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Rationals, reals, complexes! What next?

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This is a talk about what we are prepared to call **numbers**. As well as the algebraic and geometric aspects of numbers, we will look a little at their history and philosophy. History is a difficult subject. The history of mathematics is even harder: the historian needs to be a mathematician.

The broad history of geometry is probably familiar to you. It provides steps we can look for in any area of mathematics.

Step 1 **Naive subject** The 3D space that we see all around us was the starting point of geometry. Learning about it began by experience and some logical deduction. In such a nebulous framework, contradictions can arise.

Step 2 **Axiomatic method** This involves listing of basic truths. Everything else should be deduced logically from these axioms. Euclid did this for geometry.

Step 3 **Questioning of axioms** Are all the axioms "true"? Are they independent or can one be deduced from the others? Do the axioms completely characterize 3D space? In geometry there was long debate over the Parallels Axiom (that is, there exists a unique line parallel to a given line and through a given point) because it required the concept of the whole line. There were attempts to prove it from the other axioms of Euclid.

Step 4 **Non-standard models** Axioms are shown to be independent by constructing, from an assumed model of the full set of axioms, a model satisfying all but one of the axioms. Early last century, Bolyai and Lobatchewsky did this for the Parallels Axiom. They created Non-Euclidean Geometries – the Parallels Axiom is replaced by something else.

Step 5 **Vastly weakened axioms and classification problems** Curves, surfaces, manifolds, Hilbert space can be studied axiomatically as spaces of a generalised kind. This does not contradict Euclid: it enhances the theory since it tells us more about our original naive space. Many major projects in mathematics have been concerned with the classification of all structures satisfying certain axioms.

Here I am interested in discussing the first few analogous steps in the area of algebra.

Step 1 **Naive subject** Numbers are the naive subject; they are older than history.

Step 2 **Axiomatic method** This required the symbolic method taken to Europe by the Arabs (*Al Jebr* = bone-setting). Axioms therefore came to algebra much later than to geometry. The rules of arithmetic were certainly discussed by Euler (1740s). This was after a hotly disputed struggle about what a number should be: hence we are left with disparaging names such as "irrational" and "imaginary"; but even "zero" and "negatives" were vilified. The axioms for what we now call a ***field*** had become familiar by Euler's time.

FIELD AXIOMS

associative

$(a + b) + c = a + (b + c)$ $\quad\quad$ $(a\,b)\,c = a\,(b\,c)$

units

$a + 0 = a = 0 + a$ $\quad\quad$ $a\,1 = a = 1\,a$

distributive

$$a\,(b + c) = a\,b + a\,c$$
$$(a + b)\,c = a\,c + b\,c$$

cancellation

$a + c = b + c$ *implies* $a = b$ $\quad\quad$ $a\,c = b\,c$ *and* $c \neq 0$ *imply* $a = b$

$c + a = c + b$ *implies* $a = b$ $\quad\quad$ $c\,a = c\,b$ *and* $c \neq 0$ *imply* $a = b$

inverses

$a + (-a) = 0 = (-a) + a$ $\quad\quad$ $a \neq 0$ *implies* $a\,a^{-1} = 1 = a^{-1}\,a$

commutative

$a + b = b + a$ $\quad\quad$ $a\,b = b\,a$

There is some redundancy since **inverses** imply **cancellation**; also **commutative** simplifies some of the earlier axioms.

Here are the number systems you are used to, each an extension of its predecessor.

N is the collection of **natural numbers** 0, 1, 2, 3, 4, . . .

Z is the collection of **integers**[1] $0, \mp 1, \mp 2, \mp 3, \mp 4, \ldots$

Q is the collection of **rational numbers**[2] m/n for m, n integers with $n \neq 0$

R is the collection of **real numbers** represented by decimals such as

$$-\,123456789\cdot 10111213141516171819202122232425\ldots$$

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[1] The German word for number begins with Z.

[2] Presumably Q stands for quotients.

C denotes the collection of complex numbers $a + i b$ where a, b are real.

N satisfies all axioms except "**inverses**".
Z satisfies all axioms except "**inverses**" for multiplication.
Q, **R**, **C** are all fields.

