Consequences of splitting idempotents

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This note analyses ideas related to an apparently little known result of John Isbell [I]. The result is stated as Corollary 6 below.

A terminal object in a category $\mathcal{X}$ is defined to be a limit for the unique functor from the empty category into $\mathcal{X}$. However, it is easily seen that a terminal object is also a colimit of the identity functor of $\mathcal{X}$. The brings us to our first consequence of idempotents splitting.

Proposition 1 *If $\tau : 1_{\mathcal{X}} \Rightarrow K$ is a cocone over the identity functor of the category $\mathcal{X}$ (that is, τ is a natural transformation from $1_{\mathcal{X}}$ to the constant functor at the object K) and if all idempotents on K split then $\mathcal{X}$ has a terminal object.*

Proof The cocone property means that, for each object X of $\mathcal{X}$, we have an arrow $\tau_X : X \longrightarrow K$ and note $\tau_X\, h = \tau_Y$ for all arrows $h : Y \longrightarrow X$. In particular, taking $h = 1_K$, we see that $e = \tau_K$ is an idempotent on K. So there exist $r : K \longrightarrow T$, $i : T \longrightarrow K$ with $r\, i = 1_T$ and $i\, r = e$. Then T is a terminal object. For, for any X, we have $r\, \tau_X : X \longrightarrow T$, and, if $f : X \longrightarrow T$, then $f = r\, i\, r\, i\, f = r\, e\, i\, f = r\, \tau_K\, i\, f = r\, \tau_X$. **Q.E.D.**

Corollary 2 *If idempotents split in $\mathcal{X}$ and $J : \mathcal{D} \longrightarrow \mathcal{X}$ is a weakly dense functor admitting some cocone over it then $\mathcal{X}$ has a terminal object.*

Proof Since J is weakly dense, $1_{\mathcal{X}}$ is a weak[1] left Kan extension of J along J. We are told there is a cocone $J \Rightarrow K$, and, hence, since K is constant, this can be regarded as a natural transformation $J \Rightarrow KJ$. By the weak Kan property, this gives a natural transformation $1_{\mathcal{X}} \Rightarrow K$. **Q.E.D.**

Beware that a weakly terminal object in $\mathcal{X}$ is not sufficient for a cocone over $1_{\mathcal{X}}$.

From now on we deal with two functors

$$I : \mathcal{A} \longrightarrow \mathcal{X}, \; R : \mathcal{X} \longrightarrow \mathcal{A}$$

and a natural transformation

$$\eta : 1_{\mathcal{A}} \Rightarrow RI : \mathcal{A} \longrightarrow \mathcal{A} .$$

Proposition 3 *If η is invertible and idempotents split in $\mathcal{X}$ then idempotents split in $\mathcal{A}$.*

[1] We use "weak" in the sense of Peter Freyd: drop the uniqueness property from the definition of the concept being qualified.

Proof Let e be an idempotent on A in $\mathcal{A}$. Then Ie is an idempotent on IA in $\mathcal{X}$. So there exist $r : IA \longrightarrow X$, $i : X \longrightarrow IA$ with $r\,i = 1_X$ and $i\,r = e$. Put $s = (Rr)\,\eta_A : A \longrightarrow RX$ and $j = \eta_A^{-1}\,Ri$. Then

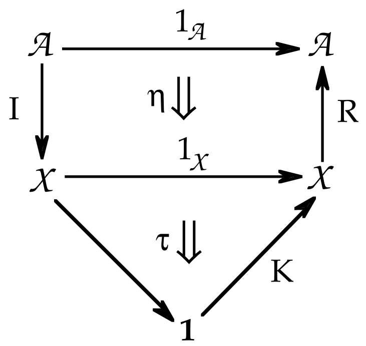
$$s\,j = (Rr)\,\eta_A\,\eta_A^{-1}\,Ri = R(r\,i) = R1_X = 1_{RX} \quad \text{and}$$

$$j\,s = \eta_A^{-1}\,(Ri)\,(Rr)\,\eta_A = \eta_A^{-1}\,(Re)\,\eta_A = \eta_A^{-1}\,\eta_A\,e = e.$$

So s, j provide a splitting for e. **Q.E.D.**

Theorem 4 *If idempotents split in $\mathcal{A}$ and $\mathcal{X}$ has a terminal object then $\mathcal{A}$ has a terminal object.*

Proof Let K denote a terminal object for $\mathcal{X}$; we then have a (universal) cocone $\tau : 1_{\mathcal{X}} \Rightarrow K$. The pasting composite



gives a cocone over $1_{\mathcal{X}}$ with vertex RK. So the result follows from Proposition 1. **Q.E.D.**

Proposition 5 *Suppose idempotents split in $\mathcal{A}$ and suppose $J : \mathcal{D} \longrightarrow \mathcal{S}et$, $S : \mathcal{D} \longrightarrow \mathcal{A}$ are functors. If $\eta S : S \Rightarrow RIS$ is invertible and the J-weighted limit $\lim(J, IS)$ of IS exists in $\mathcal{X}$ then the J-weighted limit $\lim(J, S)$ of S exists in $\mathcal{A}$.*

