{F}), V \in \mathrm{Nbhd}(y)$, we have $U \cap V = \varnothing$. Then we have $V \subseteq X \setminus U$, so that $X \setminus U \in \mathrm{Nbhd}(y)$, and as it is a down-set as well, $X \setminus U \in \mathrm{Nhd}^{\downarrow}(y) \subseteq \mathcal{F}$ – a contradiction.

Let $\mathcal{G}$ be an ultrafilter extending $\rho(\mathcal{F}) \cup \mathrm{Nbhd}(y)$, then $\mathcal{G}$ clearly converges to y. We then have $\rho(\mathcal{F}) \subseteq \rho(\mathcal{G})$, so $\mathcal{F} \lesssim \mathcal{G}$ by (3.2.7), and as ξ is monotone, $x = \xi(\mathcal{F}) \leq \xi(\mathcal{G}) = y$. □

The results in this section up to now show how $\hat{\beta}$-algebras correspond to compact pospaces. By Lemma 3.2.4, a compact pospace structure on an ordered set $(X, \leqslant)$ is endowed by a map ξ which satisfies the β-algebra axioms and is monotone. From Lemma 3.2.3, such a ξ corresponds to a monotone map $\hat{\xi}$ with commutative diagram

$$\begin{array}{ccc} \beta(X, \lesssim) & \xrightarrow{\rho_X} & \hat{\beta}(X, \leqslant) \\ \downarrow \xi & \swarrow \hat{\xi} & \\ (X, \leqslant) & & \end{array} \tag{3.2.9}$$

To complete our description of the correspondence between $\hat{\beta}$-algebras and compact pospaces, it remains to be shown that the β-algebra axioms correspond to the $\hat{\beta}$-algebra axioms. This is essentially the content of the following theorem. The original, due to Flagg, is in [8].

Theorem 3.2.5 (Flagg)**.** *The category of algebras for the monad $\hat{\beta}$ on* $\mathbf{Poset}^{\hat{\beta}}$ *is equivalent to the category* **KPos** *of compact pospaces and continuous monotone maps.*

Proof. Suppose $(X, \leqslant)$ is an ordered set endowed with $\hat{\beta}$-algebra structure; that is, we have a structure map $\hat{\xi} : \hat{\beta}(X, \leqslant) \to (X, \leqslant)$ satisfying the monad axioms. Precomposition with ρ_X yields a monotone map

$$\xi = \hat{\xi} \circ \rho : \beta(X, \lesssim) \to (X, \leqslant) \tag{3.2.10}$$

Consider the following diagrams of sets:

$$
\begin{array}{ccc}
 & X & \\
{\scriptstyle \eta_X}\swarrow & & \searrow{\scriptstyle \hat{\eta}_X} \\
\beta X & \xrightarrow{\ \rho_X\ } & \hat{\beta} X \\
{\scriptstyle \xi}\searrow & & \swarrow{\scriptstyle \hat{\xi}} \\
 & X &
\end{array}
\qquad
\begin{array}{ccccc}
\beta\beta X & \xrightarrow{\ \beta\rho_X\ } & \beta\hat{\beta} X & \xrightarrow{\ \beta\hat{\xi}\ } & \beta X \\
\downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \rho_{\hat{\beta} X}} & & \downarrow{\scriptstyle \rho_X} \\
\beta X & & \hat{\beta}\hat{\beta} X & \xrightarrow{\ \hat{\beta}\hat{\xi}\ } & \hat{\beta} X \\
 & {\scriptstyle \rho_X}\searrow & \downarrow{\scriptstyle \hat{\mu}_X} & & \downarrow{\scriptstyle \hat{\xi}} \\
 & & \hat{\beta} X & \xrightarrow{\ \hat{\xi}\ } & X
\end{array}
\tag{3.2.11a, 3.2.11b}
$$

where $\hat{\beta} X$ denotes $\hat{\beta}(X, \leqslant)$ and where $\hat{\beta}\hat{\beta} X$ denotes $\hat{\beta}(\hat{\beta}(X, \leqslant), \leqslant)$. The unit axiom $\hat{\xi} \circ \widehat{\eta_X} = 1_X$ then implies the unit axiom $\xi \circ \eta_X = 1_X$ (3.2.11a), and the multiplication axiom $\hat{\xi} \circ \widehat{\mu_X} = \hat{\xi} \circ \hat{\beta}\hat{\xi}$ implies the multiplication axiom $\xi \circ \mu_X = \xi \circ \beta\xi$. Thence $(X, \xi, =)$ is a β-algebra, and thus a compact Hausdorff space. Moreover, $\xi : \beta(X, \lesssim) \to (X, \leqslant)$ is monotone under the order induced on βX by ρ_X, which is equivalent to saying $(X, \leqslant)$ is a compact pospace by Lemma 3.2.4.