People spoke of a vague and mysterious **Principle of Permanence of Form** (PPF) stating that whenever a number system was extended, while new rules were gained, all the old rules should apply. In fact, this is not a general rule at all: in going from **R** to **C** a gain is that every polynomial of positive degree has a zero (the so-called Fundamental Theorem of Algebra), however the well-behaved order relation is lost.

The mystical introduction to complex numbers (still practised by many teachers and lecturers) is to postulate the existence of some "unknowable" number i with the property that $i^2 = -1$ which is supposed to mix in with the real numbers in such a way that the result still satisfies the field axioms. PPF still lives!

William Rowan Hamilton (Irish mathematician, 3 August 1805 – 2 September 1865) asked (around 1828): "What is next after **C**?"

He (and others) had risen above the mystical description of **C** and realised that **C** was just the Euclidean plane $\mathbf{R}^2$ with a special multiplication. [Why is the mysticism perpetuated by calling this the "Argand diagram"?] As you know, Descartes had identified points on the Euclidean line with real numbers and points in the Euclidean plane with pairs (a, b) of real numbers. There is an obvious addition for points in the plane

$$(a, b) + (c, d) = (a + c,\ b + d)$$

which can be described geometrically by saying that the four points

$$(0, 0), (a, b), (a + c,\ b + d), (c, d)$$

form a parallelogram (the degenerate case where $(0, 0), (a, b), (c, d)$ are collinear reduces to addition on the line). The unit for this addition is the origin $0 = (0, 0)$.

The similarly obvious multiplication of points in the plane is

$$(a, b)\,(c, d) = (a c,\ b d).$$

While this is of some interest, it does **NOT** give a field (the cancellation axiom fails since

$$(1, 0)\,(0, 1) = (0, 0) = 0$$

while $(1, 0) \neq 0$ and $(0, 1) \neq 0$). I shall call this the ***silly multiplication*** for the plane.

The multiplication we want is less obvious:

$$(a, b)\,(c, d) = (a c - b d,\ a d + b c).$$

Geometrically, let r be the length of the segment from 0 to (a, b), and let θ be the angle that segment makes with the x-axis; then $(a, b)\,(c, d)$ is the point obtained by rotating the point (c, d) about the origin through the angle θ and multiplying its distance from the origin by r. This is the ***clever multiplication*** for the plane.

Proposition *The obvious addition and clever multiplication for the plane satisfy the field axioms.*

Complex numbers are **defined** to be points in the plane. As usual, points $(a, 0)$ on the x-axis are identified with real numbers a. There is nothing mystical about i : it is just the point $(0, 1)$ on the y-axis. Using these identifications and our formulas for addition and multiplication, we then find:

$$\begin{aligned} a + i\,b &= (a, 0) + (0, 1)(b, 0) \\ &= (a, 0) + (0, b) \\ &= (a, b) \end{aligned}$$

and

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1.$$

So we are justified in writing complex numbers uniquely in the form $a + i\,b$ with a, b real.

Coming from this point of view, Hamilton was led to ask whether there was a multiplication for points in 3D space which, together with the obvious addition, made 3D Euclidean space $\mathbf{R}^3$ into a field. It seems that he struggled with this for 15 years and failed to find such a multiplication.

Hamilton was a great mathematician and physicist with deep contributions to the theory of rays and to mechanics.

I believe he knew about a multiplication on 4D Euclidean space $\mathbf{R}^4$ for quite a long time but he rejected it because an axiom failed.

Step 3 <u>**Questioning of axioms**</u> Finally, *he dared to question the axiom.* While walking with his wife on 16 October 1843, he carved on a wooden bridge the formulas:

$$i^2 = j^2 = k^2 = i\,j\,k = -1$$

which have the consequences:

$$\begin{aligned} &i\,j = k, \quad &&j\,k = i, \quad &&k\,i = j, \\ &j\,i = -k, \quad &&k\,j = -i, \quad &&i\,k = -j. \end{aligned}$$

In particular, $i\,j = -j\,i$: ***the multiplication is not commutative.***

It took courage to reject this hallowed axiom. Hamilton named his numbers ***quaternions*** and proceeded to market them.