Proof Let $\mathrm{Cyl}(J, S)$ be the category whose objects are pairs (A, θ) where A is an object of $\mathcal{A}$ and $\theta : J \Rightarrow \mathcal{A}(A, S)$ is a natural transformation; an arrow $f : (A, \theta) \longrightarrow (B, \phi)$ is an arrow $f : A \longrightarrow B$ in $\mathcal{A}$ such that $\theta = \mathcal{A}(f, 1_S)\,\phi$. It is clear that $\lim(J, S)$ is precisely a terminal object of $\mathrm{Cyl}(J, S)$. It is clear also that idempotents split in $\mathrm{Cyl}(J, S)$ since they do in $\mathcal{A}$. The category $\mathrm{Cyl}(J, IS)$ is defined in the obvious way and it does have a terminal object $\lim(J, IS)$.

Define a functor $I' : \mathrm{Cyl}(J, S) \longrightarrow \mathrm{Cyl}(J, IS)$ by $I'(A, \theta) = (IA, \theta')$ where θ' is the composite

$$J \xrightarrow{\ \theta\ } \mathcal{A}(A, S) \xrightarrow{\ I\ } \mathcal{X}(IA, IS),$$

and $I'f = If$. Define a functor $R' : \mathrm{Cyl}(J, IS) \longrightarrow \mathrm{Cyl}(J, S)$ by $R'(X, \xi) = (RX, \xi')$ where ξ' is the composite

$$\mathrm{J} \xrightarrow{\xi} \mathcal{X}(\mathrm{X},\mathrm{IS}) \xrightarrow{\mathrm{R}} \mathcal{A}(\mathrm{RX},\mathrm{RIS}) \xrightarrow{\mathcal{A}(\mathrm{RX},\eta_S^{-1})} \mathcal{A}(\mathrm{RX},\mathrm{S}),$$

and $\mathrm{R'h} = \mathrm{Rh}$. There is a natural transformation $\eta' : 1_{\mathrm{Cyl(J,S)}} \Rightarrow \mathrm{R'I'}$ whose component at (A,θ) is η_A. By Theorem 4, the category $\mathrm{Cyl(J,S)}$ has a terminal object. **Q.E.D.**

Corollary 6 *If* η *is invertible and* $\mathcal{X}$ *is complete or cocomplete then so is* $\mathcal{A}$.

The following standard result [ML; V.5 Exercise 3, p.116] is a consequence.

Corollary 7 *Suppose* I *is fully faithful and* R *is a right adjoint for* I *with unit* η. *If* $\mathcal{X}$ *is complete or cocomplete then so is* $\mathcal{A}$.

Of course, in the situation of Corollary 7 we know more about the construction of individual limits and colimits in $\mathcal{A}$ since I preserves colimits and R preserves limits.

The existence of a right adjoint to a functor $\mathrm{S} : \mathcal{X} \longrightarrow \mathcal{A}$ is equivalent to the existence of a terminal object in the comma categories $\mathrm{S} \downarrow \mathrm{A}$ for all $\mathrm{A} \in \mathcal{A}$. So one might investigate the application of Theorem 4 to adjoint functor theorems. However, we shall relate to adjunction in another way. We begin with a rather silly proof of an interesting observation of Robert Paré [ML; IV.1 Exercise 4, p.84].

Corollary 8 *Suppose* $\mathrm{S} : \mathcal{X} \longrightarrow \mathcal{A}$, $\mathrm{T} : \mathcal{A} \longrightarrow \mathcal{X}$ *are functors and* $\alpha : \mathrm{ST} \Rightarrow 1_\mathcal{A}$, $\beta : 1_\mathcal{X} \Rightarrow \mathrm{TS}$ *are natural transformations such that* $\mathrm{T}\alpha \,.\, \beta\mathrm{T} = 1_\mathrm{T}$. *If idempotents split in* $\mathcal{A}$ *then* T *has a left adjoint.*

Proof Let $[\mathrm{T},\mathcal{A}]/1_\mathcal{A}$ denote the comma category of the functors $[\mathcal{X},\mathcal{A}] \longrightarrow [\mathcal{A},\mathcal{A}]$ given by restriction along T and the functor $\mathbf{1} \longrightarrow [\mathcal{A},\mathcal{A}]$ which picks out the identity functor $1_\mathcal{A}$ of $\mathcal{A}$. Similarly, we have the slice category $[\mathcal{A},\mathcal{A}]/1_\mathcal{A}$ which, as with all slice categories, has a terminal object. Idempotents split in $[\mathrm{T},\mathcal{A}]/1_\mathcal{A}$ since they do in $\mathcal{A}$ and hence in $[\mathcal{X},\mathcal{A}]$.