Any map of $\hat{\beta}$-algebras, say $h : (X, \leqslant, \hat{\xi}) \to (Y, \leqslant, \hat{\sigma})$, has an underlying monotone map $(X, \leqslant) \to (Y, \leqslant)$. To see that h is also continuous; we can write h as a map of β-algebras $(X, \hat{\xi} \circ \rho_X) \to (Y, \hat{\sigma} \circ \rho_Y)$ by Eq. (3.2.10), so h is a morphism between compact Hausdorff spaces. Thus we have defined the action of a functor $\Phi :$ $\mathbf{Poset}^{\hat{\beta}} \to \mathbf{KPos}$.

On the other hand, assume $f : X \to Y$ underlies a continuous and monotone map between $\hat{\beta}$-algebras $(X, \leqslant, \hat{\xi})$ and $(Y, \leqslant, \hat{\sigma})$, and consider the following diagram:

$$
\begin{array}{ccc}
\beta X & \xrightarrow{\ \beta f\ } & \beta Y \\
{\scriptstyle \rho_X}\downarrow & & \downarrow{\scriptstyle \rho_Y} \\
\hat{\beta} X & \xrightarrow{\ \hat{\beta} f\ } & \hat{\beta} Y \\
{\scriptstyle \hat{\xi}}\downarrow & & \downarrow{\scriptstyle \hat{\sigma}} \\
X & \xrightarrow[\ f\ ]{} & Y
\end{array}
\tag{3.2.12}
$$

where we write $\hat{\beta} X$ (respectively $\hat{\beta} Y$) to mean $\hat{\beta}(X, \leqslant)$ and βX (respectively βY) here has the order induced by ρ as in Lemma 3.2.3. The outer and upper squares commute by construction, and since ρ is epimorphic, the lower square also commutes. Thus f is a map of $\hat{\beta}$-algebras, showing the functor $\Phi :$ $\mathbf{Poset}^{\hat{\beta}} \to \mathbf{KPos}$ is fully faithful.

We now show $\Phi :$ $\mathbf{Poset}^{\hat{\beta}} \to \mathbf{KPos}$ is surjective on objects. Let X be a compact pospace, *i.e.* an ordered set $(X, \leqslant)$ equipped with a topology in which the partial order is closed in the (binary) product space. By Corollary 1.3.16, X is a compact Hausdorff space, and thus by Lemma 2.2.1 the function $\xi : \beta X \to X$ which maps an ultrafilter to its unique convergent point endows X with β-algebra structure in **Set**. Furthermore, ξ is monotone as a map $\beta(X, \lesssim) \to (X, \leqslant)$ by

Lemma 3.2.4, and so by Lemma 3.2.3 we obtain a unique monotone map $\hat{\xi}: \hat{\beta}(X, \leqslant) \to (X, \leqslant)$ such that $\hat{\xi} \circ \rho_X = \xi$.

We claim $\hat{\xi}$ endows the compact pospace X with $\hat{\beta}$-algebra structure. It is clear that (3.2.11a) commutes, so the unit axiom is satisfied. For the multiplication axiom for $\hat{\beta}$-algebras, we consider (3.2.11b). The left parallelogram commutes by Lemma 3.2.2. The upper right parallelogram commutes by naturality of ρ. Observe that tracing around the outside, we have commutativity by the multiplication axiom for β-algebras. It follows that the precomposition of the maps in the bottom square with $\rho_{\hat{\beta}X} \circ \beta\rho_X$ are equal, and since $\rho_{\hat{\beta}X} \circ \beta\rho_X$ is epimorphic, we conclude the bottom square commutes. Hence $(X, \leqslant, \hat{\xi})$ is a $\hat{\beta}$-algebra, and since by construction $\hat{\xi} \circ \rho_X = \xi$, the image of the $\hat{\beta}$-algebra under Φ is the original compact pospace X. Hence we have shown the desired equivalence of categories $\mathbf{Poset}^{\hat{\beta}} \simeq \mathbf{KPos}$. □