Quotes from E.T. Bell *Men of Mathematics* (Simon & Schuster, 1937)

From page 305

To use Einstein's phrase, <u>Lobatchewsky challenged an axiom.</u> Anyone who challenges an "accepted truth" that has seemed necessary and reasonable to the great majority of sane men for 2000 years or more takes his scientific reputation, if not his life, in his hands.

From pages 360-1

Hamilton's deepest tragedy was neither alcohol nor marriage but his obstinate belief that quaternions held the key to the mathematics of the physical universe. History has shown that Hamilton tragically deceived himself when he insisted " . . . I still must assert that this discovery appears to me to be as important for the middle of the 19th Century as the discovery of fluxions [the calculus] was for the close of the seventeenth". Never was a great mathematician so hopelessly wrong.

I strongly disagree with Bell's assessment. I claim that the first quote applies at least as validly to Hamilton as to Lobatchewsky.

Back to the mathematics. Points in 4D space $\mathbf{R}^4$ can be identified with lists (w, x, y, z) of four real numbers w, x, y, z. But we could equally well think of them as pairs (a, b) of complex numbers a, b via a relationship such as $a = w + i x$, $b = y - i z$. It is not hard to believe that Hamilton tried to mimic the clever multiplication of pairs of real numbers to come up with a clever multiplication of pairs of complex numbers. The ***clever multiplication*** he probably found by experimentation is

$$(a, b)(c, d) = (a c - \bar{b} d, \ \bar{a} d + b c);$$

it only differs from the earlier formula because of the presence of a couple of complex conjugate signs.

Proposition *The obvious addition and clever multiplication for pairs of complex numbers satisfy all the field axioms except for commutativity of multiplication.*

When all field axioms except commutativity of multiplication hold, we say we have a ***division ring.*** So quaternions form a division ring which is denoted by $\mathbf{H}$ in honour of Hamilton[3] .

Identify each complex number a with the pair $(a, 0)$ and, since we already have a meaning for i, put $j = (0, 1)$. As before we can see that

[3] Q is already taken.

$$a + j\,b \;=\; (a, b)$$

for *complex numbers* a, b. If a happens to be *real,* notice that, for any pair (c, d) of complex numbers,

$$\begin{aligned} a\,(c, d) &= (a, 0)\,(c, d) \;=\; (a\,c, \bar{a}\,d) \;=\; (a\,c, a\,d) \\ &= (c\,a, d\,a) \;=\; (c\,a - \bar{d}\,0, d\,a + \bar{c}\,0) \\ &= (c, d)\,a\,. \end{aligned}$$

So the real numbers do commute with all other quaternions. We now introduce the quaternion

$$k \;=\; i\,j \qquad\qquad (\text{so } k = (i, 0)\,(0, 1) = (0, -i)\,).$$

For real numbers w, x, y, z, we see that

$$\begin{aligned} w + x\,i + y\,j + z\,k \;&=\; (w, 0) + (x, 0)\,(i, 0) + (y, 0)\,(0, 1) + (z, 0)\,(0, -i) \\ &=\; (w. 0) + (x\,i, 0) + (0, y) + (0, -z\,i) \\ &=\; (w + x\,i\,, y - z\,i) \\ &=\; (a, b) \end{aligned}$$

where $a = w + x\,i$, $\bar{b} = y + z\,i$. *So every quaternion* $q = (a, b)$ can be written uniquely in the form

$$q \;=\; w + x\,i + y\,j + z\,k.$$

The ***conjugate*** of q is defined by

$$\bar{q} \;=\; w - x\,i - y\,j - z\,k \;=\; (\bar{a}\,, -b)$$

and the ***modulus*** $|q|$ of q is defined to be the non-negative real number satisfying

$$|q|^2 \;=\; w^2 + x^2 + y^2 + z^2 \;=\; |a|^2 + |b|^2.$$

Notice that $q = 0$ if and only if $|q| = 0$. We can calculate

$$q\,\bar{q} \;=\; (a, b)\,(\bar{a}\,, -b) \;=\; (a\,\bar{a} + \bar{b}\,b, b\,\bar{a} - \bar{a}\,b) \;=\; (|a|^2 + |b|^2, 0) \;=\; |q|^2\,.$$