Define a functor $\mathrm{I} : [\mathrm{T},\mathcal{A}]/1_\mathcal{A} \longrightarrow [\mathcal{A},\mathcal{A}]/1_\mathcal{A}$ by $\mathrm{I}(\mathrm{G},\rho : \mathrm{GT} \Rightarrow 1_\mathcal{A}) = (\rho : \mathrm{GT} \Rightarrow 1_\mathcal{A})$ and a functor $\mathrm{R} : [\mathcal{A},\mathcal{A}]/1_\mathcal{A} \longrightarrow [\mathrm{T},\mathcal{A}]/1_\mathcal{A}$ by $\mathrm{R}(\theta : \mathrm{P} \Rightarrow 1_\mathcal{A}) = (\mathrm{PS},\, \theta \,.\, \mathrm{P}\alpha : \mathrm{PST} \Rightarrow 1_\mathcal{A})$. Using the equation $\mathrm{T}\alpha \,.\, \beta\mathrm{T} = 1_\mathrm{T}$ we see that $\mathrm{G}\beta : (\mathrm{G},\rho) \longrightarrow (\mathrm{GTS},\rho \,.\, \mathrm{GT}\alpha)$ is an arrow of the category $[\mathrm{T},\mathcal{A}]/1_\mathcal{A}$; so these $\mathrm{G}\beta$ are the components of a natural transformation $\eta : 1_\mathcal{A} \Rightarrow \mathrm{RI}$. By Theorem 4, the category $[\mathrm{T},\mathcal{A}]/1_\mathcal{A}$ has a terminal object $(\mathrm{L},\tau : \mathrm{LT} \Rightarrow 1_\mathcal{A})$. There is a unique arrow $\alpha' : (\mathrm{S},\alpha : \mathrm{ST} \Rightarrow 1_\mathcal{A}) \longrightarrow (\mathrm{L},\tau : \mathrm{LT} \Rightarrow 1_\mathcal{A})$ in $[\mathrm{T},\mathcal{A}]/1_\mathcal{A}$ and it is easy to see that L is a left adjoint for T with counit τ and unit $\mathrm{T}\alpha' \,.\, \beta$. **Q.E.D.**

A functor $\mathrm{T} : \mathcal{A} \longrightarrow \mathcal{X}$ is said to be *uniformly continuous with respect to a functor* $\mathrm{J} : \mathcal{C} \longrightarrow \mathcal{D}$ when, for all functors $\mathrm{K} : \mathcal{C} \longrightarrow \mathcal{A}$, $\mathrm{X} : \mathcal{D} \longrightarrow \mathcal{X}$ and all natural transformations $\theta : \mathrm{XJ} \Rightarrow \mathrm{TK}$, there exist a functor $\mathrm{A} : \mathcal{D} \longrightarrow \mathcal{A}$ and natural transformations $\xi : \mathrm{X} \Rightarrow \mathrm{TA}$, $\alpha :$

$AJ \Rightarrow K$ such that $T\alpha . \xi J = \theta$. For his (unpublished, I believe) "Most General Adjoint Functor Theorem", Peter Freyd defined T to be uniformly continuous when it was uniformly continuous with respect to the functors $\mathcal{C} \longrightarrow \mathbf{1}$ with $\mathcal{C}$ small. It is easy to see that, if $\mathcal{A}$ admits all right Kan extensions along J and T respects them, then T is uniformly continuous with respect to J.

Proposition 9 *If* $T : \mathcal{A} \longrightarrow \mathcal{X}$ *has a left adjoint then it is uniformly continuous with respect to all functors. If idempotents split in* $\mathcal{A}$ *and* $T : \mathcal{A} \longrightarrow \mathcal{X}$ *is uniformly continuous with respect to itself then* T *has a left adjoint.*

Proof Let $S : \mathcal{X} \longrightarrow \mathcal{A}$ be the left adjoint for T with unit $\eta : 1 \Rightarrow TS$. Then, for each natural transformation $\theta : XJ \Rightarrow TK$, there is a unique natural transformation $\theta' : SXJ \Rightarrow K$ such that $T\theta' . \eta XJ = \theta$. So we satisfy the definition of uniform continuity with $A = SX$, $\xi = \eta X$ and $\alpha = \theta'$.

For the second sentence of the Proposition, take K to be the identity functor of $\mathcal{A}$, take X to be the identity functor of $\mathcal{X}$, and take θ to be the identity natural transformation of T. Since T is uniformly continuous with respect to T, there exist a functor $S : \mathcal{X} \longrightarrow \mathcal{A}$ and natural transformations $\beta : 1 \Rightarrow TS$, $\alpha : ST \Rightarrow 1$ such that $T\alpha . \beta T = 1$. By Corollary 8, T has a left adjoint. **Q.E.D.**

References

[I] John R. Isbell, Structure of categories, *Bulletin of the American Math. Society* **72** (1966) 619-655.

[ML] Saunders Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Math. 5 (Springer-Verlag, Berlin 1971).