3.3 Priestley duality from the comparison functor

The comparison functor for the prime filter monad is given by

$$\begin{array}{ccc} \hat{\mathcal{K}}: \mathbf{DLat}^{\mathrm{op}} & \longrightarrow & \mathbf{Poset}^{\hat{\beta}} \\ D & & (\mathsf{P}D, \mathsf{P}\varepsilon_D) \\ \uparrow h & \longmapsto & \downarrow \mathsf{P}h \\ C & & (\mathsf{P}C, \mathsf{P}\varepsilon_C) \end{array} \tag{3.3.1}$$

We therefore have the following functors:

- The free functor $\mathsf{Up}^{\hat{\beta}}: \mathbf{Poset} \to \mathbf{Poset}^{\hat{\beta}}$ acts on an ordered set $(X, \leqslant)$ by sending it to the free algebra $(\beta X, \mu_X)$ – this is referred to in [8] as the Stone-Cěch-Nachbin compactification of $(X, \leqslant)$.
- The forgetful functor $\mathsf{P}^{\hat{\beta}}: \mathbf{Poset}^{\hat{\beta}} \to \mathbf{Poset}$ maps a $\hat{\beta}$-algebra to its underlying ordered set.

This completes our description of the adjoint triangle (1.5.1) for the prime filter monad.

3.3.1 The canonical presentation of a bounded distributive lattice

To show the comparison functor $\hat{\mathcal{K}}: \mathbf{DLat}^{\mathrm{op}} \to \mathbf{Poset}^{\hat{\beta}}$ is full and faithful, we require that the canonical presentation (definition given in Eq. (1.5.4)) of a distributive lattice D exhibits $\hat{\varepsilon}_D$ as the coequaliser of $\hat{\varepsilon}_{\mathsf{Up}\,\mathsf{P}D}$ and $\mathsf{Up}\,\mathsf{P}\hat{\varepsilon}_D$ in $\mathbf{DLat}^{\mathrm{op}}$. It was observed in Lemma 3.1.4 that $\hat{\varepsilon}_D$ is an injective homomorphism of distributive lattices (and is thus an epimorphism in $\mathbf{DLat}^{\mathrm{op}}$). For each $\mathfrak{F} \in \mathsf{P}\,\mathsf{Up}\,\mathsf{P}D$, define

$$\lim \mathfrak{F} := \mathsf{P}\hat{\varepsilon}_D(\mathfrak{F}) = \{a \in D : \hat{\varepsilon}_D(a) \in \mathfrak{F}\} \tag{3.3.1}$$

From the definitions given for the prime filter adjunction, for all $\mathcal{H} \in \mathsf{Up}\,\mathsf{P}D$, we have the following:

$$\mathsf{Up}\,\mathsf{P}\hat{\varepsilon}_D(\mathcal{H}) = \{\mathfrak{F} \in \mathsf{P}\,\mathsf{Up}\,\mathsf{P}D : \lim \mathfrak{F} \in \mathcal{H}\}$$

and

$$\hat{\varepsilon}_{\mathsf{Up}\,\mathsf{P}D}(\mathcal{H}) = \{\mathfrak{F} \in \mathsf{P}\,\mathsf{Up}\,\mathsf{P}D : \mathcal{H} \in \mathfrak{F}\} = [\![\mathcal{H}]\!]$$

We claim that $\mathcal{H} \in \mathsf{Up}\,\mathsf{P}D$ satisfies

$$(\forall \mathfrak{F} \in \mathsf{P}\,\mathsf{Up}\,\mathsf{P}D) \qquad \lim \mathfrak{F} \in \mathcal{H} \iff \mathcal{H} \in \mathfrak{F} \tag{3.3.2}$$

if and only if $\mathcal{H} = [\![a]\!]$ for some $a \in D$.

As in the case of Boolean algebras, we note that the diagram of a canonical presentation always commutes, so the 'if' direction is immediate. The next two lemmas will show the converse.