It follows that the multiplicative inverse of q is given by

$$q^{-1} \;=\; \frac{1}{|q|^2}\,\bar{q} \qquad\qquad \text{for } q \neq 0.$$

Hamilton studied 3D geometry using quaternions. Sometimes points in 3D Euclidean space are called ***vectors*** and we identify the vector (x, y, z) with the quaternion

$$\mathbf{r} \;=\; x\,i + y\,j + z\,k.$$

Notice that $|\mathbf{r}|$ is the distance from the origin to the point $\mathbf{r}$. Also, $|\mathbf{r} - \mathbf{r}¢$ is the distance from $\mathbf{r}$ to $\mathbf{r}'$ in 3D space. For $q \;=\; w + x\,i + y\,j + z\,k \;=\; w + \mathbf{r}$,

$\mathcal{Re}(q) \;=\; w$ is called the ***real part*** of q, and

$\mathcal{Ve}(q) \;=\; \mathbf{r}$ is called the ***vector part*** of q.

Hamilton defined two operations on vectors which came out of quaternion multiplication as follows. Take two quaternions $q = w + \mathbf{r}$ and $q' = w' + \mathbf{r}'$. Using the division ring axioms, we deduce that

$$q\,q' = (w + \mathbf{r})\,(w' + \mathbf{r}') = w\,w' + w\,\mathbf{r}' + w'\,\mathbf{r} + \mathbf{r}\,\mathbf{r}'.$$

Let us look at the four terms on the right-hand side:

$w\,w'$ is just the product of the two real numbers;

$w\,\mathbf{r}' = (w\,x', w\,y', w\,z')$ is the obvious vector on the line joining the origin to $\mathbf{r}'$;

$w'\,\mathbf{r} = (w'\,x, w'\,y, w'z)$ is the obvious vector on the line joining the origin to $\mathbf{r}$;

but more interestingly, $\mathbf{r}\,\mathbf{r}'$ is a quaternion having a real part and a vector part.

The **dot product** $\mathbf{r} \bullet \mathbf{r}'$ of the two vectors is defined by

$$\mathbf{r} \bullet \mathbf{r}' = -\mathcal{R}e(\mathbf{r}\,\mathbf{r}') .$$

The **vector product** $\mathbf{r} \times \mathbf{r}'$ is defined by

$$\mathbf{r} \times \mathbf{r}' = \mathcal{V}e(\mathbf{r}\,\mathbf{r}').$$

From the multiplication of quaternions, it is possible to work out explicit algebraic formulas for the dot and vector products:

$$\mathbf{r} \bullet \mathbf{r}' = x\,x' + y\,y' + z\,z'$$

$$\mathbf{r} \times \mathbf{r}' = (y\,z' - z\,y' ,\ z\,x' - x\,z' ,\ x\,y' - y\,x') .$$

There are important geometric meanings for these operations. The dot product allows us to deal with angles in 3D: if the points $\mathbf{r}$ and $\mathbf{r}'$ are at unit distance from the origin $\mathbf{0}$ then $\mathbf{r} \bullet \mathbf{r}'$ is equal to the cosine of the angle subtended by them at the origin. The vector product $\mathbf{r} \times \mathbf{r}'$ is the point whose distance from the origin is the area of the parallelogram with vertices $\mathbf{0}, \mathbf{r}, \mathbf{r} + \mathbf{r}', \mathbf{r}'$ and which lies on the line through the origin perpendicular to the parallelogram (actually there are two such points – the right-hand-screw law picks out the right one).

While we use multiplication-like symbols for dot and vector product, the rules they satisfy are not the familiar division ring axioms. It does not make sense to ask dot product to be associative. Vector product is not associative. Vector product does distribute over addition but it satisfies the funny property that $\mathbf{r} \times \mathbf{r} = \mathbf{0}$.