Lemma 3.3.1. *If $\mathcal{H} \in \mathsf{Up}\,\mathsf{P}D$ satisfies (3.3.2), then for all $\mathcal{F} \in \mathcal{H}$ there exists some $a \in \mathcal{F}$ such that $[\![a]\!] \subseteq \mathcal{H}$.*

Proof. Suppose by way of contradiction that $[\![a]\!] \nsubseteq \mathcal{H}$ for all $a \in D$. Then in particular there exists $\mathcal{F} \in \mathcal{H}$ with $[\![a]\!] \nsubseteq \mathcal{H}$ for all $a \in \mathcal{F}$. Therefore the filter generated by

$$\{[\![a]\!] \in \mathsf{Up}\,\mathsf{P}D : a \in \mathcal{F}\} \tag{3.3.3}$$

does not contain $\mathcal{H}$. By Theorem 1.2.12 (DPI), the filter in (3.3.3) is contained in a prime filter $\mathfrak{F}$ which also does not contain $\mathcal{H}$. It is easy to see that $\mathcal{F} \subseteq \lim \mathfrak{F}$, and since $\mathcal{F} \in \mathcal{H}$ we must have $\lim \mathfrak{F} \in \mathcal{H}$. But then by (3.3.2) $\mathcal{H} \in \mathfrak{F}$, which – by construction of $\mathfrak{F}$ – is a contradiction. □

Proposition 3.3.2. *If $\mathcal{H} \in \mathsf{Up}\,\mathsf{P}D$ satisfies (3.3.2), then for all $\mathcal{F} \in \mathcal{H}$ there exists some $a \in \mathcal{F}$ such that $[\![a]\!] = \mathcal{H}$.*

Proof. Consider the ideal generated by the following:

$$\{[\![a]\!] \in \mathsf{Up}\,\mathsf{P}D : [\![a]\!] \subseteq \mathcal{H}\}. \tag{3.3.4}$$

Suppose the statement in our proposition is false. Then there is a prime ideal containing (3.3.4) and a prime filter $\mathfrak{F}$ containing $\mathcal{H}$ which are disjoint, by Theorem 1.2.12 (DPI) (as usual). Now $\mathcal{H} \in \mathfrak{F}$, so by assumption (3.3.2) $\lim \mathfrak{F} \in \mathcal{H}$. It follows from Lemma 3.3.1 that there is some $a \in \lim \mathfrak{F}$ such that $[\![a]\!] \subseteq \mathcal{H}$. By construction, every such $[\![a]\!]$ is in the prime ideal disjoint from $\mathfrak{F}$, and thus $[\![a]\!] \notin \mathfrak{F}$. But $a \in \lim \mathfrak{F}$ implies that $[\![a]\!] \in \mathfrak{F}$; contradiction. □

The results of this subsection give us the following:

Lemma 3.3.3. *The diagram*

$$D \xrightarrow{\hat\varepsilon_D} \mathsf{Up}\,\mathsf{P}D \underset{\hat\varepsilon_{\mathsf{Up}\,\mathsf{P}D}}{\overset{\mathsf{Up}\,\mathsf{P}\hat\varepsilon_D}{\rightrightarrows}} \mathsf{Up}\,\mathsf{P}\,\mathsf{Up}\,\mathsf{P}D \tag{3.3.5}$$

is an equaliser of bounded distributive lattices (and thus a coequaliser in $\mathbf{DLat}^{\mathrm{op}}$*).*

Then by Proposition 1.5.4:

Corollary 3.3.4. *The comparison functor* $\hat{\mathcal{K}}: \mathbf{DLat}^{\mathrm{op}} \to \mathbf{Poset}^{\hat\beta}$ *is fully faithful.*

3.3.2 Left adjoint for the comparison functor

We use Theorem 1.5.5 (Dubuc) to construct the left adjoint in the same manner as for the ultrafilter monad.

$$\hat{\mathcal{L}}: \mathbf{Poset}^{\hat\beta} \longrightarrow \mathbf{DLat}^{\mathrm{op}}$$

$$\begin{array}{ccc} (X,\leqslant,\xi) & & \mathsf{Up}\,\mathsf{P}\,\mathsf{Up}(X,\leqslant) \underset{\varepsilon_{\mathsf{Up}(X,\leqslant)}}{\overset{\mathsf{Up}\,\xi}{\rightrightarrows}} \mathsf{Up}(X,\leqslant) \xrightarrow{q} \hat{\mathcal{L}}(X,\leqslant,\xi) \\ \Big\downarrow g & \longmapsto & \Big\uparrow \hat{\mathcal{L}}g \\ (Y,\leqslant,\sigma) & & \mathsf{Up}\,\mathsf{P}\,\mathsf{Up}(Y,\leqslant) \underset{\varepsilon_{\mathsf{Up}(Y,\leqslant)}}{\overset{\mathsf{Up}\,\sigma}{\rightrightarrows}} \mathsf{Up}(Y,\leqslant) \xrightarrow{c} \hat{\mathcal{L}}(Y,\leqslant,\sigma) \end{array} \tag{3.3.1}$$

Proposition 3.3.5. *The counit* $\hat\delta: \hat{\mathcal{L}}\hat{\mathcal{K}} \to 1_{\mathbf{DLat}}$ *is invertible at each* $D \in \mathbf{DLat}$.

Proof. From Eq. (3.3.2) and universal property of the coequaliser. □

Proposition 3.3.6. *The functor* $\hat{\mathcal{L}}: \mathbf{Poset}^{\hat\beta} \to \mathbf{DLat}^{\mathrm{op}}$ *sends each* $\hat\beta$*-algebra* $(X,\leqslant,\hat\xi)$ *to the distributive lattice of its clopen up-sets.*

Proof. Let $(X,\leqslant,\hat\xi)$ be a $\hat\beta$-algebra. By Theorem 1.5.5, the comparison functor's left adjoint $\hat{\mathcal{L}}$ sends $(X,\leqslant,\hat\xi)$ to the coequaliser diagram

$$\hat{\mathcal{L}}(X,\leqslant,\hat\xi) \xleftarrow{\hat q} \mathsf{Up}(X,\leqslant) \underset{\mathsf{Up}\,\hat\xi}{\overset{\hat\varepsilon_{\mathsf{Up}(X,\leqslant)}}{\leftleftarrows}} \mathsf{Up}\,\mathsf{P}\,\mathsf{Up}(X,\leqslant) \tag{3.3.2}$$

where $\hat q$ denotes $\mathrm{coeq}(\mathsf{Up}\,\hat\xi, \hat\varepsilon_{\mathsf{Up}(X,\leqslant)})$ in $\mathbf{DLat}^{\mathrm{op}}$. The analogous equaliser of Boolean algebras gave the clopen subsets of a β-algebra as a subalgebra of the powerset lattice (Proposition 2.3.6). In this case we are working with the up-set lattice rather than the full powerset. It is therefore unsurprising that we want to show that

$$\hat\varepsilon_{\mathsf{Up}\,X}(A) = [\![A]\!] \quad \text{and} \quad \mathsf{Up}\,\hat\xi(A) = \hat\xi^{\leftarrow}(A) \tag{3.3.3}$$

are equal precisely when A is a clopen up-set of the $\hat{\beta}$-algebra $(X, \leqslant, \hat{\xi})$.

Recall that X is equipped with β-algebra structure (namely compact Hausdorff topology) by $\hat{\xi} \circ \rho = \xi$ as in (3.2.9). So by Proposition 2.3.6 (2.3.3), we have $A^{\uparrow} \subseteq X$ clopen just when $\xi^{\leftarrow}(A^{\uparrow}) = [A^{\uparrow}]$. Now from Lemma 3.2.2, Eq. (3.2.6), for any $A \in \mathsf{Up}(X, \leqslant)$,

$$[A^{\uparrow}] = \{\mathcal{U} \in \beta X \,:\, A^{\uparrow} \in \mathcal{U} \cap \mathsf{Up}(X)\} = \rho^{\leftarrow}([\![A]\!]) \tag{3.3.4}$$

Furthermore, since $\hat{\xi} \circ \rho = \xi$, we have

$$\xi^{\leftarrow}(A) = \rho^{\leftarrow}(\hat{\xi}^{\leftarrow}(A)) \tag{3.3.5}$$

and since ρ is epimorphic by Lemma 3.2.1, $\rho^{\leftarrow}$ is monomorphic. Hence $\xi^{\leftarrow}(A^{\uparrow}) = [A^{\uparrow}]$ (*i.e.* A is a clopen up-set) if and only if $[\![A]\!] = \hat{\xi}^{\leftarrow}(A)$. □

Proposition 3.3.7. *For each $\hat{\beta}$-algebra $(X, \leqslant, \hat{\xi})$, the unit $\hat{\gamma} \colon (X, \leqslant, \hat{\xi}) \to \hat{\mathcal{K}}\hat{\mathcal{L}}(X, \leqslant, \hat{\xi})$ is surjective.*

Proof. By the same argument presented in Proposition 2.3.7, it suffices to show that $\hat{K}$ applied to $\hat{q}$ (as defined in Proposition 3.3.6) is a surjective function; *i.e.*, the function $\mathsf{P}\hat{q} \colon \mathsf{P}\,\mathsf{Up}\,X \to \mathsf{P}\hat{\mathcal{L}}(X, \leqslant, \hat{\xi})$ is surjective.