Exercise *Show that associativity of quaternion multiplication of vectors* $\mathbf{r}, \mathbf{r}', \mathbf{r}''$ *leads to the following formula*

$$(\mathbf{r} \times \mathbf{r}') \times \mathbf{r}'' - \mathbf{r} \times (\mathbf{r}' \times \mathbf{r}'') = (\mathbf{r} \bullet \mathbf{r}')\,\mathbf{r}'' - \mathbf{r}\,(\mathbf{r}' \bullet \mathbf{r}'')$$

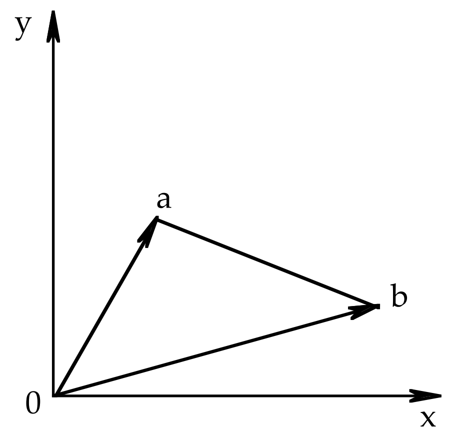
highlighting the failure of associativity for vector product.

I have heard that, in the biggest advance in Physics since Newton (and his "fluxions"), Maxwell originally used quaternions to express his fundamental equations for electromagnetic radiation; only later did he use dot and vector product (which anyway came from quaternions).

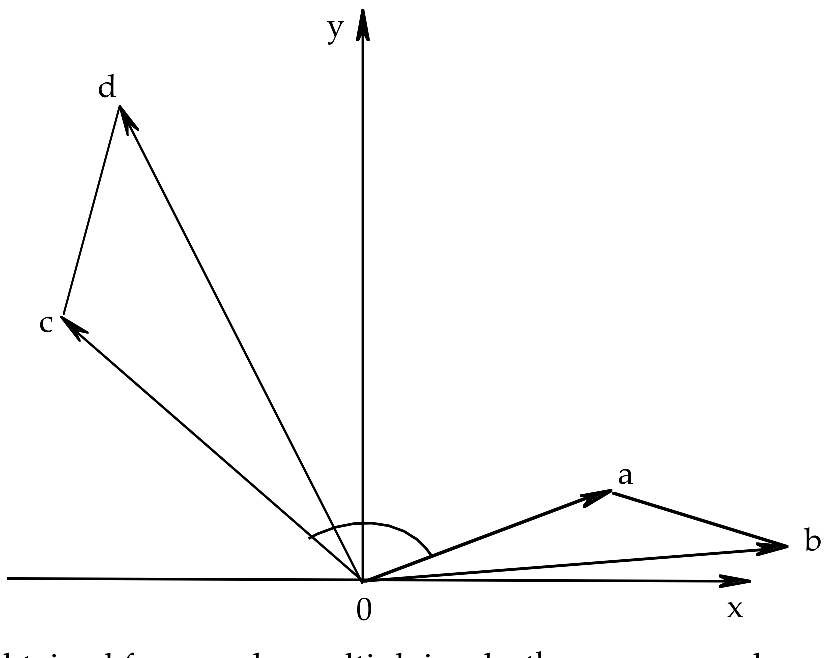
I mentioned that the geometric view of multiplication of complex numbers involves

rotation. I am indebted to my colleague Professor André Joyal FRSC for telling me of a similar interpretation for quaternion multiplication. In fact, we first go back and look at a way of thinking of complex numbers as **quotients** of points in the plane rather than points in the plane themselves. This has to do with the geometry of **similar triangles** (that is, triangles with corresponding angles equal).

Consider triangles in the Euclidean plane with one vertex at the origin 0.



We denote this triangle by a/b with the convention that we regard a/b = c/d when the two triangles are **directly similar** (that is, similar and similarly oriented).



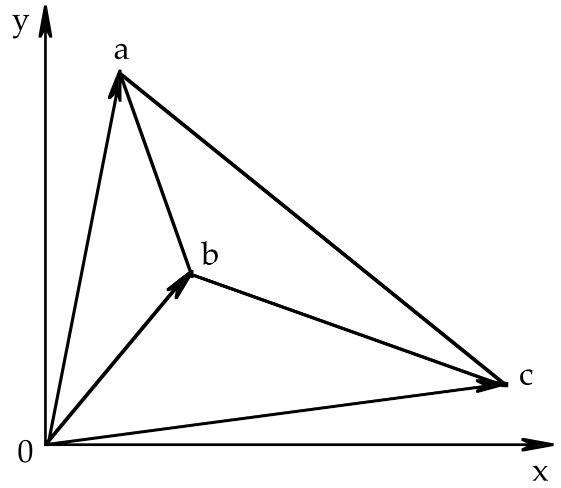
Notice that c is obtained from a by multiplying by the same complex number as needed to obtain d from b: that is,

$$c = u\,a \quad \text{and} \quad d = u\,b \quad \text{for some complex number } u \neq 0.$$

So it is like for rational numbers $u\,a \,/\, u\,b = a \,/\, b$.