Let $\mathcal{G} \in \mathsf{P}\hat{\mathcal{L}}(X, \leqslant, \hat{\xi})$. We want to show there is some $\mathcal{F} \in \mathsf{P}\,\mathsf{Up}(X, \leqslant)$ such that $\mathsf{P}\hat{q}(\mathcal{F}) = \mathcal{G}$. By Proposition 3.3.6, $\mathcal{G}$ is a prime filter on the lattice (isomorphic to the lattice) of clopen up-sets of the space $(X, \leqslant, \hat{\xi})$. Define the following:

$$\mathcal{J} \coloneqq \{b \in \hat{\mathcal{L}}(X, \leqslant, \hat{\xi}) \,:\, b \notin \mathcal{G}\} \tag{3.3.6}$$

That is, $\mathcal{J}$ is the set complement of $\mathcal{G}$ in $\hat{\mathcal{L}}(X, \leqslant, \hat{\xi})$ and is thus a prime ideal of clopen up-sets. Now $\hat{q}$ embeds $\hat{\mathcal{L}}(X, \leqslant, \hat{\xi})$ into $\mathsf{Up}(X, \leqslant)$ as a subalgebra; preservation of the lattice operations by $\hat{q}$ implies that the image of $\mathcal{G}$ under $\hat{q}$ is a filter and the image of $\mathcal{J}$ under $\hat{q}$ is an ideal. Moreover, injectivity of $\hat{q}$ means this filter and ideal are disjoint subsets of $\mathsf{Up}(X, \leqslant)$, and whence by Theorem 1.2.12 (DPI) extend to a disjoint prime ideal $\boldsymbol{\mathcal{J}}$ and prime filter $\boldsymbol{\mathcal{F}}$.

It remains only to show $\mathsf{P}\hat{q}(\boldsymbol{\mathcal{F}}) = \mathcal{G}$. By definition of P on morphisms, we have

$$\mathsf{P}\hat{q}(\boldsymbol{\mathcal{F}}) = \{c \in \hat{\mathcal{L}}(X, \leqslant, \hat{\xi}) \,:\, \hat{q}(c) \in \boldsymbol{\mathcal{F}}\} \tag{3.3.7}$$

and indeed, every clopen up-set in $\boldsymbol{\mathcal{F}}$ is of the form $\hat{q}(c)$ for some $c \in \mathcal{G}$. Every clopen up-set *not* in $\boldsymbol{\mathcal{F}}$ is of the form $\hat{q}(b)$ for some $b \in \mathcal{J}$, so is not in $\mathcal{G}$ by construction. We conclude $\mathsf{P}\hat{q}(\boldsymbol{\mathcal{F}}) = \mathcal{G}$ as required. □

Lemma 3.3.8. *The unit of the $\hat{\mathcal{K}} \dashv \hat{\mathcal{L}}$ adjunction is an order-reflecting (in particular, injective) mapping of $\hat{\beta}$-algebras*

$$\hat{\gamma} \colon (X, \leqslant, \hat{\xi}) \longrightarrow \hat{\mathcal{K}}\hat{\mathcal{L}}(X, \leqslant, \hat{\xi})$$

if and only if $(X, \leqslant, \hat{\xi})$ is a Priestley space.

Proof. We use essentially the same argument as for Lemma 2.3.8. Let $(X, \leqslant, \hat{\xi})$ be a $\hat{\beta}$-algebra. Then $(X, \leqslant, \hat{\xi})$ is (isomorphic to) a compact pospace by Theorem 3.2.5. Then for all $x \in X$,

$$\begin{aligned}\hat{\gamma}(x) &= \mathsf{P}\hat{q} \circ \hat{\eta}(x) \\ &= \{c \in \hat{\mathcal{L}}(X, \leqslant, \hat{\xi}) \,:\, \hat{q}(c) \in \mathcal{F}^x\} =: A^x.\end{aligned}$$

To say that $\hat{\gamma}$ is order-reflecting is to say that $A^x = \hat{\gamma}(x) \nleq \hat{\gamma}(y) = A^y$ for all $x \nleq y$ in $(X, \leqslant)$. This is the case just when there exists some $c \in A^x$ with $c \notin A^y$, which is equally to say there exists some clopen up-set $\hat{q}(c) \in \mathcal{F}^x$ with $\hat{q}(c) \notin \mathcal{F}^y$. Moreover, $\mathcal{F}^x \nleq \mathcal{F}^y$ in $\mathsf{P}\,\mathsf{Up}(X, \leqslant)$ precisely when $x \nleq y$ in $(X, \leqslant)$, and so every clopen up-set containing x and not y is of this form (Proposition 3.3.6). Thence $A^x \nleq A^y$ just when there exists a clopen up-set $\hat{q}(c) \ni x$ and $\hat{q}(c) \not\ni y$. Thus the Priestley condition (Definition 1.3.18) is shown satisfied by $(X, \leqslant, \hat{\xi})$ precisely when $\hat{\gamma}$ is injective. □

By Corollary 1.3.12 and Proposition 3.3.7, the unit $\hat{\gamma}$ is surjective and therefore a quotient map of compact Hausdorff spaces. Therefore $\hat{\gamma}$ is an isomorphism of the underlying spaces just in case it is injective, and an isomorphism of underlying ordered sets just in case it is order-reflecting. This yields the desired result:

Corollary 3.3.9. *By restricting the adjunction to the objects at which the unit $\hat{\gamma}$ is an isomorphism, we obtain an equivalence of categories* $\mathbf{DLat}^{\mathrm{op}} \simeq \mathbf{Pries}$.

Chapter 4

Aftermath

4.1 Conclusion

In this thesis, the ultrafilter and prime filter monads were used as a means of deriving Stone duality and Priestley duality respectively. The category of algebras for the ultrafilter monad was shown equivalent to the category of compact Hausdorff spaces, and the category of algebras for the prime filter monad was shown equivalent to the category of compact pospaces. We considered the left adjoints of the canonical comparison functors for the monads under consideration. We extracted Stone duality and Priestley duality by restricting their respective comparison functor adjunctions to the algebras at which the unit of the comparison adjunction was an isomorphism.

4.2 Further work

A direct continuation of the work presented in this thesis is obtaining Pontryagin duality of compact 0-dimensional semilattices (also known as HMS spaces [21]). This would be by way of the proper filter monad, which is induced by a contravariant adjunction between sets (or partially ordered sets – the algebras for the monad turn out to be the same) to the category of meet semilattices with identity (bottom). The algebras for this monad are continuous semilattices, or equivalently, compact Lawson semilattices.

We expect that – using the same sort of restriction of the comparison functor adjunction as for Stone and Priestley – will yield Pontryagin duality of compact 0-dimensional semilattices. See [13, 11] for the duality, and [29, 11] for a description of the relevant adjunctions, monads, and algebras.

Given a Boolean algebra B, the canonical extension is obtained from its embedding into the powerset of the ultrafilters of B; this is exactly the mapping on B given by the counit of the powerset-ultrafilter adjunction. Likewise, the canonical extension of a distributive lattice is given by the counit of the adjunction between the up-set and prime filter functors. An overview of canonical extension can be found in [9]. An account of other Stone-type dualities would automatically carry a notion of canonical extension which we hope to investigate further.

An account of canonical extension in the case of Pontryagin duality of compact 0-dimensional semilattices is presented in [21] (where they are referred to as HMS spaces) and in [12].

Generalisations of canonical extension to the categorical setting appear, for example, in the context of coherent and Heyting categories, which are a categorical generalisation of distributive lattices and Heyting algebras respectively [5]. If, indeed, a general construction for canonical extension from filter monads is possible, it would be interesting to see if there is a duality theory in this setting.

We therefore have the following as the most concrete directions for further work:

1. Extend the duality-via-comparison-functor-for-monad construction to the proper filter monads and Pontryagin duality of compact 0-dimensional semilattices (HMS spaces);

2. Give an account of canonical extension for Boolean algebras and distributive lattices using the counit of the ultrafilter and prime filter adjunctions (respectively);

3. Assuming we obtain Pontryagin duality of compact 0-dimensional semilattices, construct canonical extension in this case also.

4. Ascertain whether, or under what conditions, it is possible to extend to a general construction for obtaining canonical extension for Stone-type dualities.

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