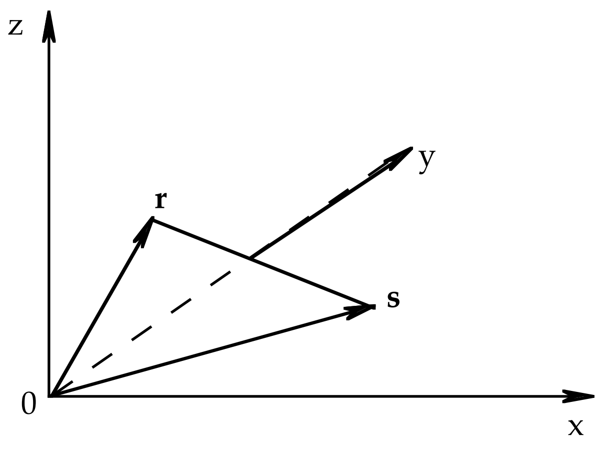
Although we were quite happy with our view of complex numbers as points in the plane, we had to be quite clever to come up with the multiplication. However, there is a one-to-one correspondence between complex numbers and classes of similar triangles. The correspondence takes the complex number a to a/1 while the triangle a/b corresponds to the complex number $a\ b^{-1}$. Under this correspondence, multiplication of complex numbers transforms as follows: rotate and dilate so that the two triangles have one side 0 b in common, then the rule is simply

$$(a/b)\ (b/c) = (a/c).$$



This multiplication of triangles was apparently known to Fermat. Does this mean that Fermat discovered complex numbers? Hardly.

There is a similar situation with quaternions. We do not go to 4D (where there would be difficulty looking at the pictures), rather, 3D. Consider triangles in 3D Euclidean space.



Again we denote such a triangle with $\mathbf{s} \neq \mathbf{0}$ by $\mathbf{r} / \mathbf{s}$ where this time we write

$$\mathbf{r} / \mathbf{s} = \mathbf{r}' / \mathbf{s}'$$

when the two triangles happen to be in the same plane and are directly similar in that plane as before. We can identify $\mathbf{r} / \mathbf{s}$ with the quaternion $q = \mathbf{r}\,\mathbf{s}^{-1}$ and the multiplication is determined by the natural rule:

$$(\mathbf{r} / \mathbf{s})\,(\mathbf{s} / \mathbf{t}) = (\mathbf{r} / \mathbf{t}).$$

Exercise *Show that* $\mathbf{r} / \mathbf{s} = \mathbf{r}' / \mathbf{s}'$ *if and only if* $\mathbf{r}\,\mathbf{s}^{-1} = \mathbf{r}'\,\mathbf{s}'^{-1}$.

When Gauss's notebooks were published posthumously, it was found that he had constructed non-Euclidean geometry and (around 1817) constructed quaternions (which he called ***mutations***). The approach Gauss took to quaternions was in terms of rotations. But Gauss did not market his ideas. I think that is an important point. Gauss's work does not detract from that of Lobatchewsky or Hamilton whose discoveries were independent (as far as we can tell). They also recognized the importance of the discoveries and had the courage to publicly dismiss an axiom. Gauss did not claim priority when he learnt of the work of the other two. Perhaps he had forgotten he had worked on those topics.

Steps 4 and 5 for algebra ran along quickly and more-or-less concurrently with the corresponding steps in geometry. A big step was taken by **George Boole** in allowing symbols to represent *predicates* not just numbers (commutativity still holds).

It took until the middle of this century (**Albert**, **Adams**, **Milnor**) to have proofs that $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ are the only division rings of the form $\mathbf{R}^n$ keeping the obvious addition. In particular, Hamilton was right to dismiss $\mathbf{R}^3$ and stop at $\mathbf{R}^4